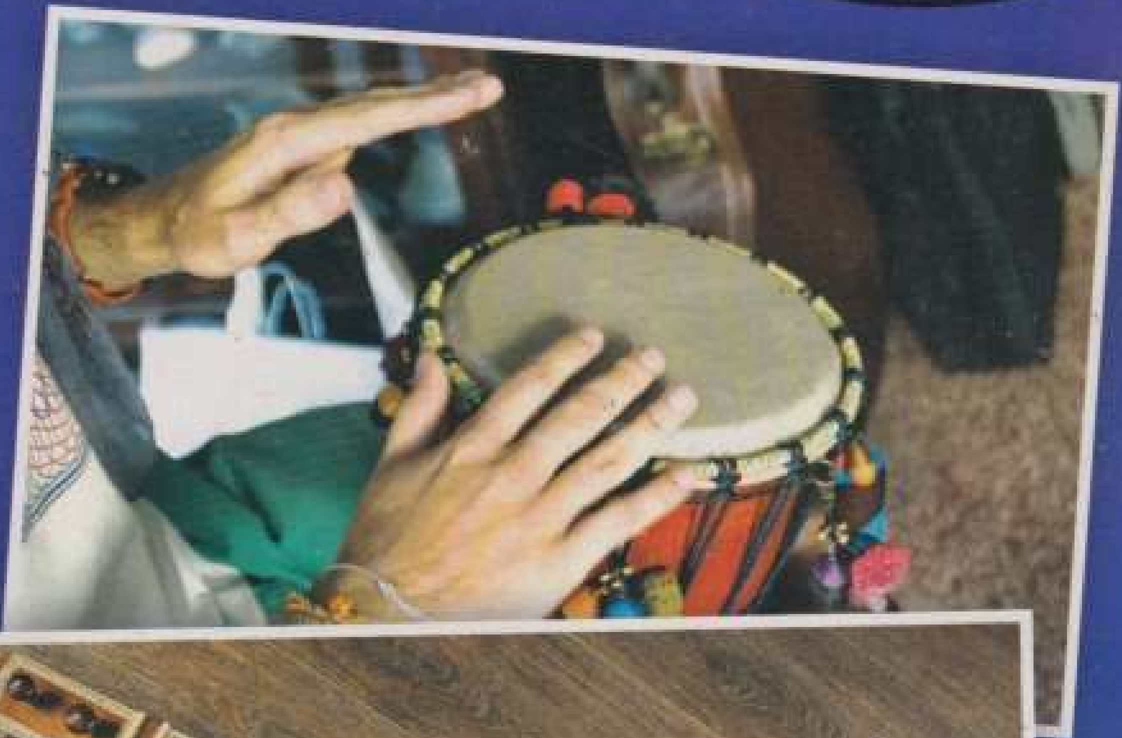
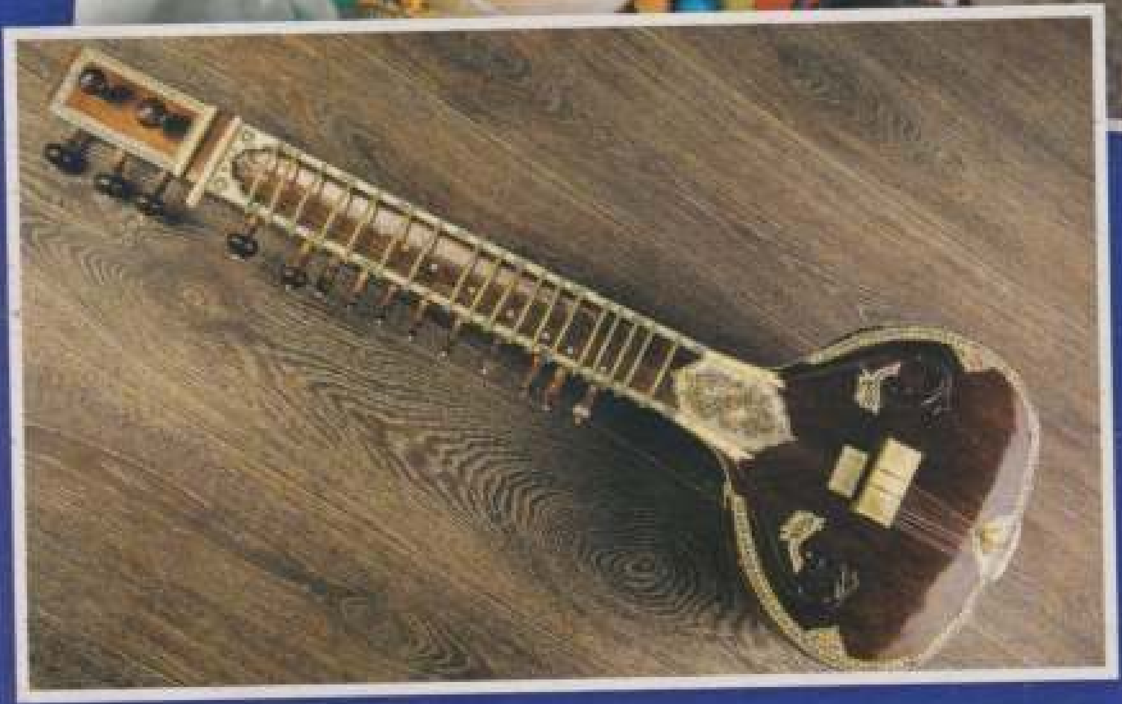
Saathi





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Quantum process tomography

Quantum operations provide a wonderful mathematical model for open quantum systems, and are conveniently visualized (at least for qubits) — but how do they relate to experimentally measurable quantities?

