Premium
SERIES

Devu's
Crown

BOXING
Champion

DEVU BOOKS
SD
Phone : 0474 - 2532617

KING SIZE NOTEBOOKS

PREMIUM QUALITY NOTEBOOK

*Bright and Fine Quality Paper
*Smooth Writing Paper
* 'A' Grade Paper For Long Lasting
*Non Transparent

soorajss1729@gmail.com +91-9400635788

INDEX

9.1

NAME: SOORAJ S. STD.:_____ SEC.:_____ ROLL NO: _____ SUB.: _____

S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		QUANTUM COMPUTATION		
		& QUANTUM INFORMATION		
		- Nielsen & Chuang		

9 DISTANCE MEASURES FOR QUANTUM INFORMATION

What does it mean to say that two items of information are similar?

What does it mean to say that information is preserved by some process?

Static measures – quantify how close two quantum states are

dynamic measures – quantify how well information has been preserved during a dynamic process

Distance measures for classical information.

Hamming distance: the # of places at which two bit strings are not equal.

Ex:-
the bits strings 00010 and 10011 differ in the 1st and last place
⇒ Hamming distance b/w them is 2.

* The Hamming distance b/w 2 objects is simply a matter of labeling, and there aren't any labels in the Hilbert space arena of quantum mechanics!

A much better place to launch the study of distance measures for quantum information is with the comparison of classical probability distributions.

In classical information theory, an information source is usually modeled as a random variable, ie., as a probability distribution over some source alphabet.

Ex:-

an unknown source of English text may be modeled as a sequence of random variables over the Roman alphabet.

What does it mean to say that two probability distributions $\{p_x\}$ and $\{q_x\}$ over the same index set, x, are similar to one another?

Fidelity

Trace distance

The trace distance (L_1 distance / Kolmogorov distance) is defined by,

$$\boxed{D(p_x, q_x) = \frac{1}{2} \sum_x |p_x - q_x|}$$

- The trace distance turns out to be a metric on probability distributions, so the use of the term 'distance' is justified.

- A metric $D(x,y)$ must be symmetric

$D(x,y) = D(y,x)$,

and satisfy the triangle inequality

$D(x,z) \le D(x,y) + D(y,z)$

Ex: 9.1. What's the trace distance b/w the probability distribution $(1,0)$ and the probability distribution $(\frac{1}{2}, \frac{1}{2})$?
B/w $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ and $(\frac{3}{4}, \frac{1}{8}, \frac{1}{8})$?

Ans:

$$D\left((1,0),(\tfrac{1}{2},\tfrac{1}{2})\right) = \frac{1}{2}\left(\left|1-\tfrac{1}{2}\right| + \left|0-\tfrac{1}{2}\right|\right)$$

$$= \frac{1}{2}\left(\left|\tfrac{1}{2}\right| + \left|-\tfrac{1}{2}\right|\right) = \frac{1}{2}$$

$$D\left((\tfrac{1}{2},\tfrac{1}{3},\tfrac{1}{6}),(\tfrac{3}{4},\tfrac{1}{8},\tfrac{1}{8})\right) = \frac{1}{2}\left(\left|\tfrac{1}{2}-\tfrac{3}{4}\right| + \left|\tfrac{1}{3}-\tfrac{1}{8}\right| + \left|\tfrac{1}{6}-\tfrac{1}{8}\right|\right)$$

$$= \frac{1}{2}\left(\left|-\tfrac{1}{4}\right| + \left|\tfrac{5}{24}\right| + \left|\tfrac{1}{24}\right|\right)$$

$$= \frac{1}{2}\left(\frac{6+5+1}{24}\right) = \frac{1}{4}$$

Ex: 9.2
probab
is $|p$

Ans: $D((p,$

Ex: 9.2 Show that the trace distance b/w probability distributions $(p, 1-p)$ and $(q, 1-q)$ is $|p-q|$

Ans: $$D\big((p, 1-p), (q, 1-q)\big) = \frac{1}{2}\big(|p-q| + |-p+q|\big)$$
$$= |p-q|$$

The fidelity of the probability distributions $\{p_x\}$ and $\{q_x\}$ is defined by,

$$\boxed{F(p_x, q_x) = \sum_x \sqrt{p_x q_x}}$$

- Fidelity is not a metric, although later we discuss a metric derived from the fidelity.

