Angel
KING

soorajss1729@gmail.com

+91-9400635788

INDEX

③

Name : SOORAJ·S· Subject :

Std. : Div. : Roll No. :

School / College :

S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		QUANTUM COMPUTATION		
		- Nielsen & Chuang		

Quantum Computation

Changes occurring to a quantum state can be described using the language of quantum computation.

Single qubit gates.

Quantum NOT gate acts linearly.

This linear behavior is a general property of Q.M.

Non linear behavior can lead to apparent paradoxes such as time travel, faster than light communication, & violations of the 2nd laws of thermodynamics.

Quantum NOT gate. (bit flip matrix)

$$\boxed{X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \quad \text{(or)} \quad X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

* Quantum gates on a single qubit can be described by 2×2 matrices.

$|\alpha|^2 + |\beta|^2 = 1$ for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\alpha'|^2 + |\beta'|^2 = 1$ for $|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$

after the gate has applied

⟹ The matrix U describing the single qubit gate be unitary, $U^{\dagger}U = I$

⇒ Any unitary matrix specifies a valid quantum gate.

There are many non-trivial single qubit gates.

Z- gate (Phase flip matrix)

$$\boxed{Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|}$$

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

$$Z\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

Hadamard gate

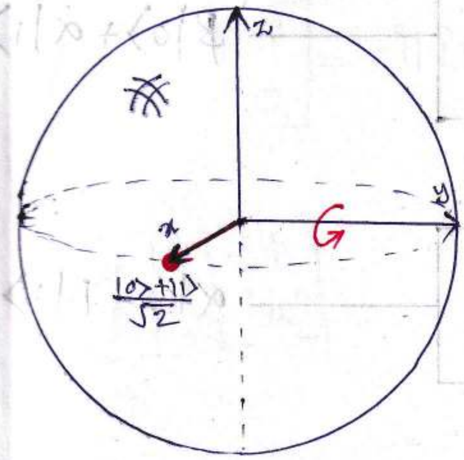
$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

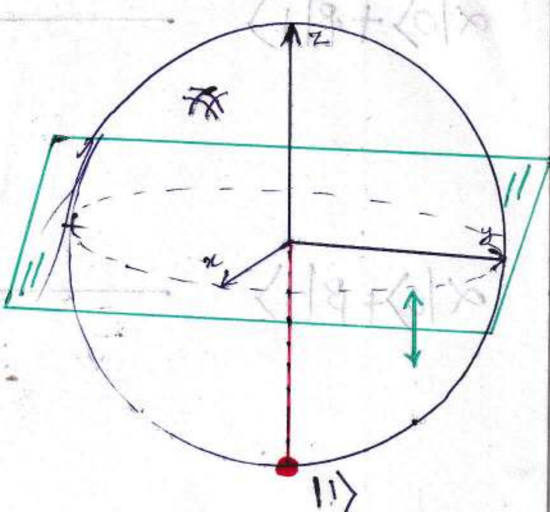
$$= F_2 \ \text{(Fourier matrix)}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

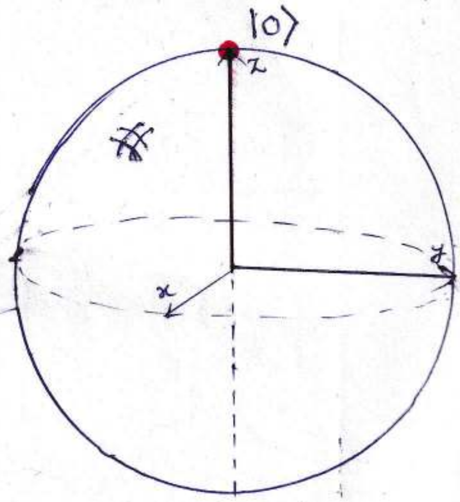
$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$H = \frac{X+Z}{\sqrt{2}}$$





* Visualization of the Hadamard on the Bloch sphere, acting on the i/p state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$

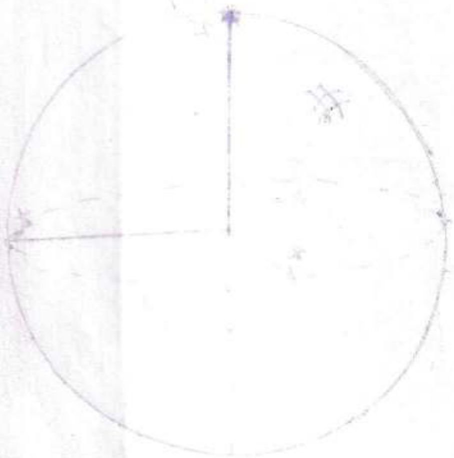


* Hadamard gate operation is just a rotation of the sphere about the y-axis by 90°, followed by a rotation about the x-axis by 180°

$$\alpha|0\rangle+\beta|1\rangle \xrightarrow{\;X\;} \beta|0\rangle+\alpha|1\rangle$$

$$\alpha|0\rangle+\beta|1\rangle \xrightarrow{\;Z\;} \alpha|0\rangle-\beta|1\rangle$$

$$\alpha|0\rangle+\beta|1\rangle \xrightarrow{\;H\;} \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}}+\beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$



Y -

Y = [

Y [α β

Y - gate

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$Y\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}$$

S - gate ($\sqrt{Z}$ gate)

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = T^2$$

T - gate ($\sqrt[4]{Z}$ gate)

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

P-gate (Phase Shift gate)

- family of single qubit gates that map the basis states $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow e^{i\phi}|1\rangle$

The probability of measuring a $|0\rangle$ or $|1\rangle$ is unchanged after applying this gate, however it modifies the phase of the quantum state.

$\Leftrightarrow$ equivalent to tracing a horizontal circle on the Bloch sphere by ϕ radians.

$$P(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

where, ϕ : phase shift with the period 2π

Ex:-

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P(\pi)$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = P(\pi/2) = \sqrt{Z}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = P(\pi/4) = \sqrt[4]{Z}$$

Hadamard	$-\boxed{H}-$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Pauli - X	$-\boxed{X}-$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli - Y	$-\boxed{Y}-$	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli - Z	$-\boxed{Z}-$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Phase	$-\boxed{S}-$	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$	$-\boxed{T}-$	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

* Unitary matrices of the common single qubit gates.

Ex: 4.1. Compute the eigenvalues, eigenvectors of the Pauli matrices. X, Y and Z. Find the points on the Bloch sphere which corresp. to the normalized eigen vectors of the different Pauli matrices.

Ans: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(1)

$$\det(X - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\Longrightarrow \lambda = \pm 1$$

$$\lambda = -1 : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} a + b = 0 \\ a = 1,\ b = -1 \end{array}$$

$$|\lambda_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right)$$

$$\lambda = 1 : \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} a - b = 0 \\ a = 1,\ b = 1 \end{array}$$

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$$

$$|X_1\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right) \quad : \quad (\pi/2, 0)$$

$$|X_{-1}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right) \quad : \quad (\pi/2, \pi)$$

$$\begin{aligned} X &= \lambda_1 |X_1\rangle\langle X_1| + \lambda_{-1} |X_{-1}\rangle\langle X_{-1}| \\ &= \tfrac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) \\ &\quad - \tfrac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \\ &= |0\rangle\langle 1| + |1\rangle\langle 0| \end{aligned}$$

② $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$\det(Y - \lambda I) = \begin{vmatrix} -\lambda & -i \\ i & \lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\Longrightarrow \lambda = \pm 1$$

$\lambda = -1,$

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a - ib = 0 \Rightarrow a = 1,\ b = -i$$

$$|\lambda_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$\lambda = 1,$

$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$ia - b = 0 \Rightarrow a = 1,\ b = i$$

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

③ Z

det (

$\lambda =$

$$Y = \lambda_1 |y_1\rangle\langle y_1| + \lambda_{-1}|y_{-1}\rangle\langle y_{-1}|$$

$$= \frac{1}{2}(|0\rangle + i|1\rangle)(\langle 0| + i\langle 1|) - \frac{1}{2}(|0\rangle - i|1\rangle)(\langle 0| - i\langle 1|)$$

$$= -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\boxed{\begin{aligned} |y_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \quad : \left(\pi/2, \pi/2\right) \\ |y_{-1}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \quad : \left(\pi/2, 3\pi/2\right) \end{aligned}}$$

③ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\det(Z - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = -(1-\lambda^2) = 0$$

$$\Longrightarrow \lambda = \pm 1$$

$\lambda = -1$, $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow$ $2a = 0$. $\boxed{a = 0}$

$$|\lambda_{-1}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$\lambda = 1, \quad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} 2b = 0 \\ \boxed{b = 0} \end{array}$

$|\lambda_{+1}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

$Z = \lambda_1 |z_1\rangle\langle z_1| + \lambda_{-1} |z_{-1}\rangle\langle z_{-1}|$

$= |0\rangle\langle 0| - |1\rangle\langle 1|$

$\lvert z_1\rangle = \lvert 0\rangle$	:	$(0, 0)$
$\lvert z_{-1}\rangle = \lvert 1\rangle$	:	$(\pi, 0)$

Ex: 4.2

matr

Ans: exp

Ex: 4.3. Let x be a real # and A a matrix such that $A^2 = I$. Show that

$$\exp(iAx) = \cos(x)\, I + i \sin(x)\, A$$

Ans: $\exp(iAx) = I + iAx - \frac{x^2}{2!} I - i\frac{x^3}{3!} A + \cdots$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] I + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] A$$

$$= \cos(x)\, I + i \sin(x)\, A$$

$\cancel{\hat{n}\cdot\sigma}$

* $(\hat{n}\cdot\vec{\sigma})^2 = I$

Proof $\hat{n} = (n_x, n_y, n_z)$ is a unit vector

$\vec{\sigma} = (X, Y, Z)$

$$(\hat{n}\cdot\vec{\sigma})^2 = (n_x X + n_y Y + n_z Z)(n_x X + n_y Y + n_z Z)$$

$$= n_x^2 X^2 + n_y Y^2 + n_z Z^2 + n_x n_y (XY + YX)$$
$$+ n_y n_z (YZ + ZY) + n_z n_x (ZX + XZ)$$

$X^2 = Y^2 = Z^2 = I$ & $\{\sigma_i, \sigma_j\} = 2 I \delta_{ij}$

$$= (n_x^2 + n_y^2 + n_z^2) I + 0 = I$$

If $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in 3D, a rotation by θ about the $\hat{n}$ axis is defined by:

$$R_{\hat{n}}(\theta) = \exp\left(-\frac{i\theta}{2}\hat{n}\cdot\vec{\sigma}\right)$$

$$= \cos\theta/2\, I - i\,(\hat{n}\cdot\vec{\sigma})\sin\theta/2$$

where, $\vec{\sigma} = (X, Y, Z)$ of Pauli matrices.

The rotation operators about the $\hat{x}, \hat{y}, \hat{z}$ axes, defined by.

$$R_x(\theta) = e^{-\frac{i\theta}{2}X} = \cos\theta/2\, I - i\sin\theta/2\, X = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_y(\theta) = e^{-\frac{i\theta}{2}Y} = \cos\theta/2\, I - i\sin\theta/2\, Y = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_z(\theta) = e^{-i\theta/2\, Z} = \cos\theta/2\, I - i\sin\theta/2\, Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

M(23)—SU(2) group

Algebra of the Pauli Matrices

$$X^2 = Y^2 = Z^2 = -iXYZ = I \qquad \left[\; X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\; Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\; Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right.$$

$$\left.\begin{aligned} XY &= -YX = iZ \\ YZ &= -ZY = iX \\ ZX &= -XZ = iY \end{aligned}\right\} \quad \begin{aligned} \sigma_i \sigma_j &= \delta_{ij} I + i \epsilon_{ijk} \sigma_k \\ &= \delta_{ij} I + i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k \end{aligned}$$

$$\boxed{(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b}) I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \qquad \text{where, } \vec{\sigma} = (X, Y, Z)}$$

Proof :-

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (a_x X + a_y Y + a_z Z)(b_x X + b_y Y + b_z Z)$$

$$= a_x b_x X^2 + a_y b_y Y^2 + a_z b_z Z^2 + a_x b_y XY + a_x b_z XZ$$
$$+ a_y b_x YX + a_y b_z YZ + a_z b_x ZX + a_z b_y ZY$$

$$= (a_x b_x + a_y b_y + a_z b_z) I + (a_y b_z - b_y a_z) YZ + (a_x b_z - b_x a_z) XZ$$
$$+ (a_x b_y - b_x a_y) XY$$

$$= (a_x b_x + a_y b_y + a_z b_z) I + (a_y b_z - b_y a_z)(iX) - (a_x b_z - b_x a_z) iY$$
$$+ (a_x b_y - b_x a_y) iZ$$

$$= (\vec{a}\cdot\vec{b}) I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{e}_1(a_y b_z - b_y a_z) - \hat{e}_2(a_x b_z - b_x a_z) + \hat{e}_3(a_x b_y - b_x a_y)$$

$$= \epsilon_{ijk}\, e_i\, a_j\, b_k$$

Proof : 2

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \left(\sum_{i=1}^{3} a_i\sigma_i\right)\left(\sum_{j=1}^{3} b_j\sigma_j\right)$$

$$= \sum_{i,j=1}^{3} a_i\sigma_i\, b_j\sigma_j$$

$$= \sum_{i,j=1}^{3} (\sigma_i\sigma_j)\, a_i b_j$$

$$= \sum_{i,j=1}^{3}\left(\delta_{ij} I + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k\right) a_i b_j$$

$$= \left(\sum_{i,j=1}^{3} a_i b_j\right) I + i\sum_{k=1}^{3}\left(\sum_{i,j=1}^{3}\epsilon_{ijk}\, a_i b_j\right)\sigma_k$$

$$= (\vec{a}\cdot\vec{b})I + i\sum_{k=1}^{3}(\vec{a}\times\vec{b})_k\,\sigma_k$$

$$= (\vec{a}\cdot\vec{b})I + (\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Cross Product :

