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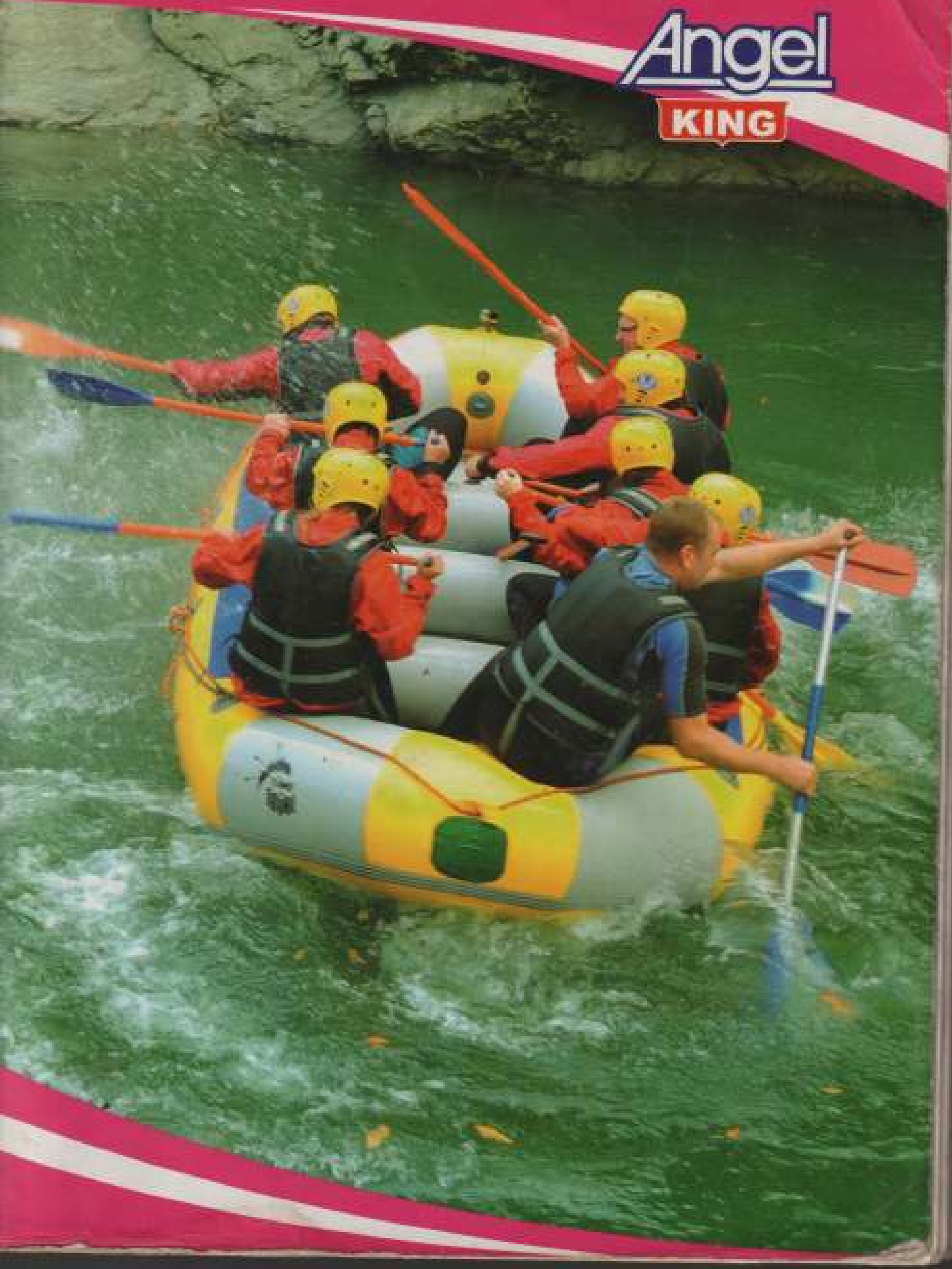

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		QUANTUM COMPUTATION		

Density Matrices

A bipartite state $|\psi\rangle$ is entangled iff its Schmidt rank is at least 2.

$|\psi\rangle$ is entangled if it cannot be written as $|\psi_1\rangle \otimes |\psi_2\rangle$ for some $|\psi_1\rangle \in \mathbb{C}^{d_1}$ and $|\psi_2\rangle \in \mathbb{C}^{d_2}$.

Intuitively,
States of the form $|\psi_1\rangle \otimes |\psi_2\rangle$, called tensor product states, are nice because one can immediately read off the state of each qubit.

qubit one is in state $|\psi_1\rangle$ &
qubit two in $|\psi_2\rangle$

Since entangled states can-not be written in tensor product form, however this raises the question :

How can we describe the stat of qubit 1 in, say, the Bell pair

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right)$$

$$\Downarrow$$

Density matrix formalism

more general formalism for describing quantum states.

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An arbitrary d-dimensional quantum state is described by a unit vector $|\psi\rangle$ in $\mathbb{C}^d$. Such states are called pure because we know exactly which state $|\psi\rangle$ we have.

Suppose,

we play a game in which with probability $\frac{1}{2}$, I give you state $|\psi_1\rangle$, and with probability $\frac{1}{2}$, I give you the state $|\psi_2\rangle$ and I don't tell you which state I have given you. How can you describe the quantum state in your posession, given that you don't know whether you actually have $|\psi_1\rangle$ or $|\psi_2\rangle$. ?

This, is done via the density operator

$$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

More generally,

if we play this game with m possible states $|\psi_i\rangle$, each with probability p_i $\left(\sum_i p_i = 1 \text{ and } p_i \geq 0\right)$,

then,

the density operator describing your system is:

$$\boxed{\rho = \sum_{i=1}^{m} p_i |\psi_i\rangle\langle\psi_i|}$$

Such a state is called mixed because you don't know with certainty which $|\psi_i\rangle$ you have your possession.

i.e.,
The quantum system is in one of a number of states $|\psi_i\rangle$, with probabilities p_i.
We call $\{p_i, |\psi_i\rangle\}$ an ensemble of pure states.

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* Sum of m

* Modes a
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* density

$\rho = \frac{1}{2}|$

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* Mixtures v/s superpositions

A mixture of states $\sum_i p_i |\psi_i\rangle\langle\psi_i|$ is different than a superposition of states $\sum_i \alpha_i |\psi_i\rangle$

- Sum of matrices
- Modes a state of ignorance about which $|\psi_i\rangle$ we actually have in our possession.
  We know our system is in precisely one such $|\psi_i\rangle$, but which one is unknown.
- density matrix,

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = I/2$$

- sum of vectors
- our system is in all of the states $|\psi_i\rangle$ simultaneously
- The state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Suppose,

the evolution of a closed quantum system is described by the unitary operator U.
If the system was initially in the state $|\psi_i\rangle$ with probability P_i then after the evolution has occurred the system will be in the state $U|\psi_i\rangle$ with probability P_i.

Thus,

the evolution of the density operator is described by the equation:

$$\boxed{\begin{array}{c} \rho = \sum_i P_i |\psi_i\rangle\langle\psi_i| \xrightarrow{U} \sum_i P_i U|\psi_i\rangle\langle\psi_i| U^\dagger \\ \hspace{8em} \parallel \\ \hspace{8em} U\rho U^\dagger \end{array}}$$

Suppose we perform a measurement described by measurement operators M_m. If the initial state was $|\psi_i\rangle$ then the probability of getting result m is:

$$\boxed{\begin{aligned} P(m|i) &= \langle \psi_i | M_m^\dagger M_m | \psi_i \rangle \\ &= tr\left(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\right) \end{aligned}}$$

The probability of obtaining result m is:

$$\boxed{\begin{aligned} p(m) &= \sum_i p(m|i)\, p_i \\ &= \sum_i p_i \, tr\left(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\right) \\ &= tr\left(M_m^\dagger M_m \rho\right) \end{aligned}}$$

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$|\psi_i^m\rangle =$

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If the initial state was $|\psi_i\rangle$ then the state after obtaining the result m is:

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}}$$

After a measurement which yields the result m we have an ensemble of states $|\psi_i^m\rangle$ with respective probabilities $p(i|m)$.

$$\boxed{\begin{aligned}\rho_m &= \sum_i p(i|m)\,|\psi_i^m\rangle\langle\psi_i^m| \\ &= \sum_i p(i|m)\,\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}\end{aligned}}$$

$$P(i|m) = \frac{p(m,i)}{p(m)} = \frac{p(m|i)\,p_i}{p(m)}$$

$$P(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = tr\left(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|\right)$$

$$\cancel{= tr\left(M_m^\dagger M_m \rho\right)}$$

$$\begin{aligned}
p(m) &= \sum_i p(m|i)\,p_i \\
&= \sum_i p_i \, tr\left(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|\right) \\
&= tr\left(M_m^\dagger M_m \rho\right) = tr\left(M_m \rho M_m^\dagger\right)
\end{aligned}$$

$$\boxed{\begin{aligned}
\rho_m &= \sum_i p(i|m)\,|\psi_i^m\rangle\langle\psi_i^m| \\
&= \sum_i p(i|m)\,\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle} \\
&= \sum_i p_i\,\frac{M_m|\psi_i\rangle\langle\psi_i|M_m^\dagger}{tr\left(M_m^\dagger M_m \rho\right)} \\
&= \frac{M_m \rho M_m^\dagger}{tr\left(M_m^\dagger M_m \rho\right)} = \frac{M_m \rho M_m^\dagger}{tr\left(M_m \rho M_m^\dagger\right)}
\end{aligned}}$$

$$\underline{tr(AB) = tr(BA)}$$

Imagine a quantum system is prepared in the state ρ_i with probability P_i.

Suppose that, ρ_i arises from some ensemble $\{(P_i)_j, |(\psi_i)_j\rangle\}$ of pure states. (i is fixed).

$\therefore$ The probability for being in the state $|(\psi_i)_j\rangle$ is $P_i (P_i)_j$.

The density matrix for the system is:

$$\rho = \sum_{i,j} P_i (P_i)_j |(\psi_i)_j\rangle\langle(\psi_i)_j|$$

$$= \sum_i P_i \rho_i$$

$\Rightarrow$ ρ is a mixture of the states ρ_i with probabilities P_i.

Ex:-

We would have a quantum system in the state ρ_m with probability $p(m)$, but no longer know the actual value of m.

The state of such a quantum system would be described by the density operator:

$$\rho = \sum_m p(m)\rho_m$$

$$= \sum_m tr\left(M_m^\dagger M_m \rho\right) \frac{M_m \rho M_m^\dagger}{tr\left(M_m^\dagger M_m \rho\right)}$$

$$= \sum_m M_m \rho M_m^\dagger$$

Ex:2.71. Let ρ be a density operator. Show that $\boxed{tr(\rho^2) \leq 1}$.

Ans: $tr(\rho^2) = tr\left(\sum_j p_j |j\rangle\langle j|\right)\left(\sum_k p_k |k\rangle\langle k|\right)$

$$= tr\left(\sum_{j,k} p_j p_k |j\rangle\underbrace{\langle j|k\rangle}\langle k|\right)$$

$$= tr\left(\sum_{j,k} p_j p_k \; \delta_{jk} \; |j\rangle\langle k|\right)$$

$$= tr\left(\sum_j p_j^2 |j\rangle\langle j|\right) = \sum_j p_j^2 \, tr(|j\rangle\langle j|)$$

$$= \sum_j p_j^2$$

ρ is +ve semi definite,

$\sum_j p_j = 1$ & $0 \leq p_j \leq 1 \Longrightarrow p_j^2 \leq p_j$

$\therefore \sum_j p_j^2 \leq \sum_j p_j = 1$

* Pure states

If there is only one state $|\psi_i\rangle$ in the mixture, ie, $P_i = 1$ for some i, then the mixture simply reads $\rho = |\psi_i\rangle\langle\psi_i|$. This state is called pure and has rank 1.

Conversely,

for any pure state $|\psi\rangle$, its density matrix is the rank 1 operator $\rho = |\psi\rangle\langle\psi|$.

$\therefore$

$\rho = |\psi\rangle\langle\psi|$ for a pure state is nothing but the projection operator onto the state $|\psi\rangle$.

* A state ρ is pure iff $\rho^2 = \rho$

Ans: ρ is +ve semi-definite

$\Longrightarrow$ Hermitian & $\lambda_i \geq 0$

$|\lambda_i\rangle$ are orthonormal.

Let $\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$

$$\rho^2 = \sum_i \lambda_i^2 |\lambda_i\rangle\langle\lambda_i| = \rho$$

$\Longrightarrow \lambda_i^2 = \lambda_i$ for any i

$\therefore \lambda_i = 0$ or 1

$$tr(\rho) = \sum_i \lambda_i = 1$$

$\Longrightarrow \lambda_p = 1$ for some p &

$\lambda_i = 0$ for $i \neq p$

$\therefore \rho = |\lambda_p\rangle\langle\lambda_p|$, is a pure state.

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* A

Ans:

* A state ρ is pure iff $tr(\rho^2) = 1$

Ans: Let $\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$

$$\rho^2 = \sum_i \lambda_i^2 |\lambda_i\rangle\langle\lambda_i|$$

$$tr(\rho^2) = \sum_i \lambda_i^2 = 1$$

Since $tr(\rho) = \sum_i \lambda_i = 1$ & $\lambda_i \geq 0$

$$\sum_i \lambda_i^2 = \sum_i \lambda_i = 1$$

$\Longrightarrow \lambda_i = 1$ or 0.