Proof

When the distributions $\{p_x\}$ and $\{q_x\}$ are identical,

$$F(p_x, q_x) = F(p_x, p_x) = \sum_x p_x = 1$$

A metric space is a set together with a notion of distance b/w its elements, usually called points. The distance is measured by a function called a metric or distance function.

Definition

A metric space is an ordered pair (M, d) where M is a set and d is a metric ie, a function $d: M \times M \rightarrow \mathbb{R}$.

satisfying the following axioms for all points $x, y, z \in M$:

① The distance from a point to itself is zero

$d(x, x) = 0$ } intuitively, it never costs anything to travel from a point to itself.

② Positivity - The distance b/w 2 distinct points is always +ve

If $x \neq y$, then $d(x, y) > 0$

③ Symmetry – the distance from x to y is always the same as the distance from y to x.

$$d(x,y) = d(y,x)$$

This excludes asymmetric notions of 'cost' which arise naturally from the observation that it's harder to walk uphill than downhill.

④ The triangle inequality holds

$$d(x,z) \leq d(x,y) + d(y,z)$$

This is a natural property of both physical and metaphorical notions of distance: you can arrive at z from x by taking a detour through y, but this will not make your journey any faster than the shortest path.

Ex: 9.3 What's the fidelity of the probability distributions $(1,0)$ and $(1/2, 1/2)$? Of $(1/2, 1/3, 1/6)$ and $(3/4, 1/8, 1/8)$?

Ans: $F\left((1,0),(1/2,1/2)\right) = \sum_x \sqrt{p_x q_x}$

$= \sqrt{1 \times 1/2} + \sqrt{0 \times 1/2} = 1/\sqrt{2}$ //.

$F\left((1/2, 1/3, 1/6),(3/4, 1/8, 1/8)\right) = \sqrt{1/2 \times 3/4} + \sqrt{1/3 \times 1/8} + \sqrt{1/6 \times 1/8}$

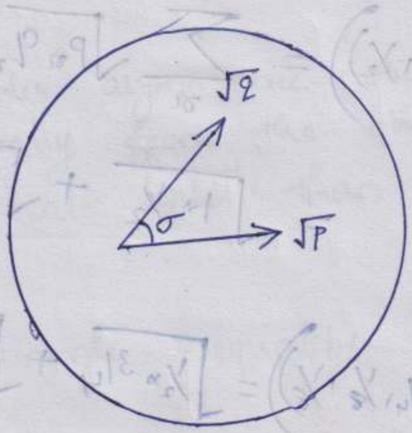
$= \sqrt{\frac{3}{8}} + \sqrt{\frac{1}{24}} + \sqrt{\frac{1}{48}}$

$= \sqrt{\frac{3 \times 6}{48}} + \sqrt{\frac{2}{48}} + \sqrt{\frac{1}{48}}$

$= \frac{\sqrt{3 \times 6 \times 3} + \sqrt{2 \times 3} + \sqrt{3}}{\sqrt{144}}$

$= \frac{3\sqrt{6} + \sqrt{6} + \sqrt{3}}{12} = \frac{4\sqrt{6} + \sqrt{3}}{12}$ //.

* Geometric interpretation of the fidelity as the inner product b/w vectors with components $\sqrt{p_x}$ and $\sqrt{q_x}$, which lie on a unit sphere.



$$F(p,q) = \sqrt{p} \cdot \sqrt{q} = \cos(\sigma)$$

The trace distance and fidelity are mathematically useful means of defining the notion of a distance b/w two probability distributions. Do these measures have physically motivated operational meanings?

trace distance : YES

$$\boxed{\begin{aligned} D(p_x, q_x) &= \max_S |p(S) - q(S)| \\ &= \max_S \left| \sum_{x\in S} p_x - \sum_{x\in S} q_x \right| \end{aligned}} \rightarrow 9.3$$

where the maximization is over all subsets S of the index set $\{x\}$.

The quantity being maximized is the difference b/w the probability that the event S occurs, according to the distribution $\{p_x\}$, and the probability that the event S occurs, according to the distribution $\{q_x\}$.