Sarru's law

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, a_{1i}\, a_{2i}\, a_{3i}$$

where,

The Levi-Civita symbol ϵ_{ijk} is anti-symmetric on each pair of indexes, which is a tensor of rank 3.

$$\epsilon_{ijk} = \begin{cases} 0 & \text{, if two labels are the same} \\ 1 & \text{, if } i,j,k \text{ is an even permutation of } 1,2,3. \\ -1 & \text{, if } i,j,k \text{ is an odd permutation of } 1,2,3. \end{cases}$$

The Levi-Civita symbol can be expressed as the determinant, or mixed triple product of any of the unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ of a normalized direct orthogonal frame of reference.

$$\epsilon_{ijk} = \det(\hat{e}_1, \hat{e}_2, \hat{e}_3) = \hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3)$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, \hat{e}_i\, a_j\, b_k$$

$$(\vec{a}\times\vec{b})_i = \epsilon_{ijk}\, a_j\, b_k$$

$$\vec{a}\times\vec{b} = [a]_\times \vec{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\vec{a}\times\vec{b} = [b]_\times^T \vec{a} = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

where,

$$[\vec{a}]_{\times,i} = \vec{a}\times\hat{e}_i \quad , \; i\in\{1,2,3\}$$

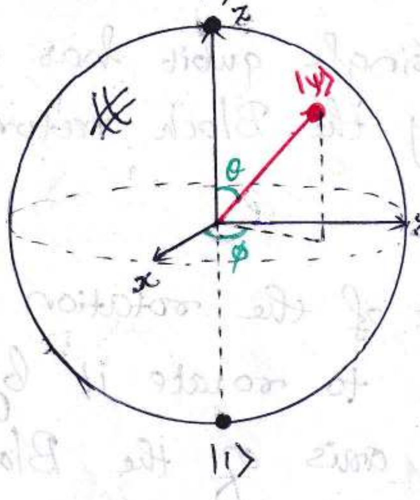
$$[\vec{a}]_\times = \sum_{i=1}^{3} (\vec{a}\times\hat{e}_i) \otimes_{\text{outer}} \hat{e}_i$$, which is a skew-symmetric matrix.

* Blach sphere interpretation of rotation

Suppose a single qubit has a state represented by the Bloch vector $\vec{\lambda}$.

Then,

the effect of the rotation $R_{\hat{n}}(\theta)$ on the state is to rotate it by an angle θ about the $\hat{n}$ axis of the Bloch sphere.



|0⟩
z
|ψ⟩
θ
φ
y
x
|1⟩

Ex:- $|\psi\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle = \begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$

Bloch vector, $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
$= (r_x, r_y, r_z)$

$$|\psi'\rangle = R_z(\delta)|\psi\rangle = \exp\left(-\frac{i\delta}{2}Z\right)\begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix}\begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$$

$$= e^{-i\delta/2}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}\begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$$

$$= e^{-i\delta/2}\begin{bmatrix} \cos\theta/2 \\ e^{i(\phi+\delta)}\sin\theta/2 \end{bmatrix}$$

$$= e^{-i\delta/2}\left[\cos\theta/2\,|0\rangle + e^{i(\phi+\delta)}\sin\theta/2\,|1\rangle\right]$$

→ The new state (up to an unimportant global phase $-\delta/2$) resulting from a ccw rotation of the state vector thro' an angle δ about the z-axis of the Bloch sphere.

⇔ cw rotation of the Bloch sphere itself thro' angle δ.

(OR)

$$R_z(\theta) X R_z^\dagger(\theta) = \left(\cos \theta/2\, I - i\sin\theta/2\, Z\right) X \left(\cos\theta/2\, I + i\sin\theta/2\, Z\right)$$

$$= \cos^2\theta/2\, X + i\sin\theta/2 \cos\theta/2\, XZ - i\sin\theta/2\cos\theta/2\, ZX + \sin^2\theta/2\, ZXZ$$

$$= \cos^2\theta/2\, X + \sin\theta/2\cos\theta/2\, Y + \sin\theta/2\cos\theta/2\, Y - \sin^2\theta/2\, X$$

$$= \left(\cos^2\theta/2 - \sin^2\theta/2\right) X + 2\sin\theta/2\cos\theta/2\, Y$$

$$= \cos\theta\, X + \sin\theta\, Y$$

Similarly,

$$R_z(\theta)\, Y R_z^\dagger(\theta) = \cos\theta\, Y - \sin\theta\, X$$

$$R_z(\theta) Z R_z^\dagger(\theta) = Z$$

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} \qquad \rho' = U\rho U^{\dagger}$$

$$\rho' = R_z(\theta)\,\rho\, R_z^{\dagger}(\theta)$$

$$= R_z(\theta)\,\frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right) R_z^{\dagger}(\theta)$$

$$= R_z(\theta)\,\frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right) R_z^{\dagger}(\theta)$$

$$= \frac{1}{2} I + \frac{1}{2} r_x R_z(\theta) X R_z^{\dagger}(\theta) + \frac{1}{2} r_y R_z(\theta) Y R_z^{\dagger}(\theta) + \frac{1}{2} r_z R_z(\theta) Z R_z^{\dagger}(\theta)$$

$$= \frac{1}{2}\left(I + r_x(\cos\theta X + \sin\theta Y) + r_y(\cos\theta Y - \sin\theta X) + r_z Z\right)$$

$$= \frac{1}{2}\left(I + (r_x\cos\theta + r_y\sin\theta) X + (r_x\sin\theta + r_y\cos\theta) Y + r_z Z\right)$$

$$= \frac{1}{2}\left(I + r_x' X + r_y' Y + r_z' Z\right) = \frac{1}{2}\left(I + \vec{r}'\cdot\vec{\sigma}\right)$$

where, $r_x' = \cos\theta\, r_x - \sin\theta\, r_y$

$r_y' = \sin\theta\, r_x + \cos\theta\, r_y$

$r_z' = r_z$

$$\vec{r}' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}$$

which is the Bloch vector of the new state and the matrix is the usual 3D rotation matrix for a rotation about the z-axis by an angle of θ.

Similarly,

$R_x(\theta)$ and $R_y(\theta)$ perform rotations of the Bloch vector about the x and y-axes by an angle θ.

Ex: 4.3. Show that, up to a global phase, the ~~$\pi/8$~~ gate satisfies $T = R_z(\pi/4)$.

Ans:
$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = e^{i\pi/8} \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} = e^{i\pi/8} R_z(\pi/4)$$

$$R_z(\pi/4) = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{+i\pi/8} \end{bmatrix} \qquad \boxed{T = e^{i\pi/8} R_z(\pi/4)}$$

Ex: 4.4 Express Hadamard gate H as a product of R_x and R_z rotations and $e^{i\phi}$ for some ϕ.

Ans:
$$R_z(\pi/2) R_x(\pi/2) R_z(\pi/2) = \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} \cos\pi/4 & -i\sin\pi/4 \\ i\sin\pi/4 & \cos\pi/4 \end{bmatrix} \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\pi/2}\cos\pi/4 & -i\sin\pi/4 \\ -i\sin\pi/4 & e^{i\pi/2}\cos\pi/4 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\pi/2}\cos\pi/4 & e^{i3\pi/2}\sin\pi/4 \\ e^{i3\pi/2}\sin\pi/4 & e^{i\pi/2}\cos\pi/4 \end{bmatrix} = \frac{e^{-i\pi/2}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boxed{H = e^{i\pi/2} R_z(\pi/2) R_x(\pi/2) R_z(\pi/2)}$$

Ex: 4.7. Show that $\boxed{XYX = -Y}$ and use this to prove that $\boxed{XR_y(\theta)X = R_y(-\theta)}$

Ans: $XY = -YX = iZ$ & $ZX = -XZ = iY$

$XYX = iZX = -Y$

$XR_y(\theta)X = Xe^{-i\frac{\theta}{2}Y}X$

$= e^{i\frac{\theta}{2}Y}X^2$ $\quad$ [$XY = -YX$

$= e^{i\theta/2\,Y} = R_y(-\theta)$ $\quad$ [$X^2 = I$

$$Xe^{-i\theta/2\,Y} = X\left(\cos\theta/2\, I - i\sin\theta/2\, Y\right)$$

$= \cos\theta/2\, X - i\sin\theta/2\, XY$

$= \cos\theta/2\, X - i\sin\theta/2\, -YX$

$= \cos\theta/2\, X + i\sin\theta/2\, YX$

$= \left(\cos\theta/2\, I + i\sin\theta/2\, Y\right)X = e^{i\theta/2\, Y}X$

Ex: 4.8. An arbitrary single qubit unitary operator can be written in the form

$$U = e^{i\alpha} R_{\hat{n}}(\theta).$$

for some real # α & θ, and a real 3D unit vector $\hat{n}$.

① Prove.

②. Find values of $\alpha, \theta, \hat{n}$ giving the Hadamard gate H.

③ Find $\alpha, \theta, \hat{n}$ giving the phase gate.

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Ans:

① Unitary operators transform a state to the other state, both are the point on the Bloch sphere.

OM (23)

$\det(U)=1$
$U = R_{\hat{n}}(\theta)$

$R_{\hat{n}}(\theta)$ can rotate a Bloch vector into any other Bloch vector, which includes all possible qubit states upto a global phase factor.

$\therefore$ An arbitrary single qubit unitary operator can be written in the form:

$\boxed{U = e^{i\alpha} R_{\hat{n}}(\theta)}$, $\alpha, \theta \in \mathbb{R}$ & $\hat{n}$: 3D unit vector

(2) $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\left[\begin{array}{l} H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{array}\right.$

$$R_{\hat{n}}(\theta) = \exp\left(\frac{-i\theta}{2}\hat{n}\cdot\vec{\sigma}\right)$$

$$= \cos\theta/2\, I - i(\hat{n}\cdot\vec{\sigma})\sin\theta/2$$

$\vec{\sigma} = (X, Y, Z)$ $\left[\begin{array}{l} X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ H = \frac{X+Z}{\sqrt{2}}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{array}\right.$

Take, $\alpha = \pi/2$, $\theta = \pi$, $\hat{n} = (1/\sqrt{2}, 0, 1/\sqrt{2})$

$$e^{i\pi/2} R_{(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})}(\pi) = i\Big(\overset{0}{\cos\pi/2}\, I - i\overset{1}{\sin\pi/2}\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right)\Big)$$

$$= \frac{X+Y}{\sqrt{2}} = H$$

(3) $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} (1+i)(1-i) & 0 \\ 0 & (1+i)(1+i) \end{bmatrix}$

$$= (1+i)\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} = (1+i)\Big(\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} - \underbrace{i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{-iZ}\Big)$$

Take $\alpha = \pi/4$, $\theta = \pi/2$, $\hat{n} = (0,0,1)$.

$$e^{i\pi/4} R_{(0,0,1)}(\pi/2) = (\cos\pi/4 + i\sin\pi/4)(\cos\pi/4\, I - i\sin\pi/4\, Z)$$

$$= \frac{1+i}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S$$

* An arbitrary unitary operator on a single qubit can be written in many ways as a combination of rotations, together with global phase shifts on the qubit.