Ex: What is the density matrix for pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$? Next, write down the 2×2 density matrix $\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|+\rangle\langle +|$

Ans:

① $\rho = |\psi\rangle\langle\psi| = (\alpha|0\rangle + \beta|1\rangle)(\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|)$

$$= |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| + \alpha\bar{\beta}\,|0\rangle\langle 1| + \bar{\alpha}\beta\,|1\rangle\langle 0|$$

$$= |\alpha|^2 \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + |\beta|^2 \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} + \alpha\bar{\beta}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} + \bar{\alpha}\beta\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}$$

$$= |\alpha|^2 \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + |\beta|^2 \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \alpha\bar{\beta}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + \bar{\alpha}\beta\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}|\alpha|^2 & \alpha\bar{\beta}\\ \bar{\alpha}\beta & |\beta|^2\end{bmatrix}$$

② $|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ & $|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$$\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|+\rangle\langle +| = \frac{1}{3}\times\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \frac{2}{3}\times\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \frac{2}{6}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}3 & -1\\-1 & 3\end{bmatrix}$$

□ General properties of the density operator:

The density operator was introduced as a means of describing ensembles of quantum states. We'll move away from this description to develop an intrinsic characterization of density operators that does not rely on an ensemble interpretation.

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* A linear operator $\rho = \mathcal{L}(\mathbb{C}^d)$ is called a density matrix if the following two properties hold:

- ρ is positive-semi definite

  ie., Hermitian & has non-negative real eigenvalues.

- ρ has trace 1

  ie., $Tr(\rho) = 1$

Proof. Suppose $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ is a density operator.

$$tr(\rho) = \sum_i p_i \, tr\left(|\psi_i\rangle\langle\psi_i|\right) = \sum p_i = 1$$

Suppose $|\phi\rangle$ is an arbitrary vector in state space, Then

$$\langle\phi|\rho|\phi\rangle = \sum_i p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle$$

$$= \sum_i p_i \left|\langle\phi|\psi_i\rangle\right|^2 \geq 0$$

Ex:- $\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$

$$Tr(\rho) = Tr\left(\frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|\right)$$

$$= \frac{1}{2}Tr\left(|\psi_1\rangle\langle\psi_1|\right) + \frac{1}{2}Tr\left(|\psi_2\rangle\langle\psi_2|\right)$$

$$= \frac{1}{2}\langle\psi_1|\psi_1\rangle + \frac{1}{2}\langle\psi_2|\psi_2\rangle = 1$$

* For any two positive semi-definite matrices A and B and real numbers $p,q \geq 0$., it holds that $pA+qB$ is also positive semi-definite.

* $|\psi\rangle\langle\psi|$ is positive-semidefinite for any $|\psi\rangle$

↳ $|\psi\rangle\langle\psi|$ has rank=1

i.e., only one eigenvalue

$$Tr(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle = \cancel{1} \geq 0$$
$$= \||\psi\rangle\|^2 =$$

$$\left.\begin{array}{l} Tr(\rho)=1 \\ \\ \rho \text{ positive-semidefinite} \end{array}\right\} \text{allow us to recover exactly the interpretation of mixed states}$$

$$\Downarrow$$

Taking the spectral decomposition of ρ

$$\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

ρ is positive-semidefinite $\longrightarrow$ $\lambda_i \geq 0$ for all i

$Tr(\rho)=1 \longrightarrow \sum_i \lambda_i = 1$

* $\{\lambda_i\}$ forms a probability distribution *

$$\left.\begin{array}{l} \text{We can interpret} \\ \rho \text{ as follows} \end{array}\right\} \begin{array}{l} \text{With probability } \lambda_i \\ \text{prepare state } |\lambda_i\rangle \end{array}$$

$\therefore$

A system in state $|x_i\rangle$ with probability x_i will have density operator ρ.

ie., the ensemble $\{x_i, |x_i\rangle\}$ is an ensemble of states giving rise to the density operator ρ.

The reformulated postulates of QM in the density operator picture:

Postulate 1: Associated to any isolated physical system is a complex vector space with inner products (i.e., Hilbert space) known as the state space of the system. The system is completely described by its density operator, which is a positive operator ρ with trace one, acting on the state space of the system. If a quantum system is in the state ρ_i with probability p_i then the density operator for the system is $\sum_i p_i \rho_i$.

$$\boxed{\rho = \sum_i p_i \rho_i}$$

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation. i.e., the state ρ of the system at time t_1 is related to the state ρ' ~~of~~ the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$\boxed{\rho' = U\rho U^{\dagger}}$$

Postulate 3: Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment.

If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by:

$$P(m) = tr\left(M_m^{\dagger} M_m \rho\right)$$

and the state of the system after the measurement is:

$$\frac{M_m \rho M_m^\dagger}{tr\left(M_m^\dagger M_m \rho\right)}$$

The measurement operators satisfy the completeness equation, $\sum_m M_m^\dagger M_m = I$.

Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state ρ_i, then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$.

Density c
approac

Density operator approach

- description of quantum systems whose state is not known
- description of subsystems of a composite quantum system

☐ Time Evolution of Density Matrices

Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \xrightarrow{\dagger} -i\hbar \frac{\partial}{\partial t}\langle\psi| = \langle\psi|H$$

For a mixed state, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$i\hbar \frac{\partial}{\partial t}\rho = i\hbar \sum_i p_i \Big(\underbrace{|\dot{\psi}_i\rangle}_{\frac{-i}{\hbar}H|\psi_i\rangle}\langle\psi_i| + |\psi_i\rangle\underbrace{\langle\dot{\psi}_i|}_{\frac{i}{\hbar}\langle\psi_i|H}\Big)$$

$$= \sum_i p_i \big(H|\psi_i\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi_i|H\big)$$

$$= H\Big(\sum_i p_i|\psi_i\rangle\langle\psi_i|\Big) - \Big(\sum_i p_i|\psi_i\rangle\langle\psi_i|\Big)H$$

$$= H\rho - \rho H = [H, \rho]$$

Density matrices satisfy the von Neumann Equation:

$$\boxed{i\hbar \frac{\partial}{\partial t} \rho = [H, \rho]}$$

☐ Unitary Freedom in the ensemble of ________ . density matrices

Suppose that a quantum system with density matrix,

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

must be in the state $|0\rangle$ with probability $3/4$ and in the state $|1\rangle$ with probability $1/4$.

However, this is not necessarily the case !

Suppose, we define

$$|a\rangle \equiv \sqrt{3/4}\,|0\rangle + \sqrt{1/4}\,|1\rangle$$

$$|b\rangle \equiv \sqrt{3/4}\,|0\rangle - \sqrt{1/4}\,|1\rangle$$

'$\equiv$' : identical to

$:=$: defined to be equal to

and the system is prepared in the state $|a\rangle$ with probability $1/2$ and in the state $|b\rangle$ with probability $1/2$. The corresp. density matrix is :

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

⟶ these 2 different ensembles of quantum states give rise to the same density matrix.

⇒ The eigenvectors and eigenvalues of a density matrix just indicate one of many possible ensembles that may give rise to a specific density matrix.

What class of ensembles does give rise to a particular density matrix ?

ANS
has applications : Quantum Noise &
Quantum Error Correction

Making

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$\rho \equiv \sum_i$

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$|\tilde{\phi}_j\rangle$ gen

$\langle \tilde{\phi}$

Making use of vectors $|\tilde{\psi}_i\rangle$ which may not be normalized to unit length.

The set $\{|\tilde{\psi}_i\rangle\}$ generates the operator

$$\rho \equiv \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$$

The connection to the usual ensemble picture of density operators is expressed by the eqn. $|\tilde{\psi}_i\rangle = \sqrt{p_i}\,|\psi_i\rangle$

When do two sets of vectors $|\tilde{\psi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ generate the same operator ρ ?

Theorem 2.6: The sets $|\tilde{\varphi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ generate the same density matrix if & only if

$$|\tilde{\varphi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle$$

where, $[u_{ij}]$ is a unitary matrix of complex numbers, with indices i and j, and we 'pad' whichever set of vectors $|\tilde{\varphi}_i\rangle$ or $|\tilde{\phi}_j\rangle$ is smaller with additional vectors 0, so that the 2 sets have the same # of elements.

$$\Longrightarrow \quad \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_j q_j |\phi_j\rangle\langle\phi_j|$$

for normalized states $|\psi_i\rangle, |\phi_i\rangle$ and probability distributions p_i, q_j iff

$$\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij} \sqrt{q_j}\,|\phi_j\rangle$$

for some unitary matrix $[u_{ij}]$

- We may
entries
the 2

Proof

Suppose

Then,

$\sum_i |\tilde{\varphi}_i\rangle\langle$

$\Longrightarrow$ |

operator

- We may 'pad' the smaller ensemble with entries having probability 0 inorder to make the 2 ensembles the same size.

Proof

Suppose $|\tilde{\psi}_i\rangle = \sum_j u_{ij}|\tilde{\phi}_j\rangle$ for some unitary u_{ij}

Then,

$$\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_{i,j,k} u_{ij} u^*_{ik} |\tilde{\phi}_j\rangle\langle\tilde{\phi}_k|$$

$$= \sum_{j,k}\left(\sum_i u^\dagger_{ki} u_{ij}\right)|\tilde{\phi}_j\rangle\langle\tilde{\phi}_k|$$

$$= \sum_{jk}\delta_{kj}|\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| \qquad \left[U^\dagger U = I \right.$$

$$= \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j|$$

$\Longrightarrow$ $|\tilde{\psi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ generate the same operator.

$\lambda_i \geq 0$ for P. & $|\phi_j\rangle$ need not have to be orthogonal

Conversely,

Suppose, $A=\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\phi_j\rangle\langle\phi_j|$

Let $A=\sum_k \lambda_k |k\rangle\langle k|$ be a decomposition for A such that the states $|k\rangle$ are orthonormal and λ_k are strictly +ve.

Relate the states $|\tilde{\psi}_i\rangle$ to states $|\tilde{k}\rangle \equiv \sqrt{\lambda_k}|k\rangle$

Relate the states $|\tilde{\phi}_j\rangle$ to states $|\tilde{k}\rangle$.

Combine

Let $|\psi\rangle$ be any vector orthonormal to the space spanned by the $|\tilde{k}\rangle$

$\therefore \langle\psi|\tilde{k}\rangle\langle\tilde{k}|\psi\rangle = 0$ for all k

---

$|\psi\rangle \in$

$\therefore$

$0 = \langle\psi|$

$\Rightarrow \langle\psi$

Each $|\tilde{\psi}$

combinatio

$|\tilde{\psi}_i\rangle =$

Since

$\therefore$

$A - \sum_{kl}($

$|\psi\rangle \in$ kernel $(A) \longrightarrow A|\psi\rangle = 0$

$\therefore$

$$0 = \langle\psi|A|\psi\rangle = \langle\psi|\sum_i |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i|\psi\rangle = \sum_i \langle\psi|\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i|\psi\rangle$$
$$= \sum_i |\langle\psi|\tilde{\varphi}_i\rangle|^2$$

$\Longrightarrow \langle\psi|\tilde{\varphi}_i\rangle = 0$ for all i, and all $|\psi\rangle$ orthonormal to the space spanned by the $|\tilde{k}\rangle$.

Each $|\tilde{\varphi}_i\rangle$ can be expressed as a linear combination of the $|\tilde{k}\rangle$.

$$|\tilde{\varphi}_i\rangle = \sum_k c_{ik}|\tilde{k}\rangle$$

Since $A = \sum_k |\tilde{k}\rangle\langle\tilde{k}| - \sum_i |\tilde{\varphi}_i\rangle\langle\tilde{\varphi}_i|$,

$$= \sum_{k,l}\left(\sum_i c_{ik}c_{il}^*\right)|\tilde{k}\rangle\langle\tilde{l}|$$

$$\therefore A - \sum_{kl}\left(\sum_i c_{ik}c_{il}^*\right)|\tilde{k}\rangle\langle\tilde{l}| = \sum_k |\tilde{k}\rangle\langle\tilde{k}| - \sum_{kl}\left(\sum_i c_{ik}c_{il}^*\right)|\tilde{k}\rangle\langle\tilde{l}| = 0$$

The operators $|\tilde{k}\rangle\langle\tilde{l}|$ are linearly independent.

$$\sum_{k=l} |\tilde{k}\rangle\langle\tilde{k}| \left(\sum_i c_{ik} c_{il}^* - 1\right) + \sum_{k\neq l} |\tilde{k}\rangle\langle\tilde{l}| \left(\sum_i c_{ik} c_{il}^*\right) = 0$$

$$\sum_i c_{ik} c_{il}^* = \delta_{kl}$$

$[C]_{ik} = c_{ik}$ & $[C^\dagger]_{li} = c_{il}^*$

$$[C^\dagger C]_{lk} = \sum_i [C^\dagger]_{li} [C]_{ik}$$

$$= \sum_i c_{il}^* c_{ik} = \sum_i c_{ik} c_{il}^* = \delta_{kl}$$

$$\Longrightarrow C^\dagger C = I$$

∴

We may append extra columns to C to obtain a unitary matrix V such that

$|\tilde{\Psi}_i\rangle$

where

the list

Similarly

we

such tha

$|\psi\rangle = V$

∴ |

$$|\tilde{\varphi}_i\rangle = \sum_k V_{ik} |\tilde{k}\rangle$$

where we have appended zero vectors to the list of $|\tilde{k}\rangle$.