$\Rightarrow$ The event S is in some sense the optimal event to examine when trying to distinguish the distributions $\{p_x\}$ and $\{q_x\}$, with the trace distance governing how well it is possible to make this distinction.

Unfortunately, a similarly clear interpretation for the fidelity is not known, but still it is a sufficiently useful quantity for mathematical purpose to justify its study, and we cannot rule out the possibility that a clear interpretation of the fidelity will be discovered in the future.

Ex: 9.4 $\quad D(p_x, q_x) = \frac{1}{2}\sum_x |p_x - q_x|$

$$= \max_S |p(S) - q(S)|$$

$$= \max_S \left|\sum_{x\in S} p_x - \sum_{x\in S} q_x\right|$$

Proof

Let $r_x = p_x - q_x$ and U be the whole index set.

$$\max_S |p(S) - q(S)| = \max_S \left|\sum_{x\in S} p_x - \sum_{x\in S} q_x\right|$$

$$= \max_S \left|\sum_{x\in S}(p_x - q_x)\right|$$

$$\therefore \quad = \max_S \left|\sum_{x\in S} r_x\right|$$

$$\sum_{x \in S} r_x = \sum_{\substack{x \in S \\ r_x \geq 0}} r_x + \sum_{\substack{x \in S \\ r_x < 0}} r_x$$

$\left| \sum_{x \in S} r_x \right|$ is maximized when $S = \{x \in U \mid r_x \geq 0\}$ or $S = \{x \in U \mid r_x < 0\}$.

Define $S_+ = \{x \in U \mid r_x \geq 0\}$ and $S_- = \{x \in U \mid r_x < 0\}$

The sum of all r_x is 0,

$$\sum_{x \in U} r_x = \sum_{x \in U} (p_x - q_x) = \sum_{x \in U} p_x - \sum_{x \in U} q_x = 1 - 1 = 0$$

$$\sum_{x \in U} r_x = \sum_{x \in S_+} r_x + \sum_{x \in S_-} r_x = 0$$

$$\therefore \sum_{x \in S_+} r_x = -\sum_{x \in S_-} r_x$$

$$\max_S \left|\sum_{x\in S} r_x\right| = \sum_{x\in S_+} r_x = -\sum_{x\in S_-} r_x$$

$$D(p_x, q_x) = \frac{1}{2}\sum_{x\in U} |p_x - q_x|$$

$$= \frac{1}{2}\sum_{x\in U} |r_x|$$

$$= \frac{1}{2}\sum_{x\in S_+} |r_x| + \frac{1}{2}\sum_{x\in S_-} |r_x|$$

$$= \frac{1}{2}\sum_{x\in S_+} r_x - \frac{1}{2}\sum_{x\in S_-} r_x$$

$$= \frac{1}{2}\sum_{x\in S_+} r_x + \frac{1}{2}\sum_{x\in S_+} r_x$$

$$= \sum_{x\in S_+} r_x$$

$$= \max_S \left|\sum_{x\in S} r_x\right|$$

$$\therefore \quad D(p_x, q_x) = \max_S \left| \sum_{x\in S} p_x - \sum_{x\in S} q_x \right| = \max_S \left| p(S) - q(S) \right|$$

$$\sum_{x\in S} \gamma_x = \sum_{\substack{x\in S \\ \gamma_x \geq 0}} \gamma_x + \sum_{\substack{x\in S \\ \gamma_x < 0}} \gamma_x$$

$$\max_S \left| \sum_{x\in S} \gamma_x \right| = \sum_{x\in S_+} \gamma_x = -\sum_{x\in S_-} \gamma_x \quad (1)$$

$\left| \sum_{x\in S} \gamma_x \right|$ is maximized when $S = \{x \in U \mid \gamma_x \geq 0\}$
or $S = \{x \in U \mid \gamma_x < 0\}$.

$\Rightarrow$ Maximizing $\left| \sum_{x\in S} \gamma_x \right|$ is equivalent to

maximizing $\sum_{x\in S} \gamma_x$

$\therefore$

$$D(p_x, q_x) = \max_S \left(\sum_{x\in S} p_x - \sum_{x\in S} q_x \right)$$

$$= \max_S \left(p(S) - q(S) \right)$$

The trace distance and fidelity are static measures of distance for comparing two fixed probability distributions. There is a 3rd notion of distance which is a dynamic measure of distance in the sense that it measures how well information is preserved by some physical process.