Z-Y decomposition for a single qubit gate

Suppose U is a unitary operation on a single qubit. Then there exists real numbers $\alpha, \beta, \gamma, \delta$ such that

$$\boxed{U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)}$$

Proof

$$e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) = e^{i\alpha} e^{-i\frac{\beta}{2}Z} e^{-i\frac{\gamma}{2}Y} e^{-i\frac{\delta}{2}Z}$$

$$= e^{i\alpha} \begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{bmatrix} \begin{bmatrix} \cos\gamma/2 & -\sin\gamma/2 \\ \sin\gamma/2 & \cos\gamma/2 \end{bmatrix} \begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix}$$

$$= e^{i\alpha} \begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{bmatrix} \begin{bmatrix} e^{-i\delta/2}\cos\gamma/2 & -e^{i\delta/2}\sin\gamma/2 \\ e^{-i\delta/2}\sin\gamma/2 & e^{i\delta/2}\cos\gamma/2 \end{bmatrix}$$

$$I=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad X=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad Y=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad Z=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$= e^{i\alpha}\begin{bmatrix} e^{-i\beta/2}e^{-i\delta/2}\cos\gamma/2 & -e^{-i\beta/2}e^{i\delta/2}\sin\gamma/2 \\ e^{i\beta/2}e^{-i\delta/2}\sin\gamma/2 & e^{i\beta/2}e^{i\delta/2}\cos\gamma/2 \end{bmatrix}$$

$$= e^{i\alpha}\begin{bmatrix} e^{i(-\beta/2-\delta/2)}\cos\gamma/2 & -e^{i(-\beta/2+\delta/2)}\sin\gamma/2 \\ e^{i(\beta/2-\delta/2)}\sin\gamma/2 & e^{i(\beta/2+\delta/2)}\cos\gamma/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cdot\cos\gamma/2 & -e^{i(\alpha-\beta/2+\delta/2)}\cdot\sin\gamma/2 \\ e^{i(\alpha+\beta/2-\delta/2)}\cdot\sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)}\cdot\cos\gamma/2 \end{bmatrix}$$

Since U is unitary, the row and columns of U are orthonormal, from which it follows that there exists real numbers $\alpha, \beta, \gamma, \delta$ such that

$$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cdot\cos\gamma/2 & -e^{i(\alpha-\beta/2+\delta/2)}\cdot\sin\gamma/2 \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\gamma/2 & e^{i(\alpha+\beta/2+\delta/2)}\cos\gamma/2 \end{bmatrix}$$

$$= e^{i\alpha} R_z(\beta)\, R_y(\gamma)\, R_z(\delta)$$

$$e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) =$$

$$= e^{i\alpha} \begin{bmatrix} \left(\cos\left(\frac{\beta+\delta}{2}\right) - i\sin\left(\frac{\beta+\delta}{2}\right)\right)\cos\gamma/2 & -\left(\cos\left(\frac{\beta-\delta}{2}\right) - i\sin\left(\frac{\beta-\delta}{2}\right)\right) \times \sin\gamma/2 \\ \left(\cos\left(\frac{\beta-\delta}{2}\right) + i\sin\left(\frac{\beta-\delta}{2}\right)\right)\sin\gamma/2 & \left(\cos\left(\frac{\beta+\delta}{2}\right) + i\sin\left(\frac{\beta+\delta}{2}\right)\right)\cos\gamma/2 \end{bmatrix}$$

$$= e^{i\alpha} \begin{bmatrix} \cos\left(\frac{\beta+\delta}{2}\right)\cos\gamma/2 & 0 \\ 0 & \cos\left(\frac{\beta+\delta}{2}\right)\cos\gamma/2 \end{bmatrix}$$

$$+ e^{i\alpha} \begin{bmatrix} 0 & i\sin\left(\frac{\beta-\delta}{2}\right)\sin\gamma/2 \\ i\sin\left(\frac{\beta-\delta}{2}\right)\sin\gamma/2 & 0 \end{bmatrix} + e^{i\alpha} \begin{bmatrix} 0 & -\cos\left(\frac{\beta-\delta}{2}\right) \times \sin\gamma/2 \\ \cos\left(\frac{\beta-\delta}{2}\right) \times \sin\gamma/2 & 0 \end{bmatrix}$$

$$+ e^{i\alpha} \begin{bmatrix} -i\sin\left(\frac{\beta+\delta}{2}\right)\cos\gamma/2 & 0 \\ 0 & i\sin\left(\frac{\beta+\delta}{2}\right)\cos\gamma/2 \end{bmatrix}$$

$$= e^{i\alpha} \Big[\cos\left(\frac{\delta+\beta}{2}\right)\cos\gamma/2 \, I - i\sin\left(\frac{\delta-\beta}{2}\right)\sin\gamma/2 \, X$$

$$- i\cos\left(\frac{\delta-\beta}{2}\right)\sin\gamma/2 \, Y - i\sin\left(\frac{\delta+\beta}{2}\right)\cos\gamma/2 \, Z$$

$$U = e^{i\alpha'} R_{\hat{n}}(\theta) = e^{i\alpha'}\left[\cos\theta/_2\, I - i(\hat{n}\cdot\vec{\sigma})\sin\theta/_2\right]$$

$$= e^{i\alpha'}\left[\cos\theta/_2\, I - i\left(n_x X + n_y Y + n_z Z\right)\sin\theta/_2\right]$$

Setting, $\alpha' = \alpha$, $\cos\theta/_2 = \cos\left(\frac{\delta+\beta}{2}\right)\cos\gamma/_2$,

$n_x \sin\theta/_2 = \sin\left(\frac{\delta-\beta}{2}\right)\sin\gamma/_2$, $n_y \sin\theta/_2 = \cos\left(\frac{\delta-\beta}{2}\right)\sin\gamma/_2$

$n_z \sin\theta/_2 = \sin\left(\frac{\delta+\beta}{2}\right)\cos\gamma/_2$

$$U = e^{i\alpha} R_{\hat{n}}(\theta) = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

Ex: 4.10

$X-$

Supp

a

num

Proof.

Ex: 4.10.

X-Y decomposition for a single qubit gate

Suppose U is a unitary operation on a single qubit. Then there exists real numbers $\alpha, \beta, \gamma, \delta$ such that

$$\boxed{U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\delta)}$$

Proof.

Suppose $\hat{m}$ and $\hat{n}$ are non-parallel real unit vectors in 3D. Then an arbitrary single qubit unitary U gate may be written as:

$$\boxed{U = e^{i\alpha} R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{\hat{n}}(\delta)}$$

for appropriate choices of $\alpha, \beta, \gamma, \delta$.

Proof.

Stark
21/9/21.

$$U' = e^{ia} R_{\hat{m}}(t) = e^{ia} R_z(b) R_y(c) R_z(d)$$

$$\Longrightarrow R_y(c) = (R_z(b))^{-1} R_{\hat{m}}(t) (R_z(d))^{-1}$$

$$\boxed{R_y(c) = R_z(-b) R_{\hat{m}}(t) R_z(-d)} \rightarrow \text{Eq. 1}$$

$$U = e^{i\alpha'} R_z(b') R_y(c) R_z(d') \quad \text{for real parameters, } \alpha', b', c', d'$$

$$= e^{i\alpha'} R_z(b') \cdot R_z(-b) R_{\hat{m}}(t) R_z(-d) \cdot R_z(d')$$

$$= e^{i\alpha'} R_z(b'-b) R_{\hat{m}}(t) R_z(d'-d)$$

$$= e^{i\alpha'} R_z(\beta'') R_{\hat{m}}(\gamma'') R_z(\delta'')$$

$$\boxed{\begin{aligned} U = e^{i\alpha'} R_{\hat{n}}(\phi) &= e^{i\alpha'} R_z(\beta') R_y(\gamma') R_z(\delta') \\ &= e^{i\alpha'} R_z(\beta'') R_{\hat{m}}(\gamma'') R_z(\delta'') \end{aligned}} \rightarrow (2.2)$$

(b) From eq^n ①, we can take

$$R_{\hat{n}}(\beta) = e^{ia} R_z(b) R_y(c) R_z(d) \quad \text{and}$$

$$R_{\hat{n}}(\delta) = e^{ia'} R_z(b') R_y(c') R_z(d')$$

and $$R_{\hat{m}}(\gamma) = e^{ih} R_z(e) R_y(f) R_z(g)$$

$$e^{i\alpha} R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{\hat{n}}(\delta) =$$

$$= e^{i\alpha} \cdot e^{ia} R_z(b) R_y(c) R_z(d) \cdot R_{\hat{m}}(\gamma) \cdot e^{ia'} R_z(b') R_y(c') R_z(d')$$

$$= e^{i(\alpha+a+a')} R_z(b) \left[R_y(c) R_z(d) R_{\hat{m}}(\gamma) R_z(b') R_y(c')\right] R_z(d')$$

$$= e^{i\alpha'} R_z(b) \left[R_y(c) R_z(d) R_{\hat{m}}(\gamma) R_z(b') R_y(c')\right] R_z(d')$$

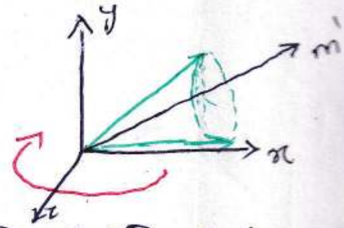
From (Eq. 2), we can take $R_z(d) R_{\hat{m}}(\gamma) R_z(b') = R_{\hat{m}'}(\theta)$

& therefore

$$R_y(c) \underbrace{R_z(d) R_{\hat{m}}(\gamma) R_z(b')}R_y(c') = R_y(c) R_{\hat{m}'}(\theta) R_y(c')$$

which corresponds to rotating the frame about y-axis by c', then rotate about the unit vector $\hat{m}'$ by θ, and rotate the frame about y-axis by c.

i.e., it is some rotation about a unit vector $\hat{N}$ by some angle ϵ.



$$R_y(c) R_z(d) R_{\hat{m}}(\gamma) R_z(b') R_y(c') = R_y(c) R_{\hat{m}'}(\theta) R_y(c')$$

$$= R_{\hat{N}}(\epsilon)$$

$$\therefore e^{i\alpha} R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{\hat{n}}(\delta) = e^{i\alpha'} R_z(b) \cdot R_{\hat{N}}(\epsilon) \cdot R_z(d')$$

$$= e^{i\alpha'} R_{\hat{\gamma}}(\eta) = U$$

i.e.,

Any single qubit unitary operator $U = e^{i\alpha'} R_{\hat{r}}(\eta)$ can be written as $U = e^{i\alpha} R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{n}(\delta)$ for some real # $\alpha, \beta, \gamma, \delta$ and $\hat{n}, \hat{m}$ are non-parallel unit vectors.

×

Suppose U is a unitary gate on a single qubit. Then there exists unitary operators A, B, C on a single qubit such that

$$ABC = I \quad \text{and} \quad U = e^{i\alpha} AXBXC$$

where, α is some overall phase factor

Proof

Set, $A = R_z(\beta) R_y(\gamma/2)$, $B = R_y(-\gamma/2) R_z\left(-\frac{(\delta+\beta)}{2}\right)$

$C = R_z\left(\frac{\delta-\beta}{2}\right)$

$$ABC = R_z(\beta) R_y(\gamma/2) R_y(-\gamma/2) R_z\left(\frac{-\delta-\beta}{2}\right) R_z\left(\frac{\delta-\beta}{2}\right)$$

$$= R_z(\beta)\, I\, R_z(-\beta) = I$$

Since $X^2 = I$,

$$XBX = X R_y(-\gamma/2) X X R_z\left(\frac{-\delta-\beta}{2}\right) X$$

$$= R_y(\gamma/2) X \cdot XX \cdot X R_z\left(\frac{\delta+\beta}{2}\right) = R_y(\gamma/2) R_z\left(\frac{\delta+\beta}{2}\right)$$

$$AXBXC = R_z(\beta) R_y(\gamma/2) \cdot R_y(\gamma/2) R_z\left(\frac{\delta+\beta}{2}\right) \cdot R_z\left(\frac{\delta-\beta}{2}\right)$$

$$= R_z(\beta) R_y(\gamma) R_z(\delta)$$

$$e^{i\alpha} AXBXC = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

$\therefore$

$U = e^{i\alpha} AXBXC$ and $ABC = I$, as required

4.

Ex: 4.12 Give A, B, C and α for the Hadamard gate.

Ans: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Ex: 4.4

$$R_x(\pi/2) R_z(\pi/2) R_x(\pi/2) = e^{i\frac{\pi}{4}X} \cdot e^{i\frac{\pi}{4}Z} \cdot e^{i\frac{\pi}{4}X}$$

$$= \frac{I+iX}{\sqrt{2}} \times \frac{I+iZ}{\sqrt{2}} \times \frac{I+iX}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \left[\left(I + i(X+Z) - XZ \right)(I+iX) \right]$$

$$= \frac{1}{2\sqrt{2}}\left[\cancel{I} + iX + iZ - \cancel{XZ} + iX - \cancel{X^2} - \cancel{ZX} - iXZX\right]$$

$$= \frac{1}{2\sqrt{2}}\left[2iX + iZ + iZX^2\right] = \frac{i(X+Z)}{\sqrt{2}} = iH$$

$$\boxed{\begin{aligned} H &= -i\, R_x(\pi/2)\, R_z(\pi/2)\, R_x(\pi/2) \\ &= e^{-i\pi/2}\, R_x(\pi/2)\, R_z(\pi/2)\, R_x(\pi/2) \end{aligned}}$$

$$A = R_x(\pi/2)\, R_z(\pi/4) = e^{-i\frac{\pi}{4}X} \cdot e^{-i\frac{\pi}{8}Z}$$

$$B = R_z(-\pi/4)\, R_x(-\pi/2) = e^{i\frac{\pi}{8}Z} \cdot e^{i\frac{\pi}{4}X}$$

$$C = R_x(0) = I$$

$$ABC = e^{-i\frac{\pi}{4}X} \cdot e^{-i\frac{\pi}{8}Z} \cdot e^{i\frac{\pi}{8}Z} \cdot e^{i\frac{\pi}{4}X} = I$$

Ex: 4.13 Circuit Identities.

$$\boxed{HXH = Z \;;\; HYH = -Y \;;\; HZH = X}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof.