Similarly,
we can find a unitary matrix W such that $|\tilde{\phi}_j\rangle = \sum_k W_{jk} |\tilde{k}\rangle$.

$|\psi\rangle = V|k\rangle$ & $|\phi\rangle = W|k\rangle$.

$$|k\rangle = W^{\dagger}|\phi\rangle.$$

$$\Longrightarrow |\psi\rangle = VW^{\dagger}|\phi\rangle = U|\phi\rangle$$

$\therefore$ $|\tilde{\varphi}_i\rangle = \sum_j u_{ij}|\tilde{\phi}_j\rangle$, where $u = VW^{\dagger}$

Start 12/9/21

* If V and W are vectorspaces with linearly independent basis $\{V_i\}$ and $\{W_j\}$, respectively. Then $V_i \otimes_{outer} W_j$ form a basis for the vectorspace $V \otimes_{outer} W$.

Proof

Theorem 1: If V_i and W_j are basis of the vectorspaces V and W respectively, then $V_i \otimes W_j$ is a basis of the vectorspace

Q.C ① . $V \otimes W$.

In the case of column vectors V_i and W_j the tensor product $V_i \otimes W_j$ can be seen as a form of flattning of the outer product, i.e., there is a one-to-one correspondence b/w the vector spaces $V \otimes W$ and $V \otimes_{outer} W$.

---

i.e.,

$V \otimes W$

Theorem

$T: V \rightarrow$

for V

$\therefore$

$V_i \otimes W_j$

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i.e.,

$V \otimes W$ is isomorphic to $V \otimes_{outer} W$

$$\Longrightarrow V \otimes W \cong V \otimes_{outer} W$$

$\boxed{\text{Theorem 2:}}$ If there is an isomorphism $T: V \rightarrow W$. The vectors $\{v_i\}$ from a Basis for V iff $T(v_i)$ from a Basis for W.

$v_i \otimes w_j$ form a basis for the vectorspace $V \otimes W$, from theorem 2, $v_i \otimes_{outer} w_j$ form a Basis for the vectorspace $V \otimes_{outer} W$.

i.e., $v_i \otimes_{outer} w_j$ are linearly independent.

Bloch Sphere

A qubit is the fundamental quantum state representing the smallest unit of quantum information containing one bit of classical information accessible by measurement.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$

with $|\alpha|^2 + |\beta|^2 = 1$

Let $\alpha = r_0 e^{i\phi_0}$ & $\beta = r_1 e^{i\phi_1}$,

We have 4 unknowns (2 phases & 2 amplitudes) that uniquely determine components. It'd seem that a qubit should have 4 free real valued parameters

$$|\psi\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle$$

A quantum state does not change if we multiply it with any # of unit norm.

i.e., $|\psi\rangle = e^{i\varphi}|\psi\rangle$

This is because the quantum property of measurement follows from identifying the moduli squared of the amplitude as an occupation probability $|\alpha|^2$ and $|\beta|^2 = 1 - |\alpha|^2$ for the qubit to occupy its logical states $|0\rangle$ and $|1\rangle$.

$$|\psi\rangle = e^{-i\phi_0}|\psi\rangle = e^{-i\phi_0}\left(r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle\right)$$

$$= r_0|0\rangle + r_1 e^{i(\phi_1 - \phi_0)}|1\rangle$$

$$= r_0|0\rangle + r_1 e^{i\phi}|1\rangle$$

where $\phi \in [0, 2\pi)$

From

3 paran

$|\alpha|^2 + |\beta|$

$|\psi\rangle$

$\Longrightarrow$ Th

parameter

From 4 parameters we end up with 3 parameters r_0, r_1 and $\phi = \phi_1 - \phi_0$

$$|\alpha|^2 + |\beta|^2 = 1 \longrightarrow r_0^2 + r_1^2 = 1$$

$$|\psi\rangle = \sqrt{1-r_1^2}\,|0\rangle + r_1 e^{i\phi}|1\rangle$$

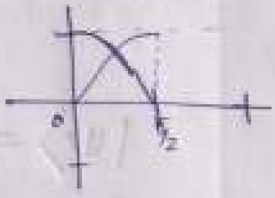
$\Rightarrow$ There are only 2 relevant free parameters to specify the state of qubit.

$r_0^2 + r_1^2 = 1$ & $0 \le r_0, r_1 \le 1$

Take, $r_0 = \sqrt{1 - r_1^2} = \cos \theta/2$ & $r_1 = \sin \theta/2$

with $0 \le \frac{\theta}{2} \le \frac{\pi}{2}$

$\Rightarrow 0 \le \theta \le \pi$



$$\boxed{|\psi(\theta, \phi)\rangle = \cos\theta/2\, |0\rangle + e^{i\phi} \sin\theta/2\, |1\rangle}$$

with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$

$\hat{n} = (n$

$\sigma = (\sigma_x$

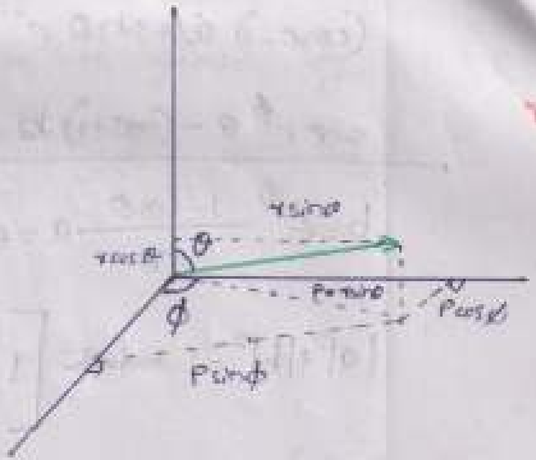
$\hat{n}(\theta, \phi) \cdot \sigma$

$\det(\hat{n} \cdot \sigma -$

$\cos^2\theta - \lambda^2$

For $\lambda =$

$\Big[\begin{matrix} \cos\theta - 1 \\ \sin\theta\, e^{i\phi} \end{matrix}$



$$\hat{n} = (n_x, n_y, n_z) = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

$$\sigma = \left(\sigma_x, \sigma_y, \sigma_z\right) \quad \text{where,} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{n}(\theta,\phi)\cdot\sigma = \hat{n}\cdot\sigma = n_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + n_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + n_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}$$

$$\det\left(\hat{n}\cdot\sigma - \lambda I\right) = 0 \Longrightarrow \begin{vmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$\cos^2\theta - \lambda^2 + \sin^2\theta = 1 - \lambda^2 = 0 \Longrightarrow \lambda = \pm 1$$

For $\lambda = +1$, solving $(\hat{n}\cdot\sigma - \lambda_1 I)\vec{x} = 0$

$$\begin{bmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$(\cos\theta - 1)\, a + \sin\theta\, e^{-i\phi}\, b = 0$$

$$\sin\theta\, e^{i\phi}\, a - (\cos\theta + 1)\, b = 0$$

$$b = e^{i\phi}\frac{1-\cos\theta}{\sin\theta}\, a = e^{i\phi}\frac{\sin\theta/2}{\cos\theta/2}\, a$$

$$|a|^2 + |b|^2 = 1 = |a|^2\left[1 + \frac{\sin^2\theta/2}{\cos^2\theta/2}\right] = |a|^2\,\frac{1}{\cos^2\theta/2}$$

$$\Longrightarrow\ |a| = |\cos\theta/2|$$

The overall phase of the eigenstate is not observable, so we take the simplest option for a.

$$a = \cos\theta/2 \quad \& \quad b = e^{i\phi}\sin\theta/2$$

$$\boxed{\begin{aligned} &\lambda_+ = 1,\ |\hat{n},+\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle \\ &\lambda_- = -1,\ |\hat{n},-\rangle = \sin\theta/2\,|0\rangle - e^{i\phi}\cos\theta/2\,|1\rangle \end{aligned}}$$

$\Rightarrow$ Eigenvalues & Eigenvectors of $\hat{n}(\theta,\phi)\cdot\sigma$

The st

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$0\le\theta\le\pi$

The state of a qubit is given by,

$$|\psi(\theta,\phi)\rangle = \cos\theta/_2|0\rangle + e^{i\phi}\sin\theta/_2|1\rangle$$

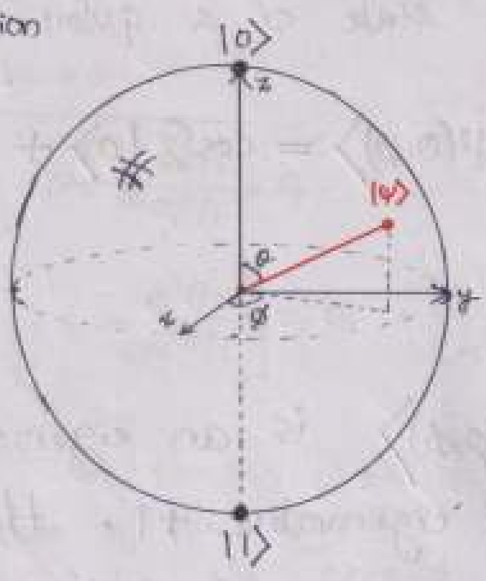
$|\psi(\theta,\phi)\rangle$ is an eigenstate of $\hat{n}(\theta,\phi)\cdot\sigma$ with eigenvalue $+1$. Here, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ and $\hat{n}(\theta,\phi) = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is a real unit vector called the Bloch vector.

→ It is natural to assign a unit vector $\hat{n}(\theta,\phi)$ to a state vector $|\psi(\theta,\phi)\rangle$.

A state $|\psi(\theta,\phi)\rangle$ is expressed as a unit vector $\hat{n}(\theta,\phi)$ on the surface of the unit sphere, called the Bloch sphere

This correspondence is one-to-one if the ranges of θ and ϕ are restricted to $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

* Bloch sphere representation of a qubit →



* A single qubit in the state, $|\psi\rangle = a|0\rangle + b|1\rangle$ can be visualized as a point (θ, ϕ) on the Bloch sphere (unit sphere), where $a = \cos\theta/2$, $b = e^{i\phi}\sin\theta/2$, and 'a' can be taken to be real because the overall phase of the state is unobservable. This is called the Bloch sphere representation and the vector $\hat{n}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is called the Bloch vector.

Maximally mixed state

$$\rho = I/d \in \mathcal{L}(\mathbb{C}^d)$$

I has eigenvalues all equal to 1 → I is positive semidefinite

$$\mathrm{Tr}(I/d) = \frac{1}{d}\,\mathrm{Tr}(I) = 1$$

$\therefore$ $\rho = I/d$ is a valid density operator

What exactly does $\rho = I/d$ represent ?

For any orthonormal basis $\{|\psi_i\rangle\}_{i=1}^{d}$ for $\mathbb{C}^d$, we have

$$\sum_{i=1}^{d} |\psi_i\rangle\langle\psi_i| = I$$

ie.,

For any orthonormal basis $\{|\psi_i\rangle\}_{i=1}^{d}$,

$\rho = \sum_i \frac{1}{d}|\psi_i\rangle\langle\psi_i| = \frac{I}{d}$ represent the following state : Pick $|\psi_i\rangle$ with probability $\frac{1}{d}$ and prepare $|\psi_i\rangle$. Since this holds for any basis, we conclude that ρ gives us absolutely no information about which state $|\psi\rangle$ we actually have — every state is an eigenvector of ρ and the eigenvalues of ρ form a uniform distribution.

⇒ the maximally mixed state represents the case where we know nothing about the state of our system.

Ex-

Ex:- Consider a beam of photons. We take a horizontally polarized state $|0\rangle = |\leftrightarrow\rangle$ and a vertically polarized state $|1\rangle = |\updownarrow\rangle$ as orthonormal basis vectors.

If the photons are a totally uniform mixture of 2 polarized states, the density matrix is given by

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$= \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I/2$$

$$tr(\rho^2) = tr(I/4) = \frac{1}{4}\,tr(I) = \frac{1}{4}\cdot 2 = \frac{1}{2}$$

This state is a uniform mixture of $|\updownarrow\rangle$ and $|\leftrightarrow\rangle$ and called a maximally mixed state.