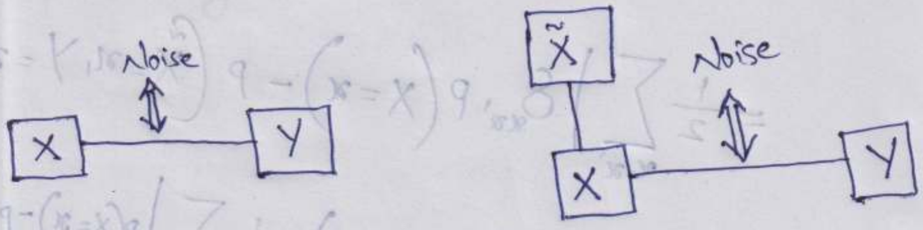
Suppose a random variable X is sent thro' a noisy channel, giving as output another random variable Y, to form a Markov process $X \rightarrow Y$.

For convenience we assume both X and Y have the same range of values, denoted by x.

Then,
the probability that Y is not equal to X, $P(X \neq Y)$, is an obvious and important measure of the degree to which information has been preserved by the process.

This dynamic measure of distance can be understood as a special case of the static trace distance!

Imagine that the random variable X is given to you, and you make a copy of X, creating a new random variable $\tilde{X} = X$. The random variable X now passes through the noisy channel, leaving as output the random variable Y



* Given a markov process $X \to Y$ we may 1st make a copy of X, $\tilde{X}$, before subjecting X to the noise which turns it into Y.

How close is the initial perfectly correlated $(X, \tilde{X})$, to the final pair $(\tilde{X}, Y)$?

Using the trace distance as our measure of closeness,

$$D\left((\tilde{X},X),(X,Y)\right) = \frac{1}{2}\sum_{x,x'}\left|p\left(\tilde{X}=x, X=x'\right) - p\left(\tilde{X}=x, Y=x'\right)\right|$$

$$= \frac{1}{2}\sum_{x,x'}\left|\delta_{xx'}\, p\left(X=x\right) - p\left(\tilde{X}=x, Y=x'\right)\right|$$

$$= \frac{1}{2}\sum_{x\neq x'} p\left(\tilde{X}=x, Y=x'\right) + \frac{1}{2}\sum_{x}\left|p(X=x) - p\left(\tilde{X}=x, Y=x\right)\right|$$

$$= \frac{1}{2}\sum_{x\neq x'} p\left(\tilde{X}=x, Y=x'\right) + \frac{1}{2}\sum_{x}\left(p(X=x) - p\left(\tilde{X}=x, Y=x\right)\right)$$

$$p(X\cap Y) = p(X)\,p(X|Y) \leq p(X)$$

$$= \frac{1}{2}\left(p\left(\tilde{X}\neq Y\right) + 1 - p\left(\tilde{X}=Y\right)\right)$$

$$= \frac{1}{2}\left(p\left(\tilde{X}\neq Y\right) + p\left(\tilde{X}\neq Y\right)\right)$$

$$= p\left(\tilde{X}\neq Y\right) = p\left(X\neq Y\right)$$

$$\boxed{D\left((\tilde{X}, X), (X, Y)\right) = P(X \neq Y)}$$

$\Longrightarrow$ the probability of an error in the channel is equal to the trace distance b/w the probability distribution for $(\tilde{X}, X)$ and $(\tilde{X}, Y)$.

□ How close are two quantum states?

(A) Trace distance

The trace distance b/w quantum states ρ and σ is,

$$\boxed{D(\rho,\sigma) = \frac{1}{2} \operatorname{tr}|\rho - \sigma|}$$

where,

$|A| = \sqrt{A^{\dagger}A}$ to be the +ve square root of $A^{\dagger}A$.

* The quantum trace distance generalizes the classical trace distance in the sense that if ρ and σ commute then the quantum trace distance b/w ρ and σ is equal to the classical trace distance b/w the eigenvalues of ρ and σ.