$$H = \frac{X+Z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$HXH = \frac{1}{2}(X+Z)X(X+Z) = \frac{1}{2}(X+Z)(I+XZ)$$

$$= \frac{1}{2}(X+Z+Z+ZXZ) = \frac{1}{2}(X+2Z-X) = Z$$

$$HYH = \frac{1}{2}(X+Z)Y(X+Z) = \frac{1}{2}(X+Z)(YX+YZ)$$

$$= \frac{1}{2}(XYX+XYZ+ZYX+ZYZ)$$

$$= \frac{1}{2}(-Y-\cancel{ZYX}+\cancel{ZYX}-Y) = -Y$$

$$HZH = \frac{1}{2}(X+Z)Z(X+Z) = \frac{1}{2}(X+Z)(ZX+I)$$

$$= \frac{1}{2}(\cancel{XZX}+X+X+\cancel{Z}) = X$$

Ex: 4.14 Show that $\boxed{HTH = R_x(\pi/4)}$

Ans: $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

Ex: 4.3

$R_z(\phi) = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$

$\boxed{T = e^{i\pi/8} R_z(\pi/4)}$

$\Rightarrow T = e^{i\pi/8} \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} = e^{i\pi/8} R_z(\pi/4)$

$HTH = H \cdot e^{i\pi/8} R_z(\pi/4) H$

$= e^{i\pi/8} H R_z(\pi/4) H$

$= e^{i\pi/8} H e^{-i\frac{\pi}{8}Z} H$

$= e^{i\pi/8} H \left[I - \frac{i\pi}{8} Z - \frac{1}{2!}\left(\frac{\pi Z}{8}\right)^2 + \cdots \right] H$

$= e^{i\pi/8} \left[H^2 - \frac{i\pi}{8} HZH - \frac{1}{2!}\left(\frac{\pi}{8}\right)^2 HZ^2H - \cdots \right]$

$HXH = Z \Longrightarrow HZH = X$

$\Longrightarrow Z^2 = HX^2H \Rightarrow HZ^2H = X^2$

$$\therefore \; HTH = e^{i\pi/8}\left[I - i\frac{\pi}{8}X - \frac{1}{2!}\left(\frac{\pi}{8}\right)^2 X^2 + \cdots\right]$$

$$= e^{i\pi/8} \cdot e^{-i\frac{\pi}{8}X} = \underline{\underline{e^{i\pi/8} R_x(\pi/4)}}$$

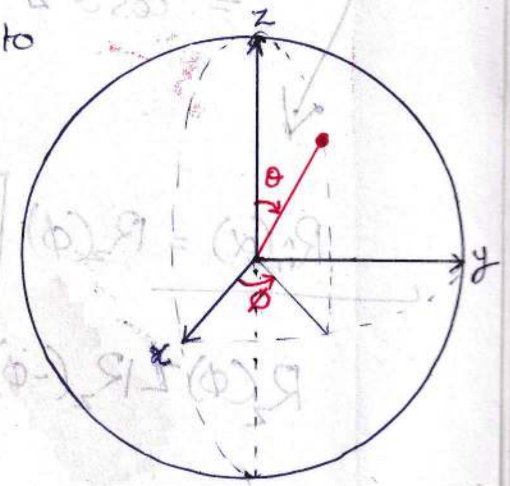
The operator for rotation by angle α about an arbitrary axis $\hat{n}$ can be contructed using the rotation operators about the y and z-axes, as:

$$R_{\hat{n}}(\alpha) = R_z(\phi) R_y(\theta) \cdot R_z(\alpha) \cdot R_y(-\theta) R_z(-\phi)$$

$$= R_z(\phi) R_y(\theta) \cdot R_z(\alpha) \cdot (R_y(\theta))^\dagger (R_z(\phi))^\dagger$$

$$= \cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2} (\hat{n} \cdot \vec{\sigma})$$

- $R_z(-\phi)$ first rotates $\hat{n}$ into the xz plane, $R_y(-\theta)$ then rotates it into the z-axis, $R_z(\alpha)$ performs the desired rotation about $\hat{n}$, and $R_y(\theta)$ and $R_z(\phi)$ rotate $\hat{n}$ back to its original orientation.



$$\hat{n} = (n_x, n_y, n_z)$$
$$= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\phi = \tan^{-1}\left(\frac{n_y}{n_x}\right)$$
$$\theta = \cos^{-1}(n_z)$$

$$R_{\hat{n}}(\alpha) = R_z(\phi)\, R_y(\theta)\, R_z(\alpha)\, R_y(-\theta)\, R_z(-\phi)$$

$$= R_z(\phi) R_y(\theta) \left[\cos\frac{\alpha}{2} I - i \sin\frac{\alpha}{2} Z\right] R_y(-\theta) R_z(-\phi)$$

$$R_y(\theta)\, Z\, R_y(-\theta) = \left[\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y\right] Z \left[\cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} Y\right]$$

$$= \cos^2\frac{\theta}{2} Z + \sin^2\frac{\theta}{2} YZY - i \sin\frac{\theta}{2}\cos\frac{\theta}{2}\big(\underbrace{YZ - ZY}_{2iX}\big)$$

$$= \cos\theta\, Z + \sin\theta\, X$$

$$R_{\hat{n}}(\alpha) = R_z(\phi) \left[\cos\frac{\alpha}{2} I - i \sin\frac{\alpha}{2} \left(\cos\theta\, Z + \sin\theta\, X\right)\right] R_z(-\phi)$$

$$R_z(\phi)\, Z\, R_z(-\phi) = R_z(\phi) R_z(-\phi) Z = Z$$

$$R_z(\phi)\, X\, R_z(-\phi) = R_z(2\phi) X = e^{-i\phi Z} X$$

$$= \left(\cos\phi\, I - i \sin\phi\, Z\right) X$$

$$= \cos\phi\, X + \sin\phi\, Y$$

$$R_{\hat{n}}(\alpha) = \cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2}\left(\cos\theta\, Z + \sin\theta\left(\cos\phi\, X + \sin\phi\, Y\right)\right)$$

$$= \cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2}\left(\sin\theta\cos\phi\, X + \sin\theta\sin\phi\, Y + \cos\theta\, Z\right)$$

$$= \cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2}\left(n_x X + n_y Y + n_z Z\right)$$

$$= \cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2}\left(\hat{n}\cdot\vec{\sigma}\right)$$

* If a rotation thro' an angle β_1 about the axis $\hat{n}_1$ is followed by a rotation thro' an angle β_2 about an axis $\hat{n}_2$, then the overall rotation is thro' an angle β_{12} about an axis $\hat{n}_{12}$ is given by:

$$c_{12} = c_1 c_2 - s_1 s_2 \, \hat{n}_1 \cdot \hat{n}_2$$

$$s_{12}\hat{n}_{12} = s_1 c_2 \hat{n}_1 + c_1 s_2 \hat{n}_2 - s_1 s_2 \, \hat{n}_2 \times \hat{n}_1$$

where, $c_i = \cos(\beta_i/2)$, $s_i = \sin(\beta_i/2)$, $c_{12} = \cos(\beta_{12}/2)$

$s_{12} = \sin(\beta_{12}/2)$

(OR)

$$\cos(\beta_{12}/2) = \cos(\beta_1/2)\cos(\beta_2/2) - \sin(\beta_1/2)\sin(\beta_2/2)\, \hat{n}_1 \cdot \hat{n}_2$$

$$\hat{n}_{12} = \frac{\sin(\beta_1/2)\cos(\beta_2/2)\hat{n}_1 + \cos(\beta_1/2)\sin(\beta_2/2)\hat{n}_2 - \sin(\beta_1/2)\sin(\beta_2/2)\,\hat{n}_2 \times \hat{n}_1}{\sin(\beta_{12}/2)}$$

Stack
23/9/21

Proof

$$R_{\hat{n}_{12}}(\beta_{12}) = R_{\hat{n}_2}(\beta_2)\, R_{\hat{n}_1}(\beta_1)$$

$$= \left[\cos(\beta_1/2)\, I - i\sin(\beta_1/2)(\hat{n}_1\cdot\vec{\sigma})\right]\left[\cos(\beta_2/2)\, I - i\sin(\beta_2/2)(\hat{n}_2\cdot\vec{\sigma})\right]$$

$$= \cos(\beta_1/2)\cos(\beta_2/2)\, I - \sin(\beta_1/2)\sin(\beta_2/2)(\hat{n}_1\cdot\sigma_1)(\hat{n}_2\cdot\vec{\sigma})$$
$$- i\left[\sin(\beta_1/2)\cos(\beta_2/2)(\hat{n}_1\cdot\vec{\sigma}) + \cos(\beta_1/2)\sin(\beta_2/2)(\hat{n}_2\cdot\vec{\sigma})\right]$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\, I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

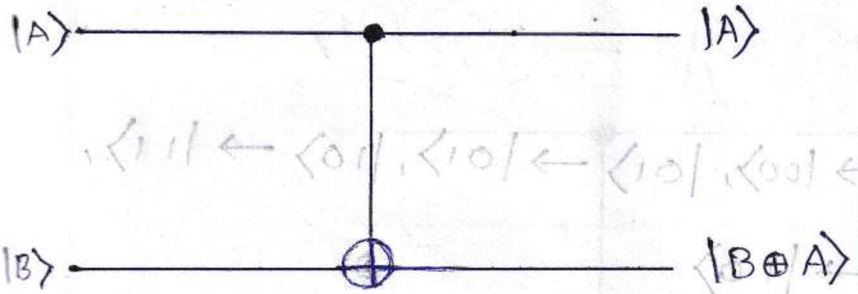
$$= \cos(\beta_1/2)\cos(\beta_2/2)\, I - \sin(\beta_1/2)\sin(\beta_2/2)\left[(\hat{n}_1\cdot\hat{n}_2)\, I + i(\hat{n}_1\times\hat{n}_2)\cdot\vec{\sigma}\right]$$
$$- i\left[\sin(\beta_1/2)\cos(\beta_2/2)(\hat{n}_1\cdot\vec{\sigma}) \quad \cos(\beta_1/2)\sin(\beta_2/2)(\hat{n}_2\cdot\sigma)\right]$$

$$= \left[\cos(\beta_1/2)\cos(\beta_2/2) - \sin(\beta_1/2)\sin(\beta_2/2)\, \hat{n}_1\cdot\hat{n}_2\right] I$$
$$- i\left[\sin(\beta_1/2)\cos(\beta_2/2)\,\hat{n} + \cos(\beta_1/2)\sin(\beta_2/2)\,\hat{n}_2 - \sin(\beta_1/2)\sin(\beta_2/2)\,\hat{n}_2\times\hat{n}_1\right]\cdot\vec{\sigma}$$

$$R_{\hat{n}_{12}}(\beta_{12}/2) = \cos(\beta_{12}/2)\, I - i \sin(\beta_{12}/2)(\hat{n}_{12}\cdot\vec{\sigma})$$

Controlled Operations

Controlled - NOT (or) CNOT gate



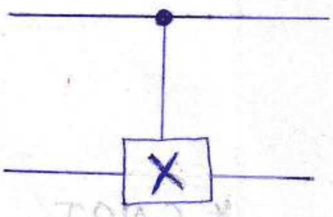
$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle$

(OR)



$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

where, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ & $X|0\rangle = |1\rangle$

$X|1\rangle = |0\rangle$

In terms of the computational basis, the action of the CNOT is given by

$|c\rangle|t\rangle \longrightarrow |c\rangle|t \oplus c\rangle$ (or) $|c,t\rangle \rightarrow |c, t \oplus c\rangle$.

i.e., if the control qubit is set to $|1\rangle$ then the target qubit is flipped, otherwise the target qubit is left alone.

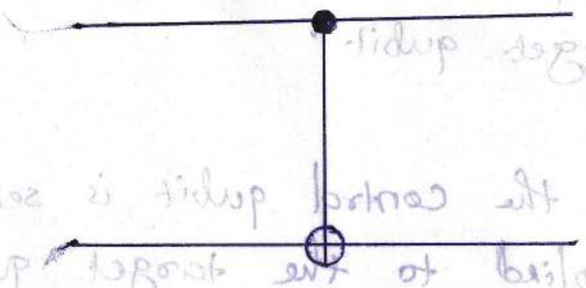
$|00\rangle \rightarrow |00\rangle$, $|01\rangle \rightarrow |01\rangle$, $|10\rangle \rightarrow |11\rangle$, $|11\rangle \rightarrow |10\rangle$

* CNOT is as a generalization of the classical XOR gate

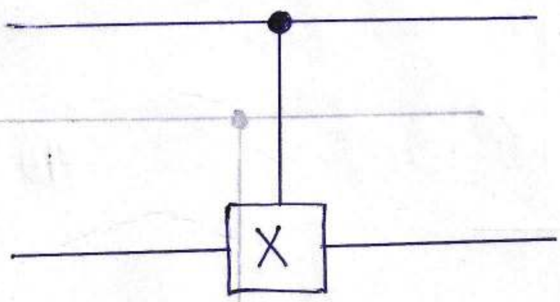
$\oplus$: addition modulo two

* Circuit representation for the controlled - NOT gate-

- The top line represents the control qubit, the bottom line the target qubit



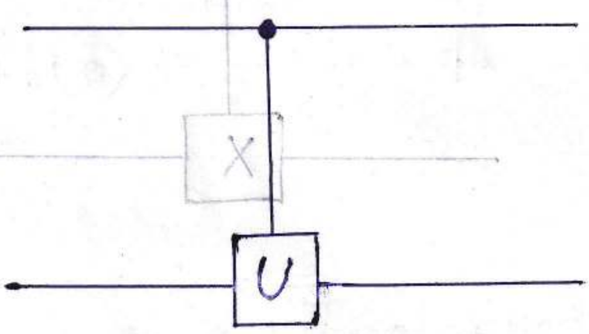
(OR)



* Suppose U is an arbitrary single qubit unitary operation.
A controlled-U operation is a two qubit operation, with a control and a target qubit.

i.e., If the control qubit is set then U is applied to the target qubit, otherwise the target qubit is left alone.