If photons are in a pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the density matrix with $\{|0\rangle, |1\rangle\}$ as basis is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

If $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ is used as a basis vector,
the other being $|\phi\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$,
the density matrix w.r.t the basis $\{|\psi\rangle, |\phi\rangle\}$
has a component expression :

☐ Bloch Sphere for Mixed States

$$S^{\dagger} = S \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \; ; \; a,b,c,d \in \mathbb{C}$$

$$a^* = a \;\&\; d^* = d \Rightarrow a,d \in \mathbb{R}$$

$$c^* = b \;\&\; b^* = c \Rightarrow b = v+iw,\; c = v-iw$$

$\begin{bmatrix} u & v+iw \\ v-iw & z \end{bmatrix}$ is 2x2 Hermitian

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho = n_0\sigma_0 + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$$

$$= \begin{bmatrix} n_0+n_3 & n_1-in_2 \\ n_1+in_2 & n_0-n_3 \end{bmatrix}$$ is Hermitian

The Pauli matrices along with the identity matrix form an orthogonal basis for the complex 2D Hilbert space.
An arbitrary density-matrix for a mixed state qubit is written as real linear combination of $I, \sigma_1, \sigma_2, \sigma_3$.

$$\mathrm{tr}(\rho)=1 \implies n_0 = \tfrac{1}{2}$$

$$\det(\rho-\lambda I) = \begin{vmatrix} n_0+n_3-\lambda & n_1-in_2 \\ n_1+in_2 & n_0-n_3-\lambda \end{vmatrix}.$$

$$= n_0^2 - n_3^2 - 2n_0\lambda + \lambda^2 - n_1^2 - n_2^2$$

$$= \lambda^2 - 2n_0\lambda + n_0^2 - (n_1^2+n_2^2+n_3^2) = 0$$

$$\lambda = n_0 \pm \sqrt{n_1^2+n_2^2+n_3^2}$$

$$= \frac{1}{2}\left[1 \pm \sqrt{(2n_1)^2+(2n_2)^2+(2n_3)^2}\right]$$

ρ is +ve semidefinite $\implies$ $\lambda \geq 0$

$$\sqrt{(2n_1)^2+(2n_2)^2+(2n_3)^2} \leq 1$$

$\sqrt{n_1^2+}$

$\vec{r} = ($

$\therefore$

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state qu

where,

$$\sqrt{n_1^2+n_2^2+n_3^2} \leq 1/2$$

$$\vec{r} = (2n_1, 2n_2, 2n_3)$$

$\therefore$

An arbitrary density matrix for a mixed state qubit may be written as:

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} \quad , \|\vec{r}\| \leq 1$$

where, $\vec{r}$ is a real 3D vector, which is known as the Bloch vector for the state ρ

A state ρ is pure iff $\rho^2 = \rho$ (or) $tr(\rho^2) = 1$

$$\vec{r}\cdot\vec{\sigma} = \|\vec{r}\|\,\hat{n}\cdot\vec{\sigma}$$

$\hat{n}\cdot\vec{\sigma}$ has eigenvalues ± 1

$\vec{r}\cdot\vec{\sigma}$ has eigenvalues $\pm\|\vec{r}\|$

I & $\vec{r}\cdot\vec{\sigma}$ are simultaneously diagonalizable

$\Rightarrow$ The eigenvalues of $\rho = \dfrac{I + \vec{r}\cdot\vec{\sigma}}{2}$ are:

$$\boxed{\lambda_+ = \frac{1}{2}\left(1 + \|\vec{r}\|\right) \quad \& \quad \lambda_- = \frac{1}{2}\left(1 - \|\vec{r}\|\right)}$$

$$tr\left(\rho^2\right) = \lambda_+^2 + \lambda_-^2 = \frac{1}{4}\left[2 + 2\|\vec{r}\|^2\right]$$

$$= \frac{1}{2}\left(1 + \|\vec{r}\|^2\right) = 1$$

$$1 + \|\vec{r}\|^2 = 2 \quad \Longrightarrow \quad \|\vec{r}\| = 1$$

$\Rightarrow$ A state ρ is pure iff $\|\vec{r}\| = 1$.

⇒ The Bloch vector for pure states can take any point on the surface of the sphere.

For mixed states the Bloch vector can take any point inside the sphere.

(Q.) For the maximally mixed state,

$$\rho = I/d$$

$\|\vec{r}\| = 0$

$\longrightarrow$ $\rho = I/d$ is at the origin of the Bloch sphere

(Q.) For pure states, $\|\vec{r}\| = 1$

$$\text{i.e., } \rho = \frac{I + \hat{n}\cdot\vec{\sigma}}{2} = |\psi\rangle\langle\psi|$$

$|\psi(\theta,\phi)\rangle$ is an eigenstate of $\rho = \frac{I + \hat{n}\cdot\vec{\sigma}}{2}$

I & $\hat{n}\cdot\vec{\sigma}$ are simultaneously diagonalizable

$\Longrightarrow$ $|\psi(\theta,\phi)\rangle$ is an eigenstate of $\hat{n}\cdot\vec{\sigma}$.

Ex: 2.73 Let ρ be a density operator.

A minimal ensemble for ρ is an ensemble $\{p_i, |\psi_i\rangle\}$ containing a # of elements equal to the rank of ρ.

Let $|\psi\rangle$ be any state in the support of ρ (The support of a Hermitian operator A is the vectorspace spanned by the eigenvectors of A with non-zero eigenvalues).

Show that there is a minimal ensemble for ρ that contains $|\psi\rangle$, and moreover that in any such ensemble $|\psi\rangle$ must appear with probability, $p_i = \dfrac{1}{\langle\psi_i|\rho^{-1}|\psi_i\rangle}$

where ρ^{-1} is defined to be the inverse of ρ, when ρ is considered as an operator acting only on the support of ρ.

(This definition removes the problem that ρ may not have an inverse).

Ans:

From the unidary freedom in the ensemble for density matrices and the spectral theorem,

$$\rho=\sum_k \lambda_k|k\rangle\langle k| = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$$

$$= \sum_i p_i|\psi_i\rangle\langle\psi_i|$$

$$\Rightarrow |\tilde{\psi}_i\rangle = \sum_k u_{ik}|\tilde{k}\rangle$$

$$\Rightarrow \sqrt{p_i}\,|\psi_i\rangle = \sum_k u_{ik}\sqrt{\lambda_k}\,|k\rangle$$

where, $k=1,2,\cdots,d$ where $d = \text{Rank}(\rho)$

$(|\tilde{\psi}_i\rangle \ldots$

$\Longrightarrow$ Th

$\{p_i,|\psi_i\rangle\}$

$|\psi_j\rangle$

i.e

i.e.,

ensemble

$|\psi_j\rangle =$

$\Rightarrow u$

$$\left(|\tilde{\psi}_1\rangle \cdots |\tilde{\psi}_d\rangle\right)^T = U\left(|\tilde{1}\rangle \cdots |\tilde{d}\rangle\right)^T$$

$\longrightarrow$ The existance of a minimal ensemble $\{p_i, |\psi_i\rangle\}$ for ρ is proved.

$|\psi_j\rangle$ be any state in the support of ρ i.e., in the span of the eigenvectors of ρ.

i.e., $$|\psi_j\rangle = \sum_k c_{jk}|k\rangle$$

If $|\psi_j\rangle$ is part of such ensemble, then

$$|\psi_j\rangle = \sum_k c_{jk}|k\rangle = \sum_k u_{jk}\frac{\sqrt{\lambda_k}}{\sqrt{p_j}}|k\rangle$$

$$\Rightarrow \quad u_{jk} = \frac{c_{jk}\sqrt{p_j}}{\sqrt{\lambda_k}}$$

and also $\sum_k |U_{jk}|^2 = 1$

$$\Longrightarrow P_j \sum_k \frac{|c_{jk}|^2}{\lambda_k} = 1$$

$$\Longrightarrow P_j = \frac{1}{\sum_k \frac{|C_{jk}|^2}{\lambda_k}}$$

With this and the given $|\psi_j\rangle = \sum_k c_{jk}|k\rangle$ we know exactly the j^{th} row of U must be., and using Gram-Schmidt orthogonalization we can construct the rest of U, and choose p_j so that the sum $\sum_j p_j = 1$.

From

ILA 10 $\rightarrow$ $\rho = \sum_k$

$\langle \psi_j | \rho^{-1}$

$\Longrightarrow$

From Moore- Penrose pseudoinverse,

ILA 10 →

$$\rho = \sum_k \lambda_k |k\rangle\langle k| \Longrightarrow \rho^{-1} = \sum_k \frac{1}{\lambda_k} |k\rangle\langle k|$$

$$\langle\psi_j|\rho^{-1}|\psi_j\rangle = \sum_k \frac{1}{\lambda_k} \langle\psi_j|k\rangle\langle k|\psi_j\rangle$$

$$= \sum_k \frac{1}{\lambda_k} \left|\langle\psi_j|k\rangle\right|^2$$

$$= \sum_k \frac{|c_{jk}|^2}{\lambda_k} = \frac{1}{p_j}$$

$$\Longrightarrow p_j = \frac{1}{\langle\psi_j|\rho^{\dagger}|\psi_j\rangle}$$

□ The partial trace operation

How to describe the state
of qubit 1 in the
Bell pair $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$?

ANS: lies in the framework of
density matrices.

$\Longrightarrow$ Partial trace operation

Intuitively,

given a bipartite density matrix ρ_{AB} on systems A and B, the partial trace operation $Tr_B(\rho_{AB})$ returns a density matrix on system A alone. Analogously, $Tr_A(\rho_{AB})$ returns a density matrix on B alone.

$$Tr_B : \mathcal{L}(\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}) \longrightarrow \mathcal{L}(\mathbb{C}^{d_1})$$

The trace of a matrix $\rho \in \mathcal{L}(\mathbb{C}^d)$ is defined as:

$$\rho = \sum_{i=1}^{d} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$\mathrm{Tr}(\rho) = \sum_{i=1}^{d} \rho(i,i)$$

$$= \mathrm{Tr}\left(\sum_i \lambda_i |i\rangle\langle i|\right)$$

$$\left[\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)\right.$$

$$= \sum_i \lambda_i \mathrm{Tr}(|i\rangle\langle i|)$$

$$\left[\mathrm{Tr}(kA) = k\,\mathrm{Tr}(A)\right.$$

$$= \sum_i \lambda_i \langle i|i\rangle = \sum_i \lambda_i$$

$$\left[\mathrm{Tr}(b a^\dagger) = a^\dagger b\right.$$

$$= \sum_{i=1}^{d} \langle i|\rho|i\rangle$$

$$\boxed{\mathrm{Tr}(\rho) = \sum_{i=1}^{d} \rho(i,i) = \sum_{i=1}^{d} \langle i|\rho|i\rangle}$$

For the partial trace, we copy this idea, except we only apply it to one subsystem as follows:

For $\rho \in \mathcal{L}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$

$$\boxed{Tr_B(\rho) = \sum_{i=1}^{d} (I_A \otimes \langle i|)\,\rho\,(I_A \otimes |i\rangle)}$$

⇒ We leave system A untouched (hence the I_A term) and "trace out" system B.

* Like the trace, the partial trace is a linear map.

---

*

$\rho \in$

defined

$Tr(\rho)$

Define $\{|j\rangle\}$

$Tr($

→

*

$\rho \in \mathcal{L}(\mathbb{C}^d)$ & [illegible] trace of ρ is defined as:

$$Tr(\rho) = \sum_{i=1}^{d} \langle i|\rho|i\rangle$$

Define the identity operator on the basis $\{|j\rangle\}$ as:

$$I = \sum_{j=1}^{d} |j\rangle\langle j|$$

$$Tr(\rho) = \sum_{i,j=1}^{d} \langle i|j\rangle\langle j|\rho|i\rangle$$

$$= \sum_{i,j=1}^{d} \langle j|\rho\underbrace{|i\rangle\langle i|}j\rangle = \sum_{j=1}^{d} \langle j|\rho \sum_{i=1}^{d} |i\rangle\langle i|j\rangle$$

$$= \sum_{j=1}^{d} \langle j|\rho|j\rangle$$

$$\longrightarrow \boxed{Tr(\rho) = \sum_{i=1}^{d} \langle i|\rho|i\rangle = \sum_{j=1}^{d} \langle j|\rho|j\rangle}$$

Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}.

The reduced density operator for system A is defined by,

$$\rho^{A} = tr_B(\rho^{AB})$$

where tr_B is a map of operators known as the partial trace over system B

Example 1 :

Suppose

of a p

$\rho = (|\psi$

$= |\psi$

The sta

Applying

subsystem

$Tr_B(\rho)$

Example 1 : Product State

Suppose that ρ is the density matrix of a pure product state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

$$\rho = \left(|\psi_1\rangle \otimes |\psi_2\rangle\right)\left(\langle\psi_1| \otimes \langle\psi_2|\right) \qquad \left[(A\otimes B)(C\otimes D) = (AC)\otimes(BD)\right.$$

$$= |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|$$

The state of $|\psi\rangle$ on qubit 1 is exactly $|\psi_1\rangle$.

Applying partial trace to ρ to trace out subsystem B :

$$Tr_B(\rho) = Tr_B\left(|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|\right)$$

$$= \sum_{i=1}^{d}\left(I \otimes \langle i|\right)|\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|\left(I \otimes |i\rangle\right)$$

$$= \sum_{i=1}^{d} |\varphi_1\rangle\langle\varphi_1| \otimes \langle i|\varphi_2\rangle\langle\varphi_2| i\rangle$$

$$= |\varphi_1\rangle\langle\varphi_1| \left(\sum_{i=1}^{d} \langle i|\underline{\varphi_2\rangle\langle\varphi_2}| i\rangle\right)$$

[Note:
$Tr(A) = \sum_i \langle\lambda_i|A|\lambda_i\rangle$
$= \sum_i \lambda_i$]

$$= |\varphi_1\rangle\langle\varphi_1| \cdot Tr\left(|\varphi_2\rangle\langle\varphi_2|\right)$$

$$= |\varphi_1\rangle\langle\varphi_1| \cdot \langle\varphi_2|\varphi_2\rangle$$

$$Tr_B(\rho) = |\varphi_1\rangle\langle\varphi_1|$$

where, $\rho = |\varphi_1\rangle\langle\varphi_1| \otimes |\varphi_2\rangle\langle\varphi_2|$

$\Rightarrow$ the state of qubit 1 is $|\varphi_1\rangle$.
as we claimed.