If ρ and σ commute then they are diagonal in the same basis.

OM 23

$$\rho = \sum_i r_i |i\rangle\langle i| \quad , \quad \sigma = \sum_i s_i |i\rangle\langle i|$$

for some orthonormal basis $|i\rangle$

- The trace distance is,

$$
\begin{aligned}
D(\rho,\sigma) &= \frac{1}{2} \operatorname{tr}|\rho - \sigma| \\
&= \frac{1}{2} \operatorname{tr}\left|\sum_i (r_i - s_i)|i\rangle\langle i|\right| \\
&= \frac{1}{2} \sum_i |r_i - s_i| = D(r_i, s_i)
\end{aligned}
$$

Ex: 9.6 What's the trace distance b/w the density operators

(a) $\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$; $\frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$?

b/w :

(b) $\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$; $\frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|$?

Ans:

(a). $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$; $\sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|$

ρ and σ are simultaneously diagonalized.

$\longrightarrow [\rho, \sigma] = 0$

$$D(\rho, \sigma) = \frac{1}{2} \operatorname{tr}|\rho - \sigma|$$

$$= D(r_i, s_i)$$

$$= D\left(\left(3/4, 1/4\right), \left(2/3, 1/3\right)\right)$$

$$= \frac{1}{2}\left(\left|\frac{3}{4} - \frac{2}{3}\right| + \left|\frac{1}{4} - \frac{1}{3}\right|\right)$$

$$= \frac{1}{2}\left(\frac{1}{12} + \frac{1}{12}\right) = 1/12$$

ⓑ $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$; $\sigma = \frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|$

$$\frac{2}{3}|+\rangle\langle +| = \frac{2}{3}\frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\frac{1}{3}|-\rangle\langle -| = \frac{1}{3}\frac{1}{2}\left(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\sigma = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{6}|0\rangle\langle 1| + \frac{1}{6}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\rho - \sigma = \frac{1}{4}|0\rangle\langle 0| - \frac{1}{6}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) - \frac{1}{4}|1\rangle\langle 1|$$

$$= \begin{bmatrix} 1/4 & -1/6 \\ -1/6 & -1/4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/2 & -1/3 \\ -1/3 & -1/2 \end{bmatrix}$$

$$(\rho-\sigma)^\dagger(\rho-\sigma) = \frac{1}{4}\begin{bmatrix} 1/2 & -1/3 \\ -1/3 & -1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -1/3 \\ -1/3 & -1/2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1/4 + 1/9 & 0 \\ 0 & 1/4 + 1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4^2 + 1/6^2 & 0 \\ 0 & 1/4^2 + 1/6^2 \end{bmatrix}$$

$$= \left(\frac{1}{4^2} + \frac{1}{6^2}\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$|\rho-\sigma| = \sqrt{(\rho-\sigma)^{\dagger}(\rho-\sigma)} = \sqrt{\frac{1}{4^2}+\frac{1}{6^2}\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)}$$

$\therefore$

$$D(\rho,\sigma) = \frac{1}{2} * 2\sqrt{1/4^2 + 1/6^2} = \sqrt{1/4^2 + 1/6^2}$$

Single qubit — Bloch sphere

Suppose ρ and σ have respective Bloch vectors $\vec{r}$ and $\vec{s}$,

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}, \quad \sigma = \frac{I + \vec{s}\cdot\vec{\sigma}}{2}$$

The trace distance b/w ρ and σ is,

$$D(\rho,\sigma) = \frac{1}{2}\,\mathrm{tr}\,|\rho - \sigma|$$

$$= \frac{1}{4}\,\mathrm{tr}\,\left|(\vec{r}-\vec{s})\cdot\vec{\sigma}\right|$$

where, $\vec{r} = (r_1, r_2, r_3)$, $\vec{s} = (s_1, s_2, s_3)$

$$\vec{\sigma} = (X, Y, Z)$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(\vec{r}-\vec{s})\cdot\vec{\sigma} = (r_1 - s_1)X + (r_2 - s_2)Y + (r_3 - s_3)Z$$

$$\vec{a}\cdot\vec{\sigma} = a_1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + a_2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + a_3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}a_3 & a_1 - i a_2\\ a_1 + i a_2 & -a_3\end{bmatrix}$$