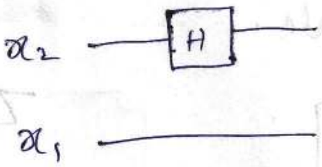
that is, $|c\rangle|t\rangle \longrightarrow |c\rangle U^{c}|t\rangle$



* Controlled-U operation

Ex: 4.16

(1) What is the 4×4 unitary matrix for the circuit



is the computational basis ?

$$|x_1\rangle|x_2\rangle \longrightarrow |x_1\rangle H |x_2\rangle$$

Ans: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Longrightarrow \begin{matrix} H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\ H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \end{matrix}$

$$I \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(2). What is the unitary matrix for the circuit

Ans :

$$H \otimes I = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Ex: 4.17 CNOT from controlled Z gates

Controlled Z gate.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

Construct a CNOT gate from one controlled-Z gate and 2 Hadamard gates, specifying the control & target qubits.

Ans:

From Ex: 4.13, $HZH = X$

Controlled H - gate

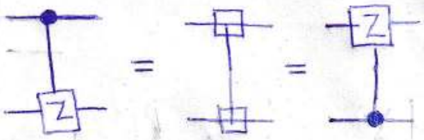
$$U_{CH} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Controlled Z-gate

$$U_{CZ} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad Z|0\rangle = |0\rangle$

$\quad Z|1\rangle = -|1\rangle$

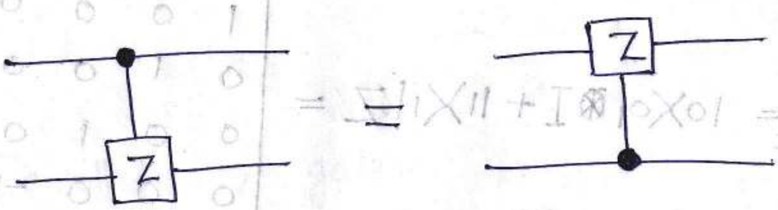
$HZH = X$

$$U_{CH}\, U_{CZ}\, U_{CH} = \left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes H\right)\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z\right)\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes H\right)$$

$$= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes HZH$$

$$= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = U_{CN}$$

Ex: 4.18. Show that



Ans: $U_{Z_{1,2}}$: controlled Z-gate which takes the 1st qubit as the controll qubit and the 2nd qubit as the target qubit.

$$U_{Z_{1,2}}\left(|\psi_1\rangle\otimes|\psi_2\rangle\right)=\left(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes Z\right)\left[\left(a_1|0\rangle+b_1|1\rangle\right)\otimes\left(a_2|0\rangle+b_2|1\rangle\right)\right]$$

$$=\left(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes Z\right)\left(a_1a_2|0\rangle\otimes|0\rangle+a_1b_2|0\rangle\otimes|1\rangle+b_1a_2|1\rangle\otimes|0\rangle+b_1b_2|1\rangle\otimes|1\rangle\right)$$

$$=a_1a_2|0\rangle\otimes|0\rangle+a_1b_2|0\rangle\otimes|1\rangle+b_1a_2|1\rangle\otimes|0\rangle-b_1b_2|1\rangle\otimes|1\rangle$$

$$U_{Z_{2,1}}\left(|\psi_1\rangle\otimes|\psi_2\rangle\right)=\left(I\otimes|0\rangle\langle 0|+Z\otimes|1\rangle\langle 1|\right)\left[\left(a_1|0\rangle+b_1|1\rangle\right)\otimes\left(a_2|0\rangle+b_2|1\rangle\right)\right]$$

Ex: 4.19

$\rho_2 =$

$\rho' =$

$$= (I \otimes |0\rangle\langle 0| + Z \otimes |1\rangle\langle 1|)\Big[(a_1 a_2 |0\rangle \otimes |0\rangle + a_1 b_2 |0\rangle \otimes |1\rangle + b_1 a_2 |1\rangle \otimes |0\rangle + b_1 b_2 |1\rangle \otimes |1\rangle)\Big]$$

$$= a_1 a_2 |0\rangle \otimes |0\rangle + a_1 b_2 |0\rangle \otimes |0\rangle + b_1 a_2 |1\rangle \otimes |0\rangle - b_1 b_2 |1\rangle \otimes |1\rangle$$

Ex: 4.19 CNOT action on density matrices

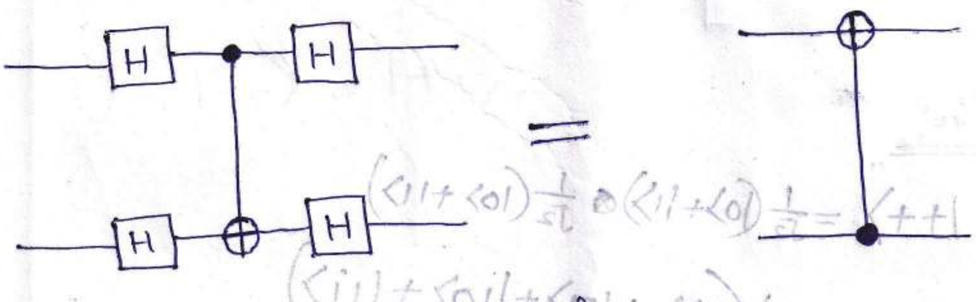
$$U_{CN} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U^{+} = U, \quad UU^{+} = I$$

$$\rho' = \frac{U_{CN} \rho U_{CN}}{Tr(U_{CN} \rho)}$$

Ex: 4.20 CNOT Basis transformation

The role of 'control' and 'target' are arbitrary – they depend on what basis we are operating with.

For the basis states, $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$



* CNOT gate in Hadamard transformed basis

$H|0\rangle = |+\rangle$ & $H|+\rangle = |0\rangle$

$H|1\rangle = |-\rangle$ & $H|-\rangle = |1\rangle$

Q.C (1) End.

The effect of a CNOT with the 1st qubit as control & the 2nd qubit as target on the basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ is as follows:

$$|+\rangle \otimes |+\rangle \longrightarrow |+\rangle \otimes |+\rangle$$

$$|-\rangle \otimes |+\rangle \longrightarrow |-\rangle \otimes |+\rangle$$

$$|+\rangle \otimes |-\rangle \longrightarrow |-\rangle \otimes |-\rangle$$

$$|-\rangle \otimes |-\rangle \longrightarrow |+\rangle \otimes |-\rangle$$

Proof

$$|++\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$U_{CN}|++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle)$$

$$= |++\rangle$$

$$|-+\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

$$U_{CN}|-+\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |11\rangle - |10\rangle).$$

$$= |-+\rangle.$$

$$|+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$U_{CN}|+-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |11\rangle - |10\rangle).$$

$$= |--\rangle$$

$$U_{CN}|--\rangle = |+-\rangle.$$

→ In the basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ the target and control have essentially interchanged roles.

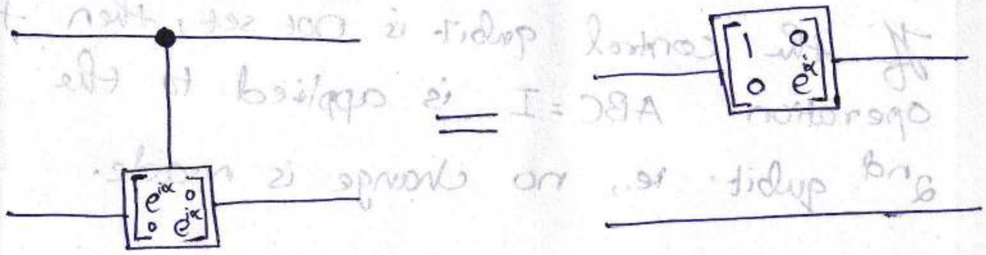
□ How to implement the controlled U operation for arbitrary single qubit U, using only single qubit operations & the CNOT gate ?

$U = e^{i\alpha} AXBXC$ & $ABC = I$

1st step: apply the phase shift $e^{i\alpha}$ on the target qubit, controlled by the control qubit.

i.e., if the control qubit is $|0\rangle$, then the target qubit is left alone, while if the control qubit is $|1\rangle$, a phase shift $e^{i\alpha}$ is applied to the target.

$|00\rangle \to |00\rangle, |01\rangle \to |01\rangle, |10\rangle \to e^{i\alpha}|10\rangle, |11\rangle \to e^{i\alpha}|11\rangle$



* Controlled phase shift gate & an equivalent circuit for 2 qubits.

$P(\alpha)|0\rangle = |0\rangle$

$P(\alpha)|1\rangle = e^{i\alpha}|1\rangle$

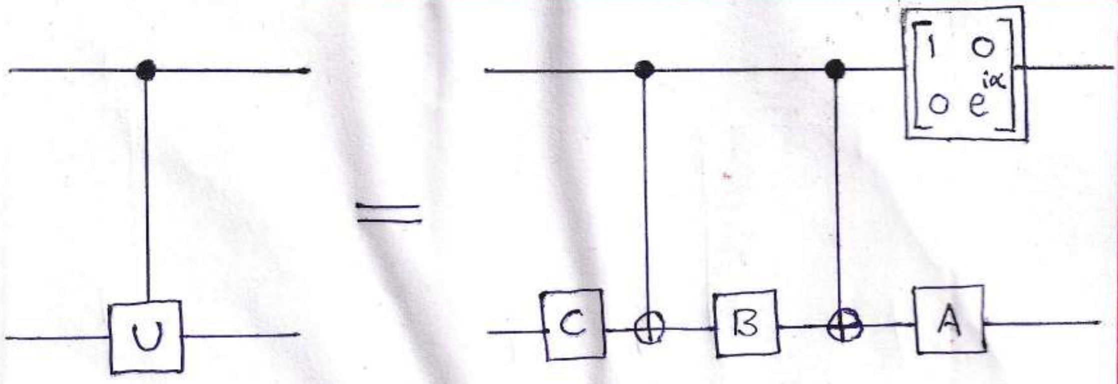
$$U = e^{i\alpha} AXBXC$$

where A, B, C are single qubit operations such as $ABC = I$.

The operation $e^{i\alpha} AXBXC = U$ is applied to the second qubit if the control qubit is set.

If the control qubit is not set, then the operation $ABC = I$ is applied to the 2nd qubit. i.e., no change is made.

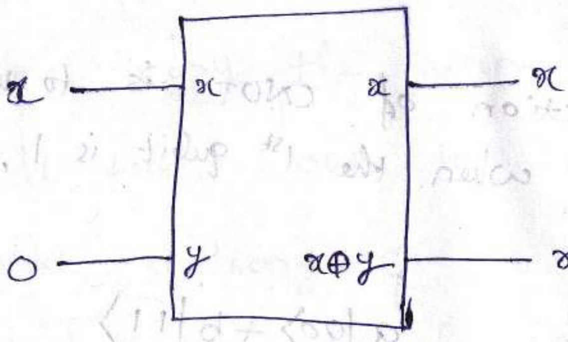
$\Rightarrow$ This circuit implements the controlled-U operation.

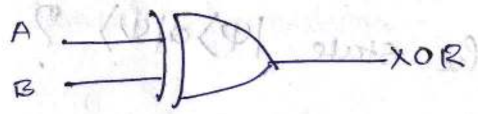


* Circuit implementation of the controlled-U operation for single qubit U.
α, A, B, C satisfy $U = e^{i\alpha} AXBXC$, $ABC = I$.

Qubit copying circuit ?

The task of copying a classical bit may be done using a classical CNOT gate, which takes in the bit to copy (in some unknown state x) and a 'scratch pad' bit initialized to zero. The o/p is two bits, both of which are in the same state x.





X = A ⊕ B

A	B	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Copy a qubit in the unknown state $|\psi\rangle = a|0\rangle + b|1\rangle$ in the same manner by using a CNOT gate. ?

~~[illegible]~~ the i/p state of the 2 qubits $\left.\right\}$ $\left[a|0\rangle + b|1\rangle\right]|0\rangle = a|00\rangle + b|10\rangle$

The function of CNOT is to negate the 2nd qubit when the 1st qubit is 1, and thus the

o/p state $\left.\right\}$ $a|00\rangle + b|11\rangle$

Have we copied $|\psi\rangle$?

Have we created the state $|\psi\rangle \otimes |\psi\rangle$?

It is possible to use quantum circuits to copy classical information encoded as a $|0\rangle$ or a $|1\rangle$.

However, for a general state $|\psi\rangle$,

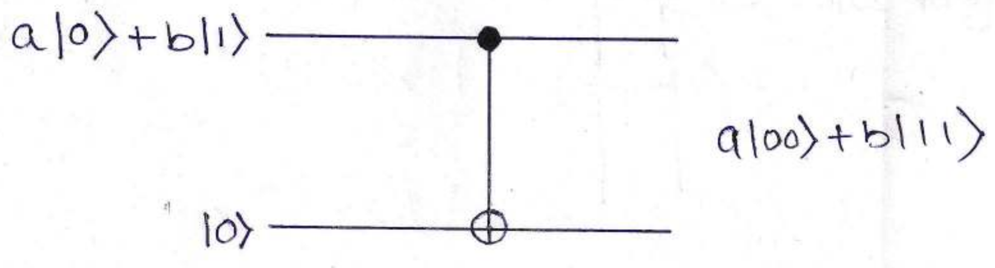
$$|\psi\rangle \otimes |\psi\rangle = a^2|00\rangle + ab\,|01\rangle + ab\,|10\rangle + b^2|11\rangle$$

Comparing with $a|00\rangle + b|11\rangle$

Unless $ab = 0$, the copying circuit below does not copy the quantum state i/p.