Note:

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Proof

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Note: The calculation (previous) did not use the fact that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is pure

For any operator, $\rho = \rho_1 \otimes \rho_2$

$$Tr_B(\rho) = \rho_1 . Tr(\rho_2)$$

$$Tr_A(\rho) = Tr(\rho_1) \cdot \rho_2$$

Proof $\rho = \rho_1 \otimes \rho_2$

$$\begin{aligned} Tr_A(\rho) &= Tr_A(\rho_1 \otimes \rho_2) \\ &= \sum_{i=1}^{d} (\langle i| \otimes I)\, \rho_1 \otimes \rho_2 \,(|i\rangle \otimes I) \\ &= \sum_{i=1}^{d} \langle i|\rho_1|i\rangle \otimes \rho_2 \\ &= \left(\sum_{i=1}^{d} \langle i|\rho_1|i\rangle\right) \rho_2 \\ &= Tr(\rho_1) \cdot \rho_2 \quad // \end{aligned}$$

Example 2 : Separable states

* A pure state $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is not entangled, or separable, iff $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for some $|\psi_1\rangle \in \mathbb{C}^{d_1}$ and $|\psi_2\rangle \in \mathbb{C}^{d_2}$.

* A bipartite density matrix $\rho \in \mathcal{L}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}\right)$ is unentangled or separable if

$$\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|$$

for some (possibly non-orthogonal) sets of vectors $\{|\psi_i\rangle\} \subseteq \mathbb{C}^{d_1}$ and $\{|\phi_i\rangle\} \subset \mathbb{C}^{d_2}$; and where the $\{p_i\}$ form a probability distribution.

ie., ρ is a probabilistic mixture of pure product states.

ie., ρ is a probabilistic mixture of pure product states.

Ex:-

$$\rho = \frac{1}{2}|0\rangle\langle 0|\otimes|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\otimes|1\rangle\langle 1|$$

Partial trace is a linear map, and $Tr_B(\rho_1 \otimes \rho_2) = \rho_1 Tr(\rho_2) = \rho_1$. for density matrices ρ_1, ρ_2 . Computing the partial trace of ρ for separable states :

$$Tr_B(\rho) = Tr_B\left(\sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|\right)$$

$$= \sum_i p_i Tr_B\left(|\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|\right)$$

$$= \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes Tr\left(|\phi_i\rangle\langle\phi_i|\right)$$

$$= \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Ex:- $\rho = \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$

Ans: $\mathrm{Tr}_B(\rho) = \frac{1}{2}|0\rangle\langle 0| \otimes \mathrm{Tr}(|0\rangle\langle 0|) + \frac{1}{2}|1\rangle\langle 1| \otimes \mathrm{Tr}(|1\rangle\langle 1|)$

$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

- It is easy to check if a bipartite pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is entangled by determining its Schmidt rank.

In stark contrast,
it turns out that determining whether a mixed state $\rho \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is separable is NP-hard!

The settings of pure v/s mixed states are indeed very difficult; the later is much more difficult to work with.

Example 3 : Pure entangled states

Pure entangled states such as

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

To compute the partial trace of any pure bipartite $|\Psi\rangle$, one approach is to simply write $|\Psi\rangle$ in the standard basis, take its density matrix $\rho = |\Psi\rangle\langle\Psi|$, and then compute $Tr_B(\rho)$.

Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$Tr_B\left(|\Phi^+\rangle\langle\Phi^+|\right) = \frac{1}{2}Tr_B\left(\left(|00\rangle+|11\rangle\right)\left(\langle 00| + \langle 11|\right)\right)$$

$$= \frac{1}{2}Tr_B\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$= \frac{1}{2}Tr_B\left(|00\rangle\langle 00|\right) + \frac{1}{2}Tr\left(|00\rangle\langle 11|\right) + \frac{1}{2}Tr\left(|11\rangle\langle 00|\right) + \frac{1}{2}Tr\left(|11\rangle\langle 11|\right)$$

$$Tr_B\left(|00\rangle\langle 11|\right) = Tr_B\left(\left(|0\rangle\otimes|0\rangle\right)\left(\langle 1|\otimes\langle 1|\right)\right)$$

$$= Tr_B\left(|0\rangle\langle 1| \otimes |0\rangle\langle 1|\right) \qquad \left[(A\otimes B)(C\otimes D) = (AC)\otimes(BD)\right]$$

$Tr_B($|

$Tr((\rho_A)^2)$

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$$Tr_B(|\Phi^+\rangle\langle\Phi^+|) = \frac{1}{2}Tr_B(|0\rangle\langle 0|\otimes|0\rangle\langle 0|) + \frac{1}{2}Tr_B(|0\rangle\langle 1|\otimes|0\rangle\langle 1|)$$

$$+ \frac{1}{2}Tr_B(|1\rangle\langle 0|\otimes|1\rangle\langle 0|) + \frac{1}{2}Tr_B(|1\rangle\langle 1|\otimes|1\rangle\langle 1|)$$

$$= \frac{1}{2}|0\rangle\langle 0|\,Tr(|0\rangle\langle 0|) + \frac{1}{2}|0\rangle\langle 1|\,Tr(|0\rangle\langle 1|)$$

$$+ \frac{1}{2}|1\rangle\langle 0|\,Tr(|1\rangle\langle 0|) + \frac{1}{2}|1\rangle\langle 1|\,Tr(|1\rangle\langle 1|)$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$= \frac{1}{2}\sum_{i=1}^{2}|i\rangle\langle i| = \underline{\underline{\frac{1}{2}\cdot I}}.$$

$Tr((\rho_A)^2) = \frac{1}{2} < 1$

$\Rightarrow$ The reduced state on qubit 1 for the Bell state is maximally mixed.

i.e., it is a completely random state about which we have zero information.

The joint state of a system can be completely known (pure state), yet a subsystem be in mixed states, a state about which we apparently do not have maximal knowledge.

Here,

we arrive at one of the most confounding aspects of quantum mechanics:

⇒ In the case of the Bell state, it is possible for us to know absolutely nothing about the states of qubit 1 and 2 individually (eg. they have reduced states $I/2$),

But,

when we bring both qubits together, we know everything about their joint state (eg. $|\Phi^+\rangle$ is a pure state, i.e., there is no uncertainty).

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□ Why the partial trace ?

Why is the partial trace used to describe part of a larger quantum system ?

Suppose M is any observable on system A. and we have some measuring device which is capable of realizing measurements of M. Let $\tilde{M}$ denote the corresponding observable for the same measurement, performed on the composite system, AB.

1, Our goal is to argue that $\tilde{M}$ is necessarily equal to $M \otimes I_B$.

If the system AB is prepared in the state $|m\rangle|\psi\rangle$, where $|m\rangle$ is an eigenstate of M with eigenvalue m, and $|\psi\rangle$ is any state of B, then the measuring device must yield the result m for the measure with probability one.

Thus, if P_m is the projector onto the m eigenspace of the observable M, then the corresp. projector for $\tilde{M}$ is $P_m \otimes I_m$.

$$P_m|m\rangle = m|m\rangle \longrightarrow M = \sum_m m P_m$$

$$(P_m \otimes I_m)|m\rangle|\psi\rangle = m|m\rangle|\psi\rangle$$

$$\Rightarrow \tilde{M} = \sum_m m P_m \otimes I_B . \quad \left[(A+B)\otimes C = A\otimes C + B\otimes C \right.$$

$$= M \otimes I_B$$

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$P(m) =$

$\langle M \rangle = \sum_m$

q, The partial trace procedure gives the correct measurement statistics for observations on part of a system.

→

Suppose we perform a measurement on system A described by the observable M. Physical consistency requires that any prescription for associating a state, ρ^A, to system A, must have the property that measurement averages be the same whether computed via ρ^A or ρ^{AB}.

Observable M has the spectral decomposition,

Pure state

$$M = \sum m P_m = \sum m |m\rangle\langle m|$$

$$|\psi\rangle = \sum_m a_m |m\rangle \longrightarrow a_m = P_m|\psi\rangle = |m\rangle\langle m|\psi\rangle$$

$$p(m) = |a_m|^2 = \| P_m|\psi\rangle\|^2 = \langle\psi|P_m|\psi\rangle = tr(P_m|\psi\rangle\langle\psi|)$$

$$\langle M\rangle = \sum_m m p(m) = \sum_m m\langle\psi|P_m|\psi\rangle = \langle\psi|\sum_m m P_m|\psi\rangle$$

$$= \langle\psi|M|\psi\rangle = tr\left(M|\psi\rangle\langle\psi|\right) = tr\left(M\rho\right)$$

Mixed state

$$P(m|i) = \langle\psi_i|P_m|\psi_i\rangle = \mathrm{tr}\left(P_m|\psi_i\rangle\langle\psi_i|\right)$$

$$P(m) = \sum_i P(m|i)\, P_i = \sum_i P_i\, \mathrm{tr}\left(P_m|\psi_i\rangle\langle\psi_i|\right)$$

$$= \mathrm{tr}\left(P_m \sum_i P_i |\psi_i\rangle\langle\psi_i|\right)$$

$$\left[\begin{array}{l} \mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B) \\ \mathrm{tr}(kA) = k\,\mathrm{tr}(A) \end{array}\right.$$

$$= \mathrm{tr}\left(P_m \rho\right)$$

$$\langle M\rangle = \sum_m m\, p(m)$$

$$= \sum_m m\, \mathrm{tr}\left(P_m \rho\right)$$

$$= \mathrm{tr}\left[\left(\sum_m m P_m\right)\rho\right]$$

$$= \mathrm{tr}\left(M\rho\right)$$

Proposition -

QC 11.2

$(M\rho$

$\mathrm{tr}(M\rho$

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$\rho^A =$

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$M = \sum_{a,a'}$

$\mathrm{tr}(M\rho^A)$

Proposition $\longrightarrow$ $tr(M\rho^A) = tr(\tilde{M}\rho^{AB})$

$$= tr((M \otimes I_B)\rho^{AB})$$

QC 11.2 $\rightarrow$

$$(M\rho^A)_{ab} = \sum_{a'} M_{aa'} \rho^A_{a'b}.$$

$$tr(M\rho^A) = \sum_{a} (M\rho^A)_{aa} = \sum_{a,a'} M_{aa'} \rho^A_{a'a}$$

(OR)

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$$\rho^A = \sum_{a,a'} \rho^A_{a,a'} |a\rangle\langle a'|$$

where $\{|a\rangle\}$ is an orthonormal basis

$$tr(M\rho^A) = \sum_{a,a'} \rho^A_{a,a'} tr(M|a\rangle\langle a'|)$$

$$= \sum_{a,a'} \rho^A_{a,a'} \langle a'|M|a\rangle$$

$$M = \sum_{a'',a'''} M_{a'',a'''} |a''\rangle\langle a'''| \rightarrow \langle a'|M|a\rangle = \langle a'|\sum_{a,a'} M_{a'',a'''}|a''\rangle\langle a'''|a\rangle$$

$$= M_{a',a}$$

$$tr(M\rho^A) = \sum_{a,a'} M_{a',a} \rho^A_{a,a'} = \sum_{a,a'} \rho^A_{a,a'} M_{a',a}$$

$$\rho^{AB}_{9\times9} = \begin{bmatrix} 0\left[\;''\;\right] & 1\left[\rho_{12}\right] & 0\left[\;''\;\right] \\ 0\left[\;''\;\right] & 0\left[\;''\;\right] & 0\left[\;''\;\right] \\ 0\left[\;''\;\right] & 0\left[\;''\;\right] & 0\left[\;''\;\right] \end{bmatrix} +$$

$$\begin{bmatrix} 0\left[\;''\;\right] & 0\left[\;''\;\right] & 0\left[\;''\;\right] \\ 0\left[\;''\;\right] & 0\left[\;''\;\right] & 0\left[\;''\;\right] \\ 0\left[\;''\;\right] & 1\left[\rho_{32}\right] & 0\left[\;''\;\right] \end{bmatrix} + \cdots\cdots$$

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\rho^{AB}_{aa'}$: aa' block entry of ρ^{AB}

$E_{aa'}$: matrix with the aa' entry 1 and 0 elsewhere.

$\rho^{AB} =$

$tr((M$

QC ①

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$$\rho^{AB} = \sum_{a,a'} E_{aa'} \otimes \rho^{AB}_{aa'}$$

$$tr\left((M \otimes I_B)\rho^{AB}\right) = tr\left((M \otimes I_B)\sum_{a,a'}\left(E_{aa'} \otimes \rho^{AB}_{aa'}\right)\right)$$

$$= tr\left(\sum_{a,a'}(M \otimes I_B)\left(E_{aa'} \otimes \rho^{AB}_{aa'}\right)\right)$$

$$= \sum_{a,a'} tr\left((M \otimes I_B)\left(E_{aa'} \otimes \rho^{AB}_{aa'}\right)\right)$$

$$= \sum_{a,a'} tr\left(ME_{aa'} \otimes \rho^{AB}_{a,a'}\right)$$

QC ① → $$= \sum_{a,a'} tr\left(ME_{aa'}\right) tr\left(\rho^{AB}_{aa'}\right) \quad \left[tr(A \otimes B) = tr(A) \cdot tr(B) \right.$$

$$(ME_{aa'})_{ij} = \sum_k M_{ik}(E_{aa'})_{kj}$$

$$tr\left(ME_{aa'}\right) = \sum_i (ME_{aa'})_{ii} = \sum_{i,k} M_{ik}(E_{aa'})_{ki}$$

$$= M_{a'a}(E_{aa'})_{aa'} = M_{a'a}$$

$$\boxed{tr\left((M \otimes I_B)\rho^{AB}\right) = \sum_{a,a'} M_{a'a}\, tr\left(\rho^{AB}_{aa'}\right)}$$

Equivalently,

$$\rho^{AB} = \sum_{a,a',b,b'} \rho^{AB}_{aba'b'} \, E_{aa'} \otimes E_{bb'}$$

$\rho^{AB}_{aba'b'}$: bb' entry of the block aa'.