$$\mathrm{tr}\left(\vec{a}\cdot\vec{\sigma}\right) = 0 = \lambda_1 + \lambda_2$$

$$\det\left(\vec{a}\cdot\vec{\sigma}\right) = -a_3^2 - \left(a_1^2 + a_2^2\right) = -|\vec{a}|^2 = \lambda_1\lambda_2$$

$$\Rightarrow \lambda_1, \lambda_2 = \pm |\vec{a}|$$

$\therefore$

$(\vec{r}-\vec{s})\cdot\vec{\sigma}$ has eigenvalues $\pm|\vec{r}-\vec{s}|$

$|(\vec{r}-\vec{s})\cdot\vec{\sigma}|$ has eigenvalues both equal to $|\vec{r}-\vec{s}|$

$$\therefore \quad \mathrm{tr}\left|(\vec{r}-\vec{s})\cdot\vec{\sigma}\right| = 2|\vec{r}-\vec{s}|$$

$$D(\rho,\sigma) = \frac{1}{4}\mathrm{tr}\left|(\vec{r}-\vec{s})\cdot\vec{\sigma}\right|$$

$$= \frac{|\vec{r}-\vec{s}|}{2}$$

$\Rightarrow$ the distance b/w 2 single qubit states is equal to ½ the ordinary Euclidean distance b/w them on the Bloch sphere.

Rotations of the Bloch sphere leave the Euclidean distance invariant.

⇒ the trace distance might be preserved under unitary transformations

$$\boxed{D(U\rho U^{\dagger}, U\sigma U^{\dagger}) = D(\rho,\sigma)}$$

(B) Fidelity

A second measure of distance b/w quantum states is the fidelity. The fidelity is not a metric on density operators, but it does give rise to a useful metric.

The fidelity of states ρ and σ is defined to be,

$$\boxed{\begin{aligned} F(\rho,\sigma) &= \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} \\ &= \mathrm{tr}\left|\sqrt{\rho}\sqrt{\sigma}\right| \end{aligned}}$$

* The fidelity is symmetric in its inputs.

Special case

When ρ
in the sa

$\rho = \sum_i r_i$

for som

$F(\rho, \sigma)$

⇒ When ρ
fidelity
fidelity F
distribution

Special case

When ρ and σ commute, ie., are diagonal in the same basis.

$$\rho = \sum_i r_i |i\rangle\langle i| \quad \text{and} \quad \sigma = \sum_i s_i |i\rangle\langle i|$$

for some orthonormal basis $|i\rangle$.

$$F(\rho,\sigma) = \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \mathrm{tr}\left|\sqrt{\rho}\sqrt{\sigma}\right|$$

$$= \mathrm{tr}\sqrt{\sum_i r_i s_i |i\rangle\langle i|}$$

$$= \mathrm{tr}\left(\sum_i \sqrt{r_i s_i}\, |i\rangle\langle i|\right)$$

$$= \sum_i \sqrt{r_i s_i} = F(r_i, s_i)$$

$\Rightarrow$ When ρ and σ commute the quantum fidelity $F(\rho,\sigma)$ reduces to the classical fidelity $F(r_i, s_i)$ b/w the eigenvalue distributions r_i and s_i of ρ and σ.

Special Case

The fidelity b/w a pure state $|\psi\rangle$ and an arbitrary state ρ,

$$F(|\psi\rangle, \rho) = tr\sqrt{|\psi\rangle\langle\psi|\rho|\psi\rangle\langle\psi|}$$

$$= tr\sqrt{\langle\psi|\rho|\psi\rangle|\psi\rangle\langle\psi|}$$

$$= tr\left(\sqrt{\langle\psi|\rho|\psi\rangle}\sqrt{|\psi\rangle\langle\psi|}\right)$$

$$= \sqrt{\langle\psi|\rho|\psi\rangle} \times tr\left(\sqrt{|\psi\rangle\langle\psi|}\right)$$

$$= \sqrt{\langle\psi|\rho|\psi\rangle}$$

⇒ The fidelity is equal to the square root of the overlap b/w $|\psi\rangle$ and ρ.