$\longrightarrow$ It turns out to be IMPOSSIBLE to make a copy of an unknown quantum state. This property, that qubits cannot be copied, is known as the no-cloning theorem.

* it is one of the chief differences b/w quantum & classical information.



The no-cloning theorem

The no-cloning theorem states that it is impossible to create an independent and identical copy of an arbitrary unknown quantum state.

Proof

Suppose,

we have a quantum machine with 2 slots labeled A and B. Slot A the data slot, starts out in an unknown but pure quantum state $|\psi\rangle$. This is the state which is to be copied into slot B, the target slot.

Assume that the target slot starts out in some standard pure state, $|s\rangle$.

$\therefore$

The initial state of the copying machine is,

$$|\psi\rangle \otimes |s\rangle$$

Some unitary evolution U now effects the copying procedure, ideally.

$$|\psi\rangle \otimes |s\rangle \xrightarrow{U} U\big(|\psi\rangle \otimes |s\rangle\big) = |\psi\rangle \otimes |\psi\rangle$$

Suppose,

the copying procedure works for 2 particular pure states $|\psi\rangle$ & $|\phi\rangle$.

Then,

$$U\big(|\psi\rangle \otimes |s\rangle\big) = |\psi\rangle \otimes |\psi\rangle$$

$$U\big(|\phi\rangle \otimes |s\rangle\big) = |\phi\rangle \otimes |\phi\rangle$$

Taking the inner products,

$$\langle\psi|\phi\rangle = \left(\langle\psi|\phi\rangle\right)^2$$

$$\Longrightarrow \langle\psi|\phi\rangle = 1 \quad (\text{or}) \quad \langle\psi|\phi\rangle = 0$$

$\longrightarrow$ Either $|\psi\rangle = |\phi\rangle$ or $|\psi\rangle$ & $|\phi\rangle$ are orthogonal.

$\Longrightarrow$ A cloning device can only clone states which are orthogonal to one another.

$\therefore$ A general quantum cloning device is impossible.

Ex:- A potential quantum cloner cannot clone the qubit states $|\psi\rangle = |0\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$.

$\therefore$ It is impossible to perfectly clone an unknown quantum state using unitary evolution.

What if we try to copy a mixed state?

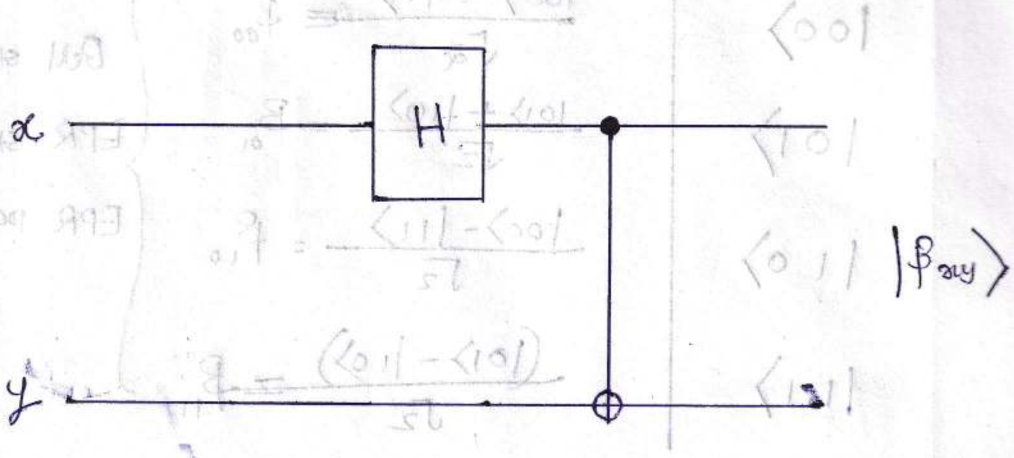
What if we allow cloning devices that are not unitary?

What if we are willing to allow imperfect copies that nevertheless are good according to some interesting measure of fedelity?

⟹ Even if one allows non-unitary cloning devices, the cloning of non-orthogonal pure states remains impossible unless one is willing to tolerate a finite loss of fedelity in the copied states.

Similar conclusions hold also for mixed states, although a somewhat more sophisticated approach is necessary to even define what is meant by the notion of cloning a mixed state.

Bell States



* Quantum circuit to create Bell states, which has a Hadamard gate followed by a CNOT.

Ex:-

The Hadamard gate takes the i/p $|00\rangle$ to $\frac{(|0\rangle+|1\rangle)}{\sqrt{2}}|0\rangle$, and then the CNOT gives the output state $\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)$.

Bell state

$$\boxed{|\beta_{xy}\rangle \equiv \frac{|0,y\rangle+(-1)^{x}|1,\bar{y}\rangle}{\sqrt{2}}}$$

$\bar{y}$: negation of y.

In	Out
$\|00\rangle$	$\frac{\|00\rangle + \|11\rangle}{\sqrt{2}} = \beta_{00}$
$\|01\rangle$	$\frac{\|01\rangle + \|10\rangle}{\sqrt{2}} = \beta_{01}$
$\|10\rangle$	$\frac{\|00\rangle - \|11\rangle}{\sqrt{2}} = \beta_{10}$
$\|11\rangle$	$\frac{(\|01\rangle - \|10\rangle)}{\sqrt{2}} = \beta_{11}$

Bell states / EPR states / EPR pairs

Conditioning on multiple qubits

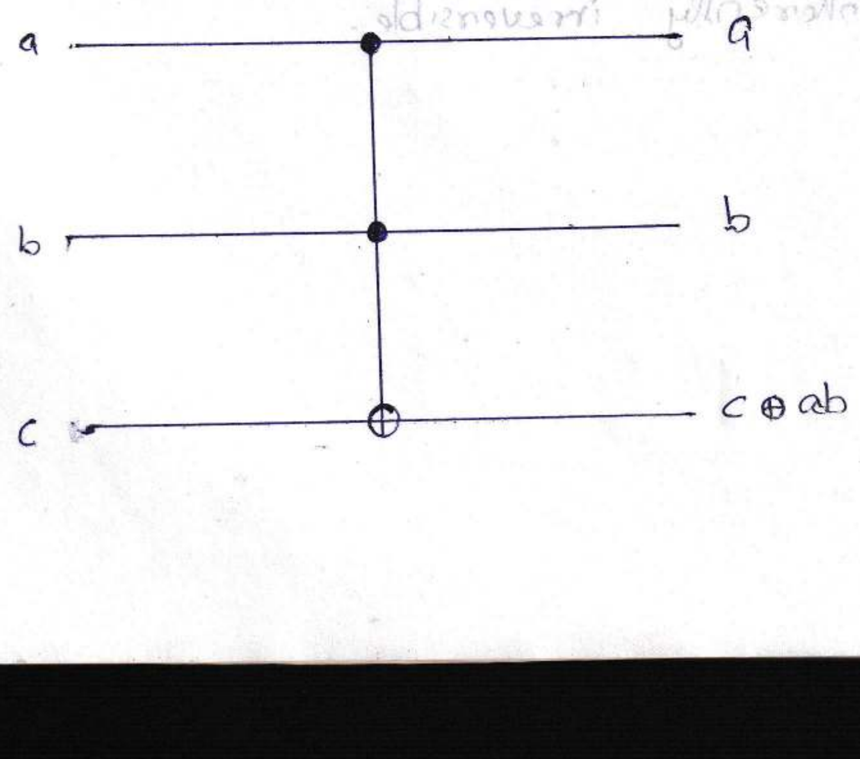
Classical computations on a quantum computer

We can simulate a classical logic circuit using a quantum circuit.

Quantum circuits cannot be used to directly simulate classical circuits since unitary quantum logic gates are inherently reversible, whereas many classical logic gates such as the NAND gate are inherently irreversible.

Any classical circuit can be replaced by an equivalent circuit containing only reversible elements, by making use of a reversible gate known as the Toffoli gate..

The Toffoli gate has 3 input bits and 3 output Bits. Two of the Bits are control bits that are unaffected by the action of the Toffoli gate. The 3^{rd} Bit is a target bit that is flipped if both control bits are set to 1, and otherwise is left alone.



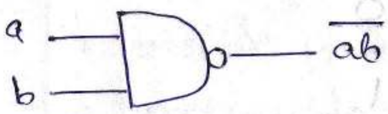
Inputs			Outputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

* Applying the Toffoli gate twice to a set of bits has the effect

$$(a,b,c) \longrightarrow (a,b,c\oplus ab) \longrightarrow (a,b,c)$$

$\Longrightarrow$ Toffoli gate is a reversible gate, since it has an inverse - itself.

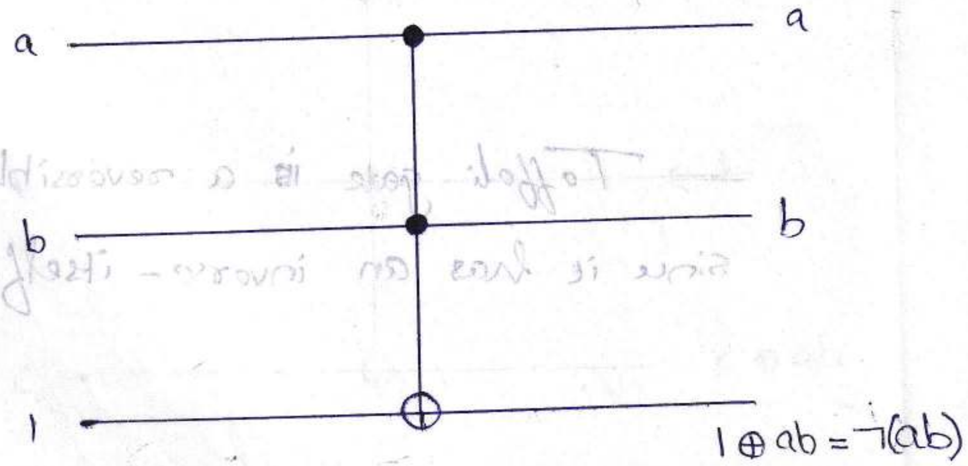
NAND gate



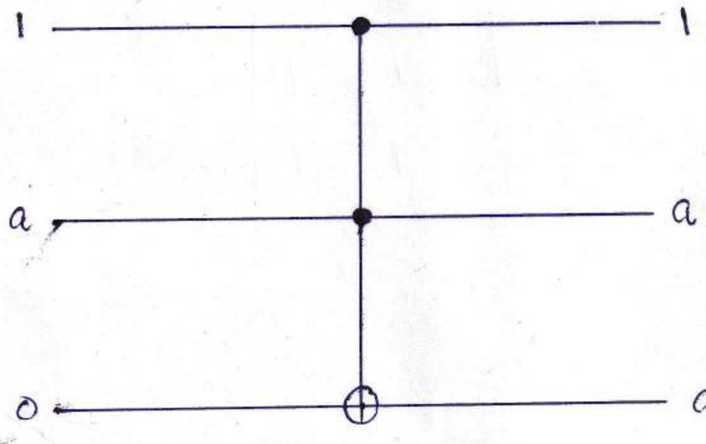
a	b	out
0	0	1
0	1	1
1	0	1
1	1	0

$$\boxed{a \text{ NAND } b \equiv \overline{a \wedge b} \equiv \neg(a \wedge b)}$$

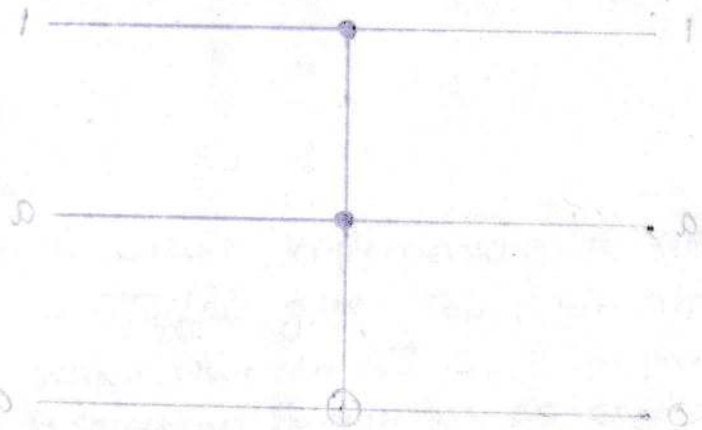
* Classical circuit implementing a NAND gate using a Toffoli gate. Top 2 bits represent the i/p to the NAND, while the 3^{rd} bit is prepared in the state 1, sometimes known as an ancilla state. The o/p from the NAND is on the 3^{rd} bit.



* FANOUT with the Toffoli gate, with the 2nd bit being the i/p to the FANOUT (and the other two bits standard ancilla states), and the o/p from FANOUT appearing the the 2nd and 3rd bits.



The Toffoli gate can be used to simulate NAND gates, and can also be used to do FANOUT. With these two operations it becomes possible to simulate all other elements in a classical circuit, and thus an arbitrary classical circuit can be simulated by an equivalent reversible circuit.