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$$tr\left((M\otimes I_B)\rho^{AB}\right) = tr\left((M\otimes I_B)\sum_{a,a',b,b'}\rho^{AB}_{aba'b'}E_{aa'}\otimes E_{bb'}\right)$$

$$= tr\left[\sum_{a,a',b,b'}\rho^{AB}_{aba'b'}(M\otimes I_B)(E_{aa'}\otimes E_{bb'})\right]$$

$$= \sum_{a,a',b,b'}\rho^{AB}_{aba'b'}\, tr\left[(M\otimes I_B)(E_{aa'}\otimes E_{bb'})\right]$$

$$= \sum_{a,a',b,b'}\rho^{AB}_{aba'b'}\, tr(ME_{aa'})\, tr(E_{bb'})$$

$$tr(ME_{aa'}) = M_{a'a} \quad \left[\text{check previous page}\right]$$

$$= \sum_{a,a',b,b'}\rho^{AB}_{aba'b'} M_{a'a}\,\delta_{bb'}$$

$$\boxed{tr\left((M\otimes I_B)\rho^{AB}\right) = \sum_{a,a',b}\rho^{AB}_{aba'b} M_{a'a}}$$

$$tr(M\rho^{A}) = tr((M\otimes I_B)\rho^{AB})$$

$$\Downarrow$$

$$\sum_{a,a'} M_{aa'}\rho^{A}_{aa'} = \sum_{a,a'} M_{a'a}\, tr(\rho^{AB}_{aa'})$$

$$= \sum_{a,a',b} \rho^{AB}_{aba'b} M_{a'a}$$

since, $tr(\rho^{AB}_{aa'}) = \sum_{b} \rho^{AB}_{aba'b}$

$\rho_{aba'b}$: bb entry of the block aa'

Since

$\rho^{A}_{aa'} = \sum$

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Since this holds for any M, we have

$$\boxed{\begin{aligned} \rho^{A}_{aa'} &= \sum_{b} \rho^{AB}_{aba'b} = tr\left(\rho^{AB}_{aa'}\right) \\ &= \left[tr_{B}\left(\rho^{AB}\right)\right]_{a,a'} \end{aligned}}$$

where, $\rho^{AB}_{aba'b}$: bb entry of the block aa'

$\rho^{AB}_{a,a'}$: aa' block entry of ρ^{AB}.

→ Ths looks like taking a trace over the B subsystem (i.e., summing over the b=b' entries) while leaving the A system alone.

∴ We call the map from $\rho^{AB} \rightarrow \rho^{A}$ the "partial trace", i.e., $\boxed{\rho^{A} = tr_{B}\left(\rho^{AB}\right)}$

(OR) $$\rho^{AB} = \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'|$$

Q.C① unitary freedom in the ensemble of density matrices}.

where, $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthonormal basis.

$$tr((M \otimes I_B)\rho^{AB}) = \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'}\, tr\left[M|a\rangle\langle a'| \otimes |b\rangle\langle b'|\right]$$

$$= \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'}\, tr(M|a\rangle\langle a'|)\, tr(|b\rangle\langle b'|)$$

$$= \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} \langle a'|M|a\rangle\, \delta_{b'b}$$

$$= \sum_{a,a',b} \rho^{AB}_{a,a',b,b} \langle a'|M|a\rangle$$

$$M = \sum_{a,a'} M_{a',a} |a'\rangle\langle a| \Longrightarrow \langle a'|M|a\rangle = \langle a'|\sum_{a,a'} M_{a',a}|a'\rangle\langle a'|a\rangle = M_{a',a}$$

$$tr((M \otimes I_B)\rho^{AB}) = \sum_{a,a',b} \rho^{AB}_{a,a',b,b} M_{a'a}$$

Similarly,

$$\left(\rho^{B}_{bb'}\right) = \left[tr_{A}\left(\rho^{AB}\right)\right]_{b,b'} = \sum_{a} \rho^{AB}_{abab'}$$

$$\rho_{B} = tr_{A}\left(\rho^{AB}\right)$$

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Define,

$$\rho^{AB} = \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'|$$

where $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthonormal basis for $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively.

$$\cdot\ tr_B(\rho^{AB}) = \sum_{i=1}^{d_2} (I_A \otimes \langle i|)\rho^{AB}(I_A \otimes |i\rangle)$$

$$= \sum_{j=1}^{d_2} (I_A \otimes \langle j|)\rho^{AB}(I_A \otimes |j\rangle)$$

where $\{|i\rangle\}$ and $\{|j\rangle\}$ are any set of orthonormal basis for $\mathbb{C}^{d_2}$.

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orthono

$tr_B(\rho^{AB})$

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* $\rho^{AB}_{aa'bb'}$

we

- Taking the partial trace of system B in the orthonormal Basis $\{|b\rangle\}$,

$$tr_B(\rho^{AB}) = \sum_{b''} (I_A \otimes \langle b''|)\, \rho^{AB} (I_A \otimes |b''\rangle)$$

$$= \sum_{b''} (I_A \otimes \langle b''|) \left(\sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'| \right) (I_A \otimes |b''\rangle)$$

$$= \sum_{a,a',b,b',b''} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes \langle b''|b\rangle\langle b'|b''\rangle$$

$$= \sum_{a,a',b,b',b''} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes \delta_{b''b}\, \delta_{b'b''}$$

$$= \sum_{a,a',b} \rho^{AB}_{a,a',b,b} |a\rangle\langle a'|$$

* $\rho^{AB}_{a,a',b,b'}$ depend on the Basis
we can always choose the standard Basis.

*

$$tr_B\left(\rho^{AB}\right) = \sum_b \left(I_A \otimes \langle b|\right)\rho^{AB}\left(I_A \otimes |b\rangle\right)$$

$$= \sum_{a,a',b} \rho^{AB}_{a,a',b,b} |a\rangle\langle a'|$$

where,

$$\rho^{AB} = \sum_{a,a',b,b'} \rho^{AB}_{aa',b,b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'|$$

☐ Partial trace to detect entanglement

An arbitrary bipartite pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ can be written in terms of its Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |e_{1,i}\rangle \otimes |e_{2,i}\rangle \leftarrow |\psi\rangle = \sum_{ij} c_{ij} |e_{1,i}\rangle \langle e_{2,j}|$$

$$= \sum_i \alpha_i |a_i\rangle |b_i\rangle$$

where $\alpha_i \geq 0$ and $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ are orthonormal sets.

$|\psi\rangle$ is entangled iff its Schmidt rank (i.e., # of non-zero Schmidt coefficients α_i) is atleast 2.

* Schmidt rank of $|\psi\rangle \in \mathbb{C}^{d} \otimes \mathbb{C}^{d}$

$$||$$

$$\text{rank of } \rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$$

Proof

For an arbitrary bipartite pure state

$$|\psi\rangle = \sum_{i,j} c_{ij} |e_{1,i}\rangle \otimes |e_{2j}\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}:$$

can be written in terms of its Schmidt decomposition:

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i}\, |f_{1i}\rangle \otimes |f_{2i}\rangle$$

where, s_i : Schmidt coefficients.

r : Schmidt # of $|\psi\rangle$

$\{|f_{1,i}\rangle$

$|f_{1,k}\rangle$

$|f_{2,k}\rangle$

Coeff. C

SVD

$|\psi\rangle =$

$=$

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$=$

$\{|f_{1,i}\rangle\}$, $\{|f_{2,j}\rangle\}$ are orthonormal set.

$$|f_{1,k}\rangle = \sum_{i=1}^{d_1} U_{ik} |e_{1,i}\rangle$$

$$|f_{2,k}\rangle = \sum_{j=1}^{d_2} V^*_{jk} |e_{2,j}\rangle$$

Coeff. C_{ij} form a matrix $C_{d_1 d_2}$.

SVD of C is, $C = U\Sigma V^\dagger$

U, V : unitary matrices

$$|\psi\rangle = \sum_{ijkl} U_{ik} \Sigma_{kl} V^*_{jl} |e_{1,i}\rangle \otimes |e_{2,j}\rangle$$

$$= \sum_{i,j,k,l} \Sigma_{kl} U_{ik} |e_{1,i}\rangle \otimes V^*_{jl} |e_{2,j}\rangle$$

$$= \sum_{k,l} \Sigma_{kl} \left(\sum_i U_{ik} |e_{1,i}\rangle \otimes \sum_j V^*_{jl} |e_{2,j}\rangle \right)$$

$$= \sum_{kl} \Sigma_{kl} |f_{1,k}\rangle \otimes |f_{2,l}\rangle$$

$\Sigma_{kl} = d_k \delta_{kl}$

$$|\psi\rangle = \sum_{i=1}^{r} d_i |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

r: # of non-vanishing diagonal elements in Σ.

$d_i = \sqrt{s_i}$

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i} |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

Normalization condition,

$\langle\psi|\psi\rangle = \sum s_i = 1$

$\Longrightarrow$ A bipartite state is separable iff its Schmidt number r is 1.

i.e., # of non-vanishing diagonal elements in Σ is 1.

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In the definition of the partial trace, the standard basis $\{|i\rangle\}$ on B can be replaced by an arbitrary orthonormal basis $\{|k_{B,i}\rangle\}$ on B, which is the Schmidt basis for $|\psi\rangle$ on system B.

$$Tr_B(\rho) = Tr_B\left(|\psi\rangle\langle\psi|\right)$$

$$= Tr_B\left(\left(\sum_{i=1}^{r}\sqrt{s_i}\,|f_{1,i}\rangle\otimes|f_{2,i}\rangle\right)\left(\sum_{j=1}^{r}\sqrt{s_j}\,\langle f_{1,j}|\otimes\langle f_{2,j}|\right)\right)$$

$$= Tr_B\left(\sum_{i,j=1}^{r}\sqrt{s_i s_j}\left(|f_{1,i}\rangle\otimes|f_{2,i}\rangle\right)\left(\langle f_{1,j}|\otimes\langle f_{2,j}|\right)\right)$$

$$(A\otimes B)(C\otimes D) = AC \otimes BD$$

$$= Tr_B\left(\sum_{i,j=1}^{r}\sqrt{s_i s_j}\,|f_{1,i}\rangle\langle f_{1,j}|\otimes|f_{2,i}\rangle\langle f_{2,j}|\right)$$

$$Tr(A+B) = Tr(A) + Tr(B)$$

$$= \sum_{i,j=1}^{r} Tr_B\left(\sqrt{s_i s_j}\,|f_{1,i}\rangle\langle f_{1,j}|\otimes|f_{2,i}\rangle\langle f_{2,j}|\right)$$

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Basis,

$$Tr_B(\rho)$$

$$Tr_B(\rho) =$$

$$= \sum$$

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Partial trace defined on the Schmidt Basis,

$$Tr_B(\rho) = \sum_i \left(I_A \otimes \langle f_{2,i}|\right) \rho \left(I_A \otimes |f_{2,i}\rangle\right)$$

$$\left[\begin{aligned} Tr(\rho) &= \sum_i \langle i|\rho|i\rangle \\ &= \sum_j \langle j|\rho|j\rangle \end{aligned}\right.$$

$$Tr_B(\rho) = \sum_{i,j=1}^{r} \left(\sum_k \left(I_A \otimes \langle f_{2,k}|\right)\left(\sqrt{s_i s_j}\, |f_{1,i}\rangle\langle f_{1,j}| \otimes |f_{2,i}\rangle\langle f_{2,j}|\right) \left(I_A \otimes |f_{2,k}\rangle\right) \right)$$

$$= \sum_{i,j=1}^{r} \left(\sum_{k=1}^{r} \sqrt{s_i s_j}\, |f_{1,i}\rangle\langle f_{1,j}| \otimes \langle f_{2,k}|f_{2,i}\rangle\langle f_{2,j}|f_{2,k}\rangle \right)$$

$$\langle f_{2,j}|f_{2,k}\rangle = \delta_{jk}$$

$$= \sum_{i=1}^{r} s_i\, |f_{1,i}\rangle\langle f_{1,i}|$$

Schmidt decomposition of an arbitrary bipartite pure state

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i}\, |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$ is :

$$Tr_B(\rho) = Tr_B(|\psi\rangle\langle\psi|) = \sum_{i=1}^{r} s_i |f_{1,i}\rangle\langle f_{1,i}|$$

* The Schmidt coefficients $\sqrt{s_i} \geq 0$ are the square roots of the eigenvalues of the two partial traces of the density matrix $\rho = |\psi\rangle\langle\psi|$.

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* Schmidt basis $\{|f_{1,i}\rangle\}$ for system A are the eigenvectors of $tr_B(\rho)$.

•→ Let $|\psi\rangle$ be a pure state of a composite system AB. Then by the Schmidt decomposition, $\rho_A = \sum_i d_i^2 |f_{1,i}\rangle\langle f_{1,i}|$ and $\rho_B = \sum_i d_i^2 |f_{2,i}\rangle\langle f_{2,i}|$, so the eigenvalues of ρ_A and ρ_B are identical, namely d_i^2 for both density operators.

* To determine whether a bipartite pure state $|\psi\rangle$ is entangled —

Compute $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$

If $\mathrm{rank}(\rho_A) > 1$, then $|\psi\rangle$ is entangled.

otherwise,

$|\psi\rangle$ is a product state.