For the case of a qubit we were able to explicitly evaluate the trace distance b/w 2 states, and give it a simple geometrical interpretation as half the Euclidean distance b/w points on the Bloch sphere.

Unfortunately, no similarly clear geometric interpretation is known for the fidelity b/w 2 states of a qubit.

However, the fidelity does satisfy many of the same properties as the trace distance.

Ex:-

* Fidelity is invariant under unitary transformations

$$\boxed{F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma)}$$

Proof

For a positive operator A,

'A' is hermitian & have non-negative eigenvalues.

A is diagonalizable $\Rightarrow A = VDV^\dagger$

$$A = V\sqrt{D}\sqrt{D}V^\dagger = V\sqrt{D}(V^\dagger V)\sqrt{D}V^\dagger \qquad [V^\dagger V = I$$

$$= (V\sqrt{D}V^\dagger)(V\sqrt{D}V^\dagger)$$

$$= \sqrt{A}\sqrt{A}$$

where $\sqrt{A} = V\sqrt{D}V^\dagger$

$$UAU^\dagger = U\sqrt{A}\sqrt{A}U^\dagger = U\sqrt{A}U^\dagger U\sqrt{A}U^\dagger$$

$$= (U\sqrt{A}U^\dagger)(U\sqrt{A}U^\dagger)$$

$$\Rightarrow \sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger$$

$$F(U\rho U^\dagger, U\sigma U^\dagger) = \operatorname{tr}\sqrt{(U\rho U^\dagger)^{1/2}\, U\sigma U^\dagger\, (U\rho U^\dagger)^{1/2}}$$

$$= \operatorname{tr}\sqrt{U\rho^{1/2}U^\dagger U\sigma U^\dagger U\rho^{1/2}U^\dagger}$$

$$= \operatorname{tr}\sqrt{U\rho^{1/2}\sigma\rho^{1/2}U^\dagger}$$

$$= \operatorname{tr}\left(U\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\,U^\dagger\right)$$

$$= \operatorname{tr}\left(\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\,U^\dagger U\right)$$

$$= \operatorname{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$$

$$= F(\rho, \sigma)$$

Relationship b/w distance measures

The trace distance and the fidelity are closely related, despite their very different forms.

In the case of pure states, the trace distance and the fidelity are completely equivalent to one another.

Consider the trace distance b/w a pure states, $|a\rangle$ and $|b\rangle$.

Using the Gram-Schmidt procedure, we may find orthonormal states $|0\rangle$ and $|1\rangle$ such that,

$$|a\rangle = |0\rangle \quad \text{and} \quad |b\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$$

$$F(|a\rangle, |b\rangle) = \sqrt{\langle a|\rho|a\rangle} = \sqrt{\langle a|b\rangle\langle b|a\rangle}$$
$$= |\langle a|b\rangle| = |\cos\theta|$$

$$\rho = |a\rangle\langle a| = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\sigma = |b\rangle\langle b| = \cos^2\theta |0\rangle\langle 0| + \sin\theta\cos\theta \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) + \sin^2\theta |1\rangle\langle 1|$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$\rho - \sigma = \begin{bmatrix} 1-\cos^2\theta & -\cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix}$$

$$D(|a\rangle, |b\rangle) = \frac{1}{2} tr \left| |a\rangle\langle a| - |b\rangle\langle b| \right|$$

$$= \frac{1}{2} tr \left| \begin{bmatrix} 1-\cos^2\theta & -\cos\theta\sin\theta \\ -\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix} \right|$$

$$= \frac{1}{2} tr \left| \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \right|$$

$D(|a\rangle,|$

$$= \frac{1}{2} tr \sqrt{\sin^2\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{bmatrix}}$$

$$= \frac{1}{2} tr \sqrt{\sin^2\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$= \frac{1}{2} tr \begin{bmatrix} |\sin\theta| & 0 \\ 0 & |\sin\theta| \end{bmatrix}$$

$$= |\sin\theta|$$

$$= \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{1 - F(|a\rangle, |b\rangle)^2}$$

$$\boxed{D(|a\rangle, |b\rangle) = \sqrt{1 - F(|a\rangle, |b\rangle)^2}}$$