The Toffoli gate can be implemented as a quantum logic gate. The quantum logic implementation of the Toffoli gate simply permutes computational basis states in the same way as the classical Toffoli gate.

Ex:-

The quantum Toffoli gate acting on the state $|110\rangle$ flips the 3^{rd} qubit because the 1^{st} two are set, resulting in the state $|111\rangle$. The quantum Toffoli gate has an 8×8 unitary transformation matrix.

The quantum Toffoli gate can be used to simulate irreversible classical logic gates, just as the classical Toffoli gate was, and ensures that quantum computers are capable of performing any computation which a classical (deterministic) computer may do.

What if the classical computer is non-deterministic, i.e., has the ability to generate random bits to be used in the computation. ?

To perform such a simulation, it turns out to be sufficient to produce random fair coin tosses, which can be done by preparing a qubit in the state $|0\rangle$, sending it through a Hadamard gate to produce $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, and then measuring the state. The result will be $|0\rangle$ or $|1\rangle$ with 50/50 probability.

Conditioning on multiple qubits – Cont....

Suppose we have,

$n+k$ qubits and U is a k-qubit unitary operator. Then we define the controlled operation $C^n(U)$ by the equation,

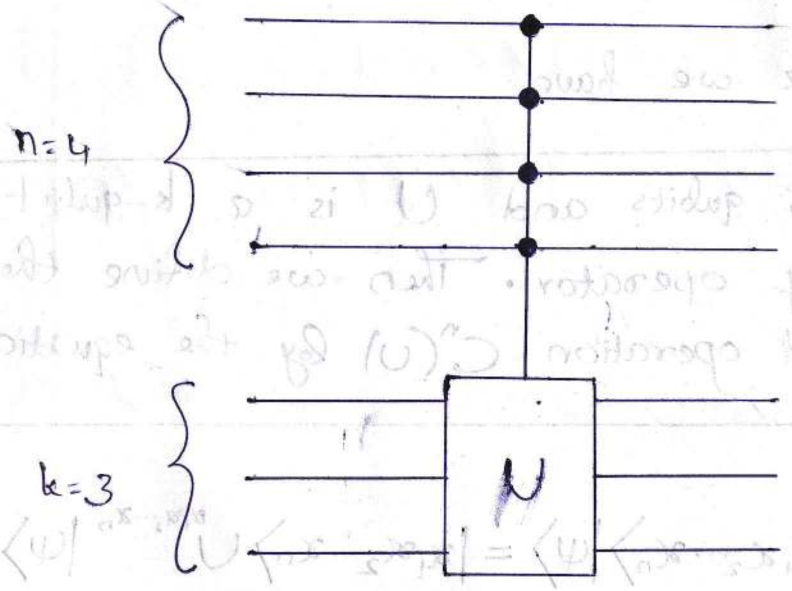
$$\boxed{C^n(U)|x_1 x_2 \cdots x_n\rangle|\psi\rangle = |x_1 x_2 \cdots x_n\rangle U^{x_1 x_2 \cdots x_n}|\psi\rangle}$$

where,

$x_1 x_2 \cdots x_n$ in $U^{x_1 x_2 \cdots x_n}$ means: the product of the bits $x_1 \cdot x_2 \cdot \cdots x_n$.

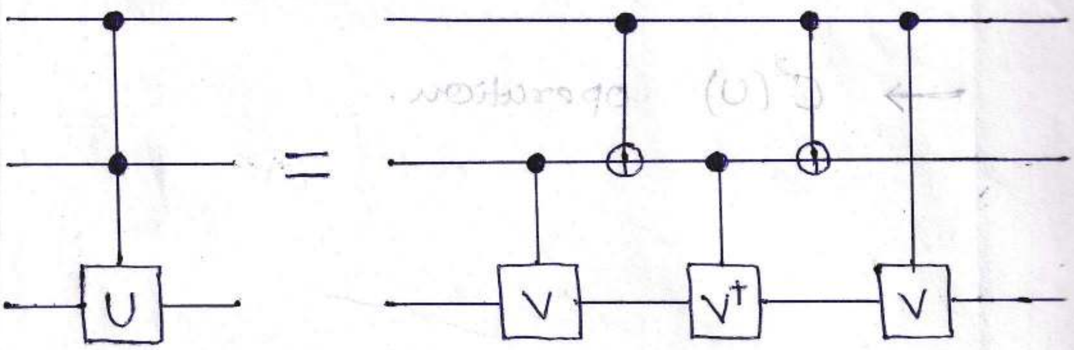
i.e., U is applied to the last k qubits if the 1st n qubits are all equal to 1, and otherwise, nothing is done.

* Sample circuit representation for the $C^n(U)$ operation, where U is a unitary operator on k qubits, for $n=4$ & $k=3$.



Suppose,

U is a single qubit unitary operator, and V is a unitary operator chosen so that $V^2 = U$. Then the operation $C^2(U)$ may be implemented using the circuit shown in the Fig. below:



* Circuit for the $C^2(U)$ gate.

V is any unitary operator satisfying $V^2 = U$.

$V = (1-i)(I+iX)/2$ corresponds to the Toffoli gate.

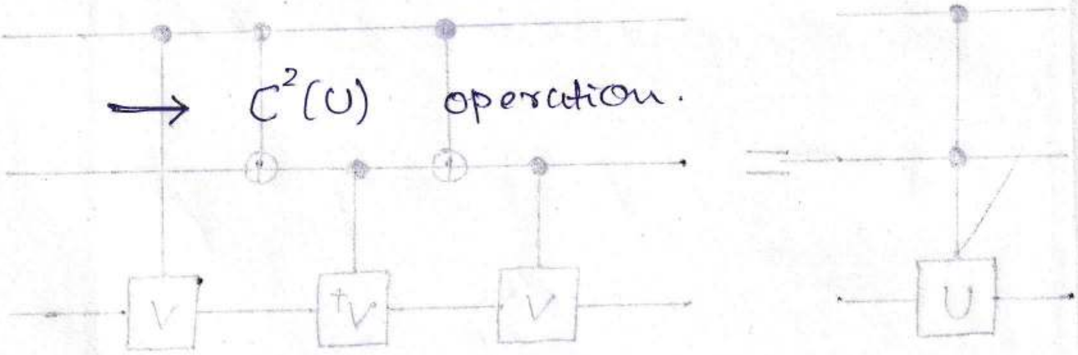
& $V^2 = X$

$$|00\rangle|\phi\rangle \longrightarrow |00\rangle|\phi\rangle$$

$$|01\rangle|\phi\rangle \longrightarrow |01\rangle V^{\dagger}V|\phi\rangle \rightarrow |01\rangle|\phi\rangle$$

$$|10\rangle|\phi\rangle \longrightarrow |10\rangle VV^{\dagger}|\phi\rangle \longrightarrow |10\rangle|\phi\rangle$$

$$|11\rangle|\phi\rangle \longrightarrow |11\rangle V^{2}|\phi\rangle \longrightarrow |11\rangle U|\phi\rangle$$



$\rightarrow$ $C^2(U)$ operation.

Special Case : $V = (1-i)(I+iX)/2$

$\Rightarrow U = V^2 = X$

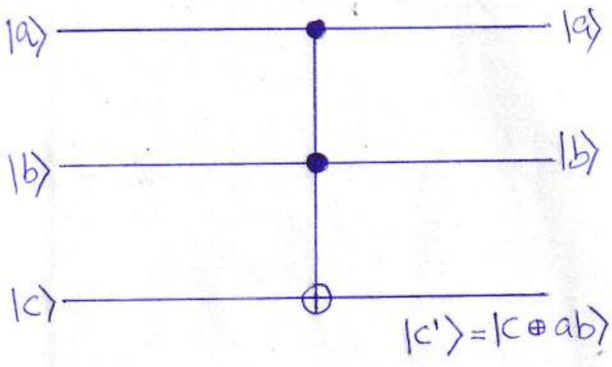
Fig : implementation of the Toffoli gate in terms of one and two qubit operations.

* From a classical viewpoint this is a remarkable result (Refer to Problem 3.5) that one and two bit classical reversible gates are not sufficient to implement the Toffoli gate, or, more generally, universal computation.

By contrast,

in the quantum case we see that one and two qubit reversible gates are sufficient to implement the Toffoli gate, and will eventually prove that they suffice for universal computation.

Toffoli gate



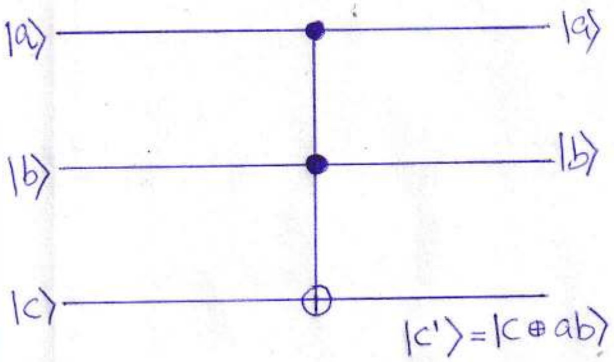
$|000\rangle \rightarrow |000\rangle$
$|001\rangle \rightarrow |001\rangle$
$|010\rangle \rightarrow |010\rangle$
$|011\rangle \rightarrow |011\rangle$
$|100\rangle \rightarrow |100\rangle$
$|101\rangle \rightarrow |101\rangle$
$|110\rangle \rightarrow |111\rangle$
$|111\rangle \rightarrow |110\rangle$

$$U_{CCNOT} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes I + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes X$$

(Toffoli)

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Toffoli gate



$|000\rangle \rightarrow |000\rangle$
$|001\rangle \rightarrow |001\rangle$
$|010\rangle \rightarrow |010\rangle$
$|011\rangle \rightarrow |011\rangle$
$|100\rangle \rightarrow |100\rangle$
$|101\rangle \rightarrow |101\rangle$
$|110\rangle \rightarrow |111\rangle$
$|111\rangle \rightarrow |110\rangle$

$$U_{CCNOT} = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes I + |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes X$$

(Toffoli)

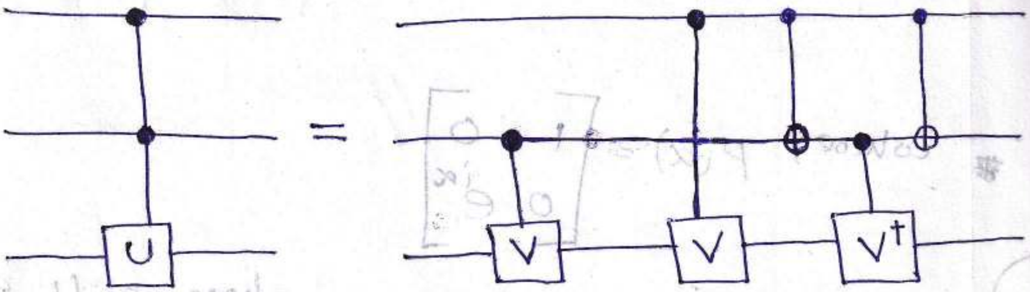
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

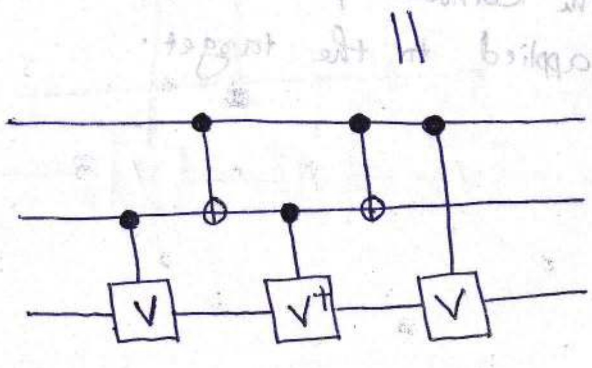
Ex: 4.22 Prove that a $C^2(U)$ gate (for any single qubit unitary U) can be constructed using at most eight one qubit gates, and 6 controlled-NOTs.