$\mathrm{rank}(\rho_A) = 1$

Tr

This only works for pure states $|\psi\rangle$, for mixed bipartite, detecting entanglement is NP-hard, so such a simple criterion for entanglement is highly unlikely to exist!

* $|\psi\rangle$ is a product state iff ρ^A (and ρ^B) are pure states.

Proof

$\boxed{\text{Product state} \Longleftrightarrow \rho^A \,\&\, \rho^B \text{ are pure states}}$

Using the Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$$

$$\rho_{AB} = |\psi\rangle\langle\psi| = \sum_{i,j} \lambda_i \lambda_j \left(|i_A\rangle \otimes |i_B\rangle\right)\left(\langle j_A| \otimes \langle j_B|\right)$$

$$= \sum_{i,j}$$

$$\rho^A = tr_B(\rho^{AB}) = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$$

$$\rho^B = tr_A(\rho^{AB}) = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

If the Schmidt rank of $|\psi\rangle$ is 1.

Then,

$$\rho^A = |i_A\rangle\langle i_A|$$

$\Longrightarrow$ ρ^A & ρ^B are pure states.

Suppose ρ^A & ρ^B are pure states. Then,

$$\rho^A = |i_A\rangle\langle i_A|$$

☐ Purification

Suppose we are given a state ρ^A of a quantum system A.
It is possible to introduce another system R, and define a pure state $|AR\rangle$ for the joint system AR such that

$$\rho^A = tr_R\left(|AR\rangle\langle AR|\right)$$

ie.,
the pure state $|AR\rangle$ reduces to ρ^A when we look at system A alone.

This is a purely mathematical procedure known as purification, which allows us to associate pure states with mixed states.

∴ We call system R a reference system: it is a fictitious system, without a direct physical significance.

Purification can be done for any state.

How to construct a system R and purification $|AR\rangle$ for ρ^A ?

Suppose ρ^A has orthonormal decomposition,

$$\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$$

[Any density operator may be expanded in its eigenbasis $|i^A\rangle$.

To purify ρ^A we introduce a system R which has the same state space as system A, with orthonormal basis states $|i^R\rangle$

We can define a pure state for the combined system:

$$|AR\rangle \equiv \sum_i \sqrt{p_i}\, |i^A\rangle \otimes |i^R\rangle$$

The reduced density operator for system A corresponding to the state $|AR\rangle$:

$$|AR\rangle\langle AR| = \left(\sum_i \sqrt{p_i}\, |i^A\rangle \otimes |i^R\rangle\right)\left(\sum_j \sqrt{p_j}\, \langle j^A| \otimes \langle j^R|\right)$$

$$\sum_{i,j} \sqrt{p_i p_j}\, \left(|i^A\rangle \otimes |i^R\rangle\right)\left(\langle j^A| \otimes \langle j^R|\right)$$

$$= \sum_{ij} \sqrt{p_i p_j}\, \left(|i^A\rangle\langle j^A|\right) \otimes \left(|i^R\rangle\langle j^R|\right)$$

$$tr_R\left(|AR\rangle\langle AR|\right) = \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A| \cdot tr\left(|i^R\rangle\langle j^R|\right)$$

$$= \sum_{ij} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A| \cdot \langle j^R|i^R\rangle$$

$$= \sum_{i,j} \sqrt{p_i p_j}\, |i^A\rangle\langle j^A| \cdot \delta_{ij}$$

$$= \sum_i p_i\, |i^A\rangle\langle i^A|$$

$$= \rho^A$$

$\longrightarrow$ $|AR\rangle = \sum_i \sqrt{p_i}\, |i^A\rangle \otimes |i^R\rangle$ is a purification of ρ^A

Schmidt decomposition:

Suppose $|\psi\rangle = \sum_{j,k} a_{jk}\, |j\rangle \otimes |k\rangle$ is a pure state of a composite systems, AB. Then there exists orthonormal states $|i_A\rangle$ for system A and orthonormal states $|i_B\rangle$ of system B such that:

$$|\psi\rangle = \sum_i \lambda_i\, |i_A\rangle \otimes |i_B\rangle$$

where λ_i: non-negative real # called Schmidt coeffs with $\sum_i \lambda_i^2 = 1$

$$|\psi\rangle = \sum_{j,k} a_{jk}|j\rangle\otimes|k\rangle$$

where $|j\rangle, |k\rangle$: any fixed orthonormal basis for systems A, B, respectively.

$a_{jk} \in \mathbb{C}$

Schmidt decomposition

$$\Longrightarrow \quad |\psi\rangle = \sum_i \lambda_i |i_A\rangle\otimes|i_B\rangle$$

$$\rho^A = tr_B(\rho^{AB}) = tr_B(|\psi\rangle\langle\psi|) = \sum_i \lambda_i^2 |i_A\rangle\langle i_A|$$

$$\rho^B = tr_B(\rho^{AB}) = tr_A(|\psi\rangle\langle\psi|) = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

$$\Longrightarrow \lambda_i \geq 0 \quad \& \quad \sum_i \lambda_i^2 = 1$$

$|i_A\rangle, |i_B\rangle$: Schmidt orthonormal basis for systems A and B, respectively.

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Purification (cont...)

$$\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$$

$$|AR\rangle = \sum_i \sqrt{p_i}\, |i^A\rangle \otimes |i^R\rangle$$

$$tr_R(|AR\rangle\langle AR|) = \sum_i p_i |i^A\rangle\langle i^A| = \rho^A$$

→ The procedure used to purify a mixed state of system A, is to define a pure state $|AR\rangle$ whose Schmidt basis $|i_A\rangle$ for system A is just the basis in which the mixed state is diagonal, with the Schmidt coeff. being the square roots of the eigenvalues of the density operator being purified.

LA⑪

ILA ⑪

$$[V]_C = P_{C\leftarrow B}[V]_B$$

$$P_{C\leftarrow B} = \Big[[b_1]_C \cdots [b_n]_C\Big] = C^{-1}B$$

$\boxed{2}$

$$[T]_B = \Big[[T(b_1)]_B \cdots\cdots [T(b_n)]_B\Big]$$

Let $\{|b_i\rangle\}$ be an eigenbasis

$$T|b_i\rangle = b_i|b_i\rangle \implies [T|b_i\rangle]_B = \begin{bmatrix} \cdot \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \text{ } i^{th} \text{ value is } b_i$$

$$[T]_B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix}$$

⇒ Matrix corresp. to a linear transformation $T: V \to V$ is diagonal w.r.t its eigen basis.

Ex: 2.79 Consider composite system consisting of 2 qubits. Find the Schmidt decompositions of the states

ⓐ $$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Ans. $$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$$

$$|\psi\rangle = \sum_{j,k} a_{jk}|j\rangle \otimes |k\rangle$$

$$\Longrightarrow |\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$$

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \sum_{i=1}^{2} \frac{1}{\sqrt{2}}|i\rangle \otimes |i\rangle$$

ⓑ $$|\psi_2\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle \otimes |1\rangle}{\sqrt{2}}$$

$$= |+\rangle \otimes |+\rangle \Longleftarrow$$

Ⓒ $|\psi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}$

Ans:

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$$

$$\rho^{AB} = |\psi_3\rangle\langle\psi_3| = \frac{1}{3}\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} & \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\\ \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} & \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\end{bmatrix}$$

$$(\rho^A)_{a,a'} = \left(tr_B(\rho^{AB})\right)_{aa'} = tr\left(\rho^{AB}_{aa'}\right)$$

where, $\rho^{AB}_{a,a'}$: a,a' block entry of ρ^{AB}

$$\Longrightarrow \rho^A = \frac{1}{3}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} = \rho^B$$

$$\det(\rho^A - \lambda I) = 0 \Rightarrow 9\lambda^2 - 9\lambda + 1 = 0$$

$$\rightarrow \quad \lambda_\pm = \frac{3 \pm \sqrt{5}}{6}$$

$\lambda_+ =$

$${}^{2}|\psi_3\rangle = \sum_{i=\{+,-\}} \sqrt{\lambda_i}\, |\lambda_i\rangle \otimes |\lambda_i\rangle$$

E. 2.80 Suppose $|\psi\rangle$ and $|\phi\rangle$ are 2 pure states of a composite quantum system with components A and B, with identical Schmidt coeff. Show that there are unitary transformation U on system A and V on system B such that

$$|\psi\rangle = (U \otimes V)|\phi\rangle .$$

Ans: The Schmit decomposition of $|\phi\rangle$,

$$|\phi\rangle = \sum_i \lambda_i \, |i_A\rangle \otimes |i_B\rangle$$

$$|\psi\rangle = (U \otimes V)|\phi\rangle = \sum_i \lambda_i \, U|i_A\rangle \otimes V|i_B\rangle$$

$$= \sum_i \lambda_i \, |i'_A\rangle \otimes |i'_B\rangle$$

Since U & V are unitary matrices, then $U|i_A\rangle = |i'_A\rangle$ and $V|i_B\rangle = |i'_B\rangle$ from two set of orthonormal vectors.

Ex: 2.81 Freedom in purification

Let $|AR_1\rangle$ and $|AR_2\rangle$ be 2 purifications of a state ρ^A to a composite system AR. Prove that there exists a unitary transformation U_R acting on system R such that

$$|AR_1\rangle = (I_A \otimes U_R)|AR_2\rangle$$

Ans: $\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$

$$|AR_1\rangle = \sum_i \sqrt{p_i}\, |i^A\rangle \otimes |i^{R}\rangle$$

such that,

$$tr_{R_1}(|AR_1\rangle\langle AR_1|) = \sum_i p_i |i^A\rangle\langle i^A| = \rho^A$$

$$|AR_2\rangle = \sum_i \sqrt{p_i}\, |i^A\rangle \otimes |i^{R_2}\rangle$$

For any orthonormal bases $\{|i^{R_1}\rangle\}$ and $\{|i^{R_2}\rangle\}$, there exists a unitary U_R mapping from $\{|i^{R_2}\rangle\}$ to $\{|i^{R_1}\rangle\}$ given by:

$$U_R = \sum_i |i^{R_1}\rangle\langle i^{R_2}| \text{ such that } U|i^{R_2}\rangle = |i^{R_1}\rangle .$$

$$(I_A \otimes U_R)|AR_2\rangle = \sum_i p_i |i^A\rangle \otimes U_R |i^{R_2}\rangle$$

$$= \sum_i p_i |i^A\rangle \otimes |i^{R_1}\rangle = |AR_1\rangle$$

Ex: 2.82 Suppose $\{p_i, |\psi_i\rangle\}$ is an ensemble of states generating a density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ for a quantum system A. Introduce a system R with orthonormal basis $|i\rangle$.

① Show that $\sum_i \sqrt{p_i}\,|\psi_i\rangle|i\rangle$ is a purification of ρ.

Ans: Let $|\psi\rangle = \sum_i \sqrt{p_i}\,|\psi_i\rangle|i\rangle$

$$Tr_R(|\psi\rangle\langle\psi|) = Tr_R\left(\sum_{i,j}\sqrt{p_i p_j}\,|\psi_i\rangle\langle\psi_j| \otimes |i\rangle\langle j|\right)$$

$$= \sum_{i,j}\sqrt{p_i p_j}\,|\psi_i\rangle\langle\psi_j|\,Tr(|i\rangle\langle j|)$$

$$= \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$\therefore$ $|\psi\rangle$ is a purification of ρ.

② Suppose we measure R in the basis $|i\rangle$, obtaining outcome i. With what prob. do we obtain the result i? What is the corresp. state of system A?

Ans: $$p(i) = Tr\left[(I \otimes |i\rangle\langle i|)\,|\psi\rangle\langle\psi|\right]$$

$$= \langle\psi|(I \otimes |i\rangle\langle i|)|\psi\rangle$$

And

$$= \langle\psi|(I\otimes|i\rangle\langle i|)\sum_j \sqrt{P_j}\,|\psi_j\rangle\otimes|j\rangle$$

$$= \left(\sum_k \sqrt{P_k}\langle\psi_k|\otimes\langle k|\right)\sqrt{P_i}\,|\psi_i\rangle\otimes|i\rangle$$

$$= \sum_k \sqrt{P_k P_i}\langle\psi_k|\psi_i\rangle\,\delta_{ki}$$

$$= P_i\,\langle\psi_i|\psi_i\rangle = P_i$$

And the post-measurement state of system A,

$$\frac{(I\otimes|i\rangle\langle i|)|\psi\rangle}{\sqrt{P_i}} = \frac{(I\otimes|i\rangle\langle i|)\left(\sum_k \sqrt{P_k}\,|\psi_k\rangle\otimes|k\rangle\right)}{\sqrt{P_i}}$$

$$= \frac{\sum_k \sqrt{P_k}\,|\psi_k\rangle\otimes|i\rangle\,\delta_{ik}}{\sqrt{P_i}}$$

$$= \frac{\sqrt{P_i}\,|\psi_i\rangle\otimes|i\rangle}{\sqrt{P_i}} = |\psi_i\rangle\otimes|i\rangle$$

$$|\psi_i\rangle$$

☐ EPR and Bell inequality

According to QM, an unobserved particle does not possess physical properties that exist independent of observation. Rather, such physical properties arise as a consequence of measurements performed upon the system.

Suppose we prepare the two qubit state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

a state sometimes known as the spin singlet, which is an entangled pair of qubits belonging to Alice and Bob, respectively.