Ans:

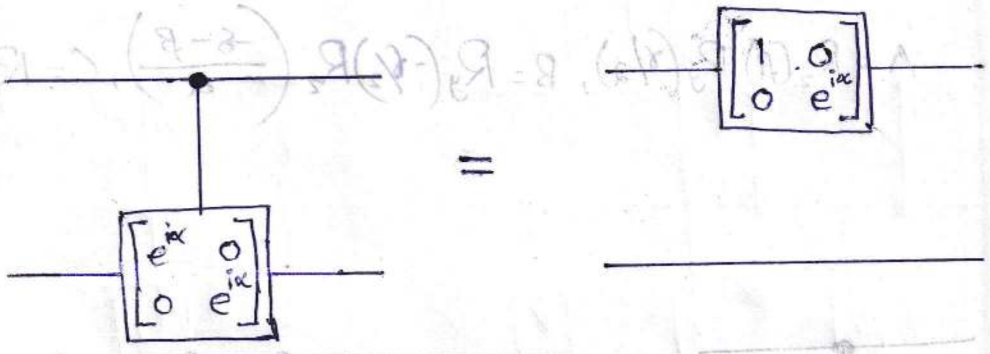
For a single qubit gate V, there exists unitary operators A, B, C on a single qubit such that $ABC = I$ and $U = e^{i\alpha} AXBXC$ *.

$$A = R_z(\beta) R_y(\gamma/2),\ B = R_y(-\gamma/2) R_z\left(\frac{-\delta-\beta}{2}\right),\ C = R_z\left(\frac{\delta-\beta}{2}\right)$$





* Controlled phase shift gate



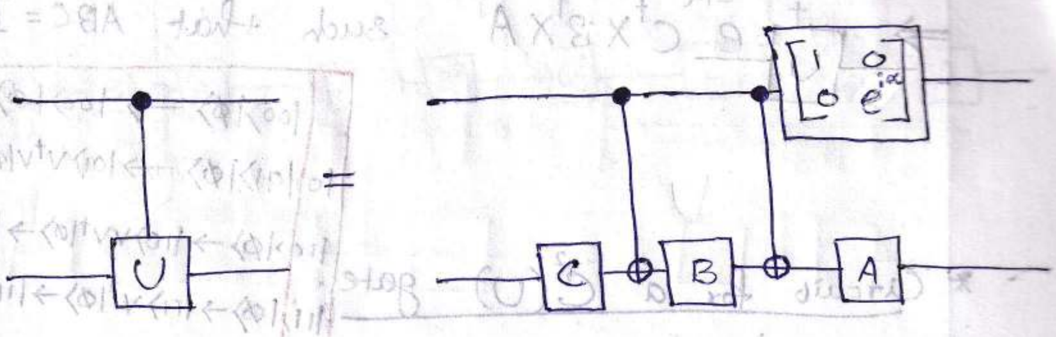
* where, $P(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{bmatrix}$

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If the control qubit is $|1\rangle$, a phase shift $e^{i\alpha}$ is applied to the target.

* Circuit implementing the controlled-U operation for single qubit U.

α, A, B, C satisfy $U = e^{i\alpha} AXBXC$, $ABC = I$



Take,

$$V = e^{i\alpha} AXBXC \quad \text{and} \quad V^2 = U$$

$$\Rightarrow V^\dagger = e^{-i\alpha} C^\dagger X B^\dagger X A^\dagger \quad \text{such that } ABC = I$$

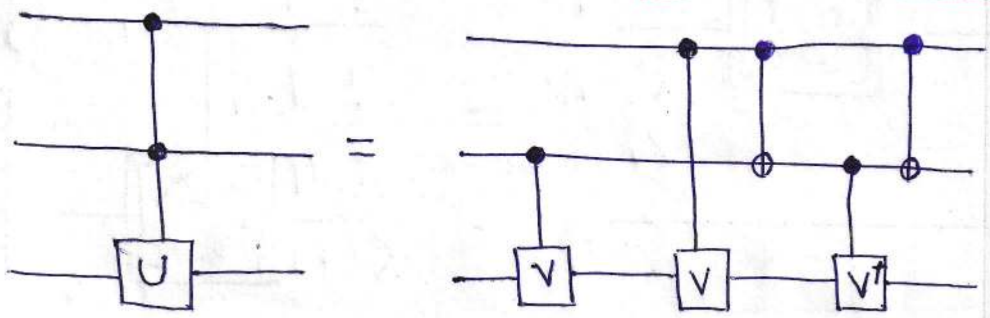
$|00\rangle|\phi\rangle \rightarrow |00\rangle|\phi\rangle$

$|01\rangle|\phi\rangle \rightarrow |01\rangle V^\dagger V|\phi\rangle \rightarrow |01\rangle|\phi\rangle$

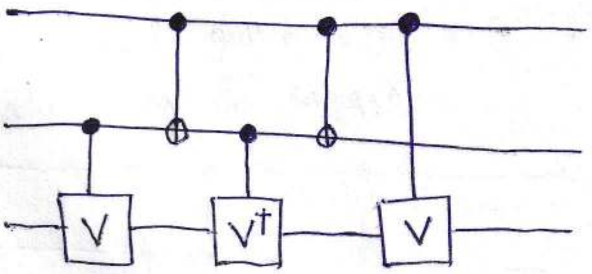
$|10\rangle|\phi\rangle \rightarrow |10\rangle V V^\dagger|\phi\rangle \rightarrow |10\rangle|\phi\rangle$

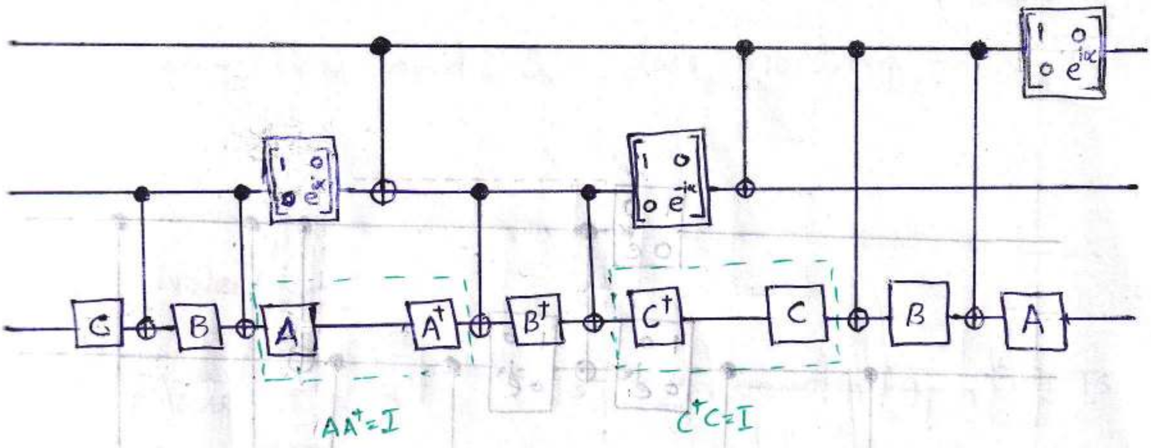
$|11\rangle|\phi\rangle \rightarrow |11\rangle V^2|\phi\rangle \rightarrow |11\rangle U|\phi\rangle$

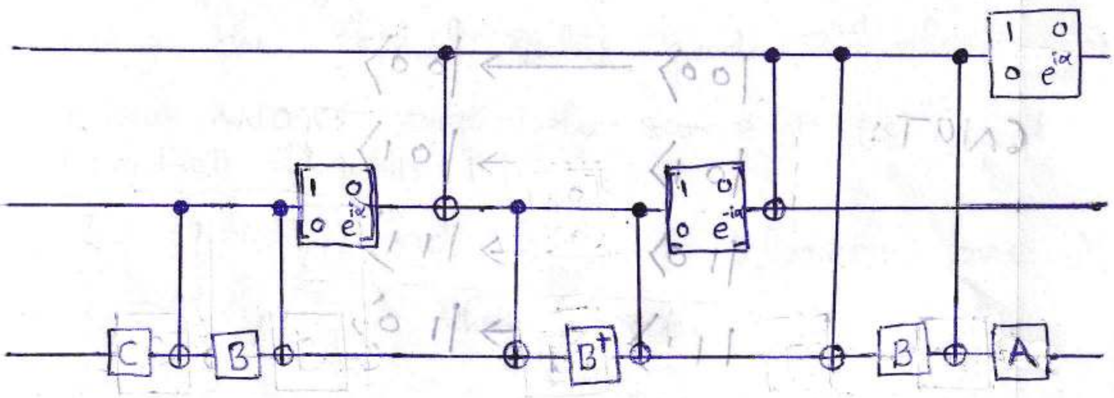
* Circuit for a $C^2(U)$ gate.

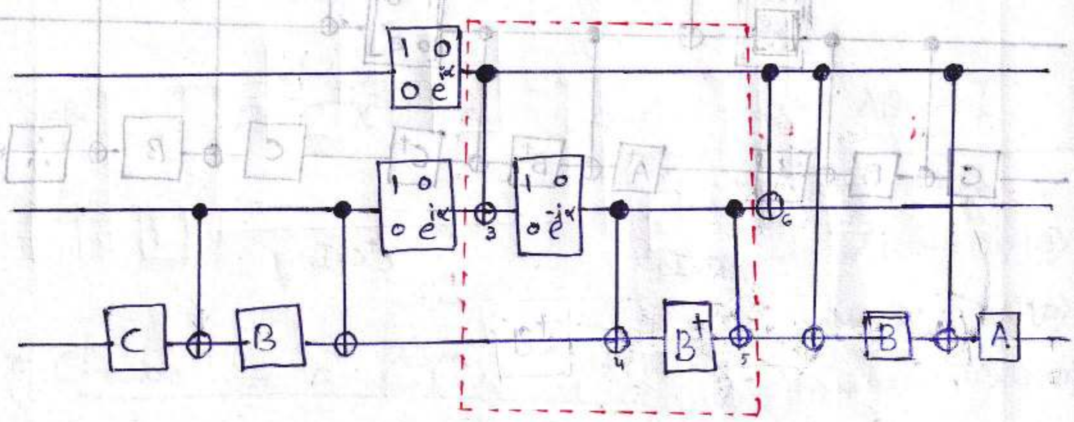


//




C
B
A
A†
B†
C†
C
B
A
AA†=I
C†C=I


C
B
B†
B
A



CNOT: $|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |11\rangle$

$|11\rangle \longrightarrow |10\rangle$

The 3rd CNOT is computing the parity of the 1st two qubits (which is uncomputed by the 6th), so that CNOTs 4 and 5 are controlled off this parity.

i.e.,
When 1st two qubits are different ($|01\rangle$ or $|10\rangle$) CNOTs 4 & 5 are activated.

When 1^{st} two qubits are the same ($|00\rangle$ or $|11\rangle$)

CNOTs 4 and 5 does nothing.

Equivalently,

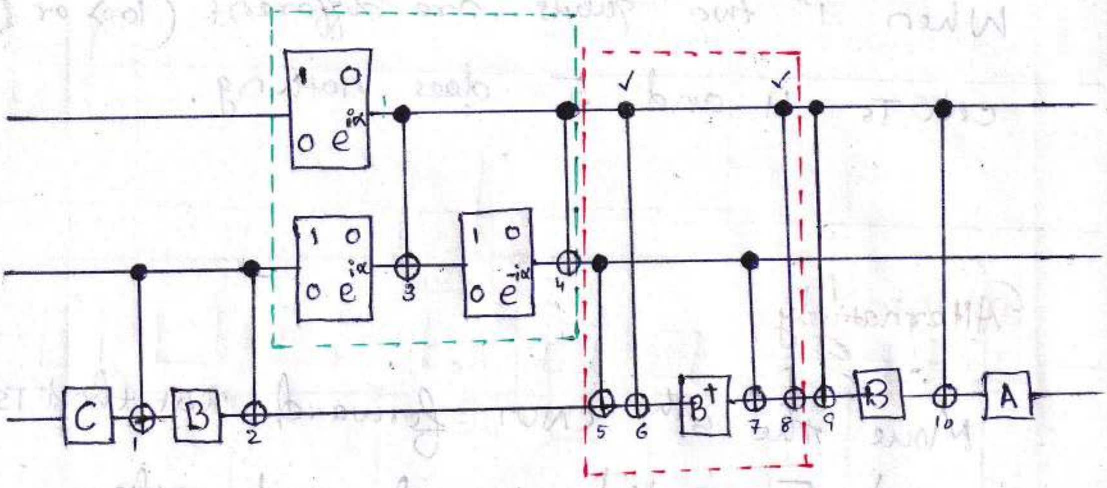
Move the 6^{th} CNOT forward, past CNOTS 4 and 5, which cancels each other.

Introducing 2 new CNOTs with 1^{st} qubit being the control qubit and 3^{rd} the target.

[two CNOTS with the same target, and controll off qubits 1 and 2.

If qubits 1 and 2 are different, one X is applied to the target.

If the two are the same, an even # of Xs are applied to the target, with the net effect being identity.

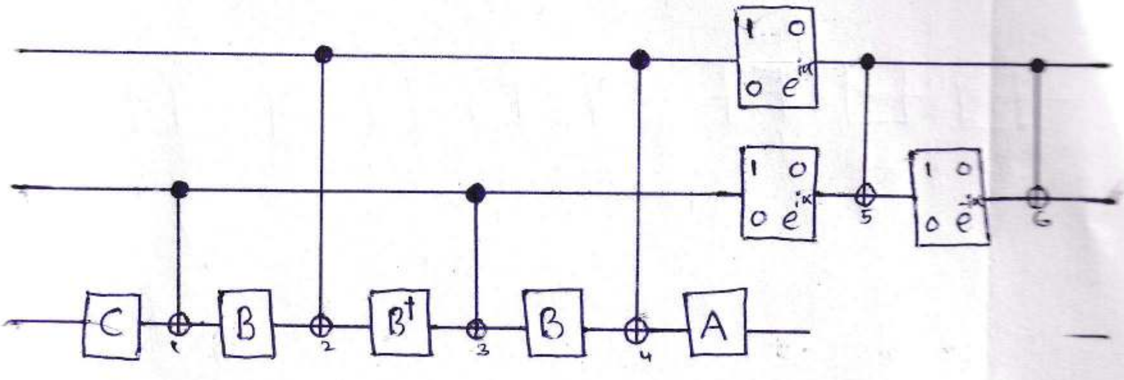


The green dashed region is diagonal, and so we can commute it past any controls

$\therefore$ We can move it all the way to the end.

Then,

the CNOTs 2 and 5 cancel each other.



$$AXBXB^\dagger XBXC = V(C^\dagger B^\dagger A^\dagger)V = V(ABC)^\dagger V = V^2 = U$$