Suppose Alice & Bob are a long way away from one another.
Alice performs a measurement of spin along the $\hat{n}$ axis, i.e., she measures the observable $\hat{n}\cdot\vec{\sigma}$.

$$\hat{n}\cdot\vec{\sigma} = \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}$$

$$\lambda_+ = +1 \;;\; |\hat{n},+\rangle = \cos\theta/2\,|0\rangle + \sin\theta/2\, e^{i\phi}|1\rangle$$

$$\lambda_- = -1 \;;\; |\hat{n},-\rangle = \sin\theta/2\,|0\rangle - \cos\theta/2\, e^{i\phi}|1\rangle$$

Suppose Alice receives the result +1.

She can predict with certainty that Bob will measure −1 on his qubit if he also measure spin along the $\hat{n}$ axis.

Similarly, if Alice measured −1, then she can predict with certainty that Bob will measure +1 on his qubit.

i.e., it turns out that no matter what choice of $\hat{n}$ we make, the results of the 2 measurements are always opposite to one another.

It is as though the 2^{nd} qubit knows the result of the measurement on the 1^{st}, no matter how the 1^{st} qubit is measured.

Why ?

Suppose $|a\rangle$ and $|b\rangle$ are the eigenstates of $\hat{n}\cdot\vec{\sigma}$. Then there exists complex numbers $\alpha, \beta, \gamma, \delta$ such that

$$|0\rangle = \alpha|a\rangle + \beta|b\rangle$$

$$|1\rangle = \gamma|a\rangle + \delta|b\rangle$$

Substituting,

$$|01\rangle = |0\rangle\otimes|1\rangle = \left(\alpha|a\rangle + \beta|b\rangle\right)\otimes\left(\gamma|a\rangle + \delta|b\rangle\right)$$

$$= \underline{\alpha\gamma|a\rangle\otimes|a\rangle} + \alpha\delta|a\rangle\otimes|b\rangle + \beta\gamma|b\rangle\otimes|a\rangle + \underline{\beta\delta|b\rangle\otimes|b\rangle}$$

$$|10\rangle = |1\rangle\otimes|0\rangle = \left(\gamma|a\rangle + \delta|b\rangle\right)\otimes\left(\alpha|a\rangle + \beta|b\rangle\right)$$

$$= \underline{\alpha\gamma|a\rangle\otimes|a\rangle} + \beta\gamma|a\rangle\otimes|b\rangle + \alpha\delta|b\rangle\otimes|a\rangle + \underline{\beta\delta|b\rangle\otimes|b\rangle}$$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha\delta - \beta\gamma)\,\frac{|ab\rangle - |ba\rangle}{\sqrt{2}}$$

$P_{w\leftarrow v} =$

ILA (11)

$P^{\dagger}_{w\leftarrow v} P_{w\leftarrow v} =$

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is unitary

$\Longrightarrow P_{w\leftarrow v} =$

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$$P_{W\leftarrow V} = W^{-1}V = \begin{bmatrix} |a\rangle & |b\rangle \end{bmatrix}^{-1} \begin{bmatrix} |0\rangle & |1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} |a\rangle & |b\rangle \end{bmatrix}^{\dagger} \begin{bmatrix} |0\rangle & |1\rangle \end{bmatrix} = \begin{bmatrix} \langle a| \\ \langle b| \end{bmatrix} \begin{bmatrix} |0\rangle & |1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle a|0\rangle & \langle a|1\rangle \\ \langle b|0\rangle & \langle b|1\rangle \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

$$P^{\dagger}_{W\leftarrow V} P_{W\leftarrow V} = \begin{bmatrix} \overline{\langle a|0\rangle} & \overline{\langle b|0\rangle} \\ \overline{\langle a|1\rangle} & \overline{\langle b|1\rangle} \end{bmatrix} \begin{bmatrix} \langle a|0\rangle & \langle a|1\rangle \\ \langle b|0\rangle & \langle b|1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} |\alpha|^2 + |\beta|^2 & \bar{\alpha}\gamma + \bar{\beta}\delta \\ \alpha\bar{\gamma} + \beta\bar{\delta} & |\alpha|^2 + |\beta|^2 \end{bmatrix}$$

$$\langle 0|1\rangle = (\bar{\alpha}\langle a| + \bar{\beta}\langle b|) \cdot (\gamma|a\rangle + \delta|b\rangle)$$

$$= \bar{\alpha}\gamma + \bar{\beta}\delta$$

$$\langle 1|0\rangle = \overline{\langle 0|1\rangle} = \alpha\bar{\gamma} + \beta\bar{\delta}$$

$$= \begin{bmatrix} 1 & \langle 0|1\rangle \\ \langle 1|0\rangle & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

* The change of basis matrix for orthonormal basis is unitary.

$$\Longrightarrow P_{W\leftarrow V} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \text{ is unitary.}$$

with determinant $(\alpha\delta - \beta\gamma)$.

$\therefore |\lambda| = 1$ for a unitary matrix U.

$$|\det U| = |\prod \lambda_i| = \prod |\lambda_i| = 1$$

$$\Longrightarrow \det U = e^{i\theta}$$

⇒ $\alpha\delta - \beta\gamma = e^{i\theta}$, phase factor for some real θ.

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{|ab\rangle - |ba\rangle}{\sqrt{2}}$$

upto an unobservable global phase factor.

⇒ If a measurement of $\hat{n}\cdot\vec{\sigma}$ is performed on both qubits, then we can see that a result of +1(−1) on the 1st qubit implies a result of −1(+1) on the 2nd qubit.

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Because it is always possible for Alice to predict the value of the measurement result recorded when Bob's qubit is measured in the $\hat{n}$ direction, that physical property must correspond to an element of reality, by the EPR criterion, and should be represented in any complete physical theory.

However, standard QM certainly does not include any fundamental element intended to represent the value of $\hat{n}\cdot\vec{\sigma}$ for all unit vectors $\hat{n}$.

Standard QM merely tells one how to calculate the probabilities of the respective measurement outcomes if $\hat{n}\cdot\vec{\sigma}$ is measured.

The goal of EPR was to show that QM is incomplete, by demonstrating that QM lacked some essential 'element of reality', by their criterion. They hoped to force a return to a more classical view of the world, one in which systems could be ascribed properties which existed independently of measurements performed on these systems.

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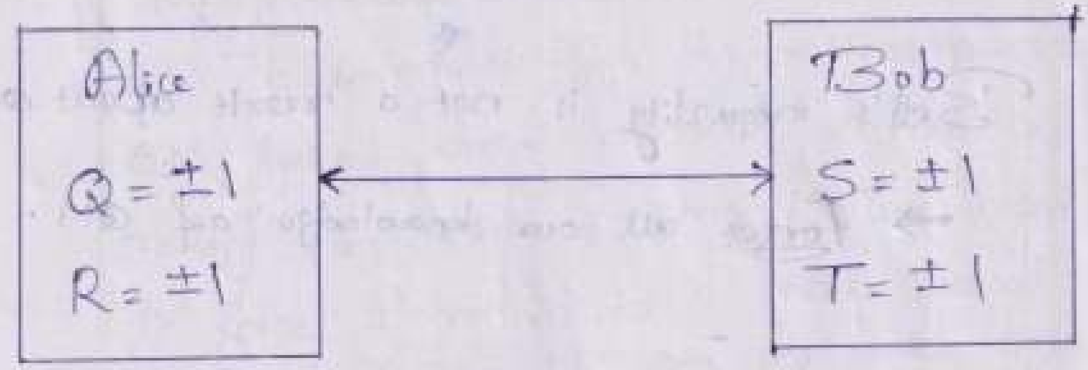
Bell's inequality is not a result about Q.M.

→ Forget all our knowledge of QM.

Thought Expt to obtain Bell's inequality, which we will analyze using our common sense notions of how the world works – the sort of notions Einstein and his collaborators thought Nature ought to obey.

After we have done the common sense analysis, we will perform a quantum mechanical analysis which we can show is not consistent with the common sense analysis.

Nature can then be asked, by means of a real expt., to decide b/w our common sense notions of how the world works, and Q.M.



* Schematic experimental setup for the Bell inequalities. Alice can choose to measure either Q or R, and Bob chooses to measure either S or T. They perform their measurements simultaneously. Alice & Bob are assumed to be far enough apart that performing a measurement on one system can not have any effect on the result of measurements on the other.

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Charlie prepares 2 particles, and he is capable of repeating the experimental procedure which he uses. Once he has performed the preparation, he sends one particle to Alice and the 2nd particle to Bob.

Once Alice receives her particle she performs a measurement on it. Imagine that she has available 2 different measurement apparatuses, so she could choose to do one of 2 different measurements. These measurements are of physical properties which we shall label P_Q and P_R, respectively. Alice doesn't know in advance which measurement she will choose to perform. Rather when she receives the particle she flips a coin or uses some other random method to decide which measurement to perform.

Suppose for simplicity that the measurements can each have one of 2 outcomes +1 or -1.

Suppose, Alice's particle has a value Q for the property P_Q. Q is assumed to be an objective property of Alice's particle, which is merely revealed by the measurement, much as we imagine the position of a tennis ball to be revealed by the particles of light being scattered off it.

Similarly, let R denote the value revealed by a measurement of the property P_R.

Similarly,

suppose that Bob is capable of measuring one of 2 properties, P_S or P_T, revealing an objectively existing value S or T for the property, each taking values +1 or −1.

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Bob does not decide beforehand which property he will measure, but wait until he has recieved the particle and then chooses randomly.

The timing of the expt. is arranged so that Alice & Bob do their measurement at the same time (in the language of relativity, in a causally disconnected manner).

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The measurement which Alice performs **cannot** disturb the result of Bob's measurement (or vice versa), since physical influences cannot propagate faster than light.

$$QS+RS+RT-QT=(Q+R)S+(R-Q)T$$

$$R,Q=\pm 1 \Longrightarrow (Q+R)S=0 \text{ (or) } (R-Q)T=0$$

$$(R-Q)T=\pm 2 \qquad (Q+R)S=\pm 2$$

$\therefore$

$$QS+RS+RT-QT=\pm 2$$

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$E(QS+RS+R$

Suppose,

$p(q,r,s,t)$: probability that, before the measurements are performed, the system is in a state where $Q=q, R=r, S=s, T=t$.

These probabilities may depend on how Charlie performs his preparation, and on experimental noise.

$E(\cdot)$: mean value of a quantity.

$$E(QS+RS+RT-QT) = \sum_{q,r,s,t} p(q,r,s,t)(qs+rs+rt-qt)$$

$$\leq \sum_{q,r,s,t} p(q,r,s,t) \times 2 = 2$$

$$E(QS+RS+RT-QT) = \sum_{q,r,s,t} p(q,r,s,t)\, qs + \sum_{q,r,s,t} p(q,r,s,t)\, rs$$

$$+ \sum_{q,r,s,t} p(q,r,s,t)\, rt - \sum_{q,r,s,t} p(q,r,s,t)\, qt$$

$$= E(QS) + E(RS) + E(RT) - E(QT)$$

Comparing,

$$\boxed{E(QS) + E(RS) + E(RT) - E(QT) \leq 2}$$

→ Bell inequality/ CHSH inequality

It is part of a larger set of inequalities known generically as Bell inequalities.

Q.M →

Imagine we perform the following quantum mechanical expt.

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Charlie prepares a quantum system of 2 qubits in the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

He passes the 1st qubit to Alice, and the 2nd qubit to Bob. They perform measurements of the following observables.

$Q = Z_1$ $\qquad$ $S = \frac{-Z_2 - X_2}{\sqrt{2}}$

$R = X_1$ $\qquad$ $T = \frac{Z_2 - X_2}{\sqrt{2}}$

Note: $Z_1, X_1, \frac{-Z-X}{\sqrt{2}}, \frac{Z-X}{\sqrt{2}}$ all have eigenvalues ±1

$Z =$

$X =$

$\frac{-Z-X}{\sqrt{2}} =$

$\frac{Z-X}{\sqrt{2}} =$

$|\psi\rangle =$

$\rho = |\psi\rangle$

$QS = Z_1 \otimes$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : \quad Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \quad X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

$$\frac{-Z-X}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{Z-X}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$QS = Z_1 \otimes \frac{-Z_2 - X_2}{\sqrt{2}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\langle QS\rangle = tr\left((Q\otimes S)\rho\right)$$

$$= \frac{1}{\sqrt{2}} tr\left(\begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2\sqrt{2}} tr\left(\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2\sqrt{2}}(1+1) = \frac{1}{\sqrt{2}}$$

Similarly,

$$\langle RS\rangle = \frac{1}{\sqrt{2}},\ \langle RT\rangle = \frac{1}{\sqrt{2}},\ \langle QT\rangle = \frac{-1}{\sqrt{2}}$$

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2}$$

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$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2$$

Yet here,

QM predicts that it is $2\sqrt{2}$!

Expt. using photons.

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⟹ The Bell inequality is NOT obeyed by Nature.

It means that one or more of the assumptions that went into the derivation of the Bell inequality must be incorrect.

There are 2 assumptions made in the proof of the Bell inequality/CHSH inequality which are questionable:

(1) The assumptions that the physical properties P_Q, P_R, P_S, P_T have definite values Q, R, S, T which exist independent of observation. This is sometimes known as the assumption of realism.

(2) The assumption that Alice performing her measurement does not influence the result of Bob's measurement. This is sometimes known as the assumption of locality.

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→ These 2 assumptions together are known as the assumptions of local realism.