What experiments should an experimentalist do if they wish to characterize the dynamics of a quantum system?

Quantum state tomography is the procedure of experimentally determining an unknown quantum state.

Suppose, we are given an unknown state ρ of a single qubit. How can we experimentally determine what the state of ρ is ?

If we are given just a single copy of ρ then it turns out to be impossible to charatterize ρ.

The basic problem is that there is no quantum measurement which can distinguish non-orthogonal quantum states like $|0\rangle$ and $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ with certainty.

However, it is possible to estimate ρ if we have a large # of copies of ρ. For instance, if ρ is the quantum state produced by some experiment, then we simply repeat the experiment many times to produce many copies of the state ρ.

Suppose we have many copies of a single qubit density matrix ρ.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The set $I/\sqrt{2}, X/\sqrt{2}, Y/\sqrt{2}, Z/\sqrt{2}$ forms an orthonormal set of matrices w.r.t the Hilbert-Schmidt inner product.

QC②

$$\rho = \begin{bmatrix} u & v+iw \\ v-iw & 1-u \end{bmatrix}$$

$$X\rho = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u & v+iw \\ v-iw & 1-u \end{bmatrix} = \begin{bmatrix} v-iw & 1-u \\ u & v+iw \end{bmatrix}$$

$$\Rightarrow tr(X\rho) = 2v$$

$$Y\rho = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} u & v+iw \\ v-iw & 1-u \end{bmatrix} = \begin{bmatrix} -iv-w & -i-iu \\ iu & iv-w \end{bmatrix}$$

$$\Rightarrow tr(Y\rho) = -2w$$

$$Z\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u & v+iw \\ v-iw & 1-u \end{bmatrix} = \begin{bmatrix} u & v+iw \\ -v+iw & -1+u \end{bmatrix}$$

$$\Rightarrow tr(Z\rho) = 2u-1$$

$\therefore$

$$2\rho = \begin{bmatrix} 2u & 2v+iw \\ 2v-iw & 2-2u \end{bmatrix} = \begin{bmatrix} 1+(2u-1) & 2v+i2w \\ 2v-i2w & 1-(2u-1) \end{bmatrix}$$

$$= I + 2vX + (-2w)Y + (2u-1)Z$$

$$= tr(\rho)I + tr(X\rho)X + tr(Y\rho)Y + tr(Z\rho)Z$$

$\therefore$

The single qubit density matrix ρ may be expanded as,

$$\boxed{\rho = \frac{tr(\rho)I + tr(X\rho)X + tr(Y\rho)Y + tr(Z\rho)Z}{2}}$$

$$\langle A \rangle = tr(A\rho) = \sum_m m\, p(m)$$

which has an interpretation as the average value of observables.

Ex.- $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

OM(22)

To estimate $tr(Z\rho)$ we measure the observable Z a large # of times, m, obtaining outcomes $z_1, z_2, \ldots, z_m$, all equal to $+1$ or -1.

The empirical average of these quantities, $\sum_i z_i/m$ is an estimate for the true value of $tr(Z\rho)$.

Standardization

Given a random variable X with mean μ and standard deviation σ, we define its standardization of X as the new random variable, $Z = \frac{X-\mu}{\sigma}$.

Z has mean 0 and standard deviation 1.

If X has a normal distribution, then the standardization of X is the standard normal distribution Z with mean 0 and variance 1.

Central limit theorem

OM(22)

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d random variables each having mean μ and standard deviation σ.

* A collection of random variables is i.i.d (independent and identically distributed) if each random variable has the same probability distribution as the others and all are mutually independent.

For each n, let S_n denote the sum and let $\overline{X}_n$ be the average of $X_1, X_2, \dots, X_n$.

$$S_n = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{S_n}{n}$$

The properties of mean and variance are,

$$E(S_n) = n\mu \ , \ Var(S_n) = n\sigma^2 \ , \ \sigma_{S_n} = \sqrt{n}\,\sigma$$

$$E(\overline{X}_n) = \mu \ , \ Var(\overline{X}_n) = \frac{\sigma^2}{n} \ , \ \sigma_{\overline{X}_n} = \frac{\sigma}{\sqrt{n}}$$

Since they are multiples of each other, S_n and $\overline{X}_n$ have the same standardization.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$$

Central limit theorem : For large n,

$$\overline{X}_n \approx N(\mu, \sigma^2/n) \quad \text{and} \quad S_n \approx N(n\mu, n\sigma^2)$$

$$Z_n \approx N(0,1)$$

⇒ The central limit theorem allows us to approximate a sum or average of i.i.d random variables by a normal random variable.

We can use the central limit theorem to determine how well the estimate for $\text{tr}(Z\rho)$ behaves for large m, where it becomes approximately Gaussian with mean equal to $\text{tr}(Z\rho)$ and with standard deviation $\Delta(Z)/\sqrt{m}$, where $\Delta(Z)$ is the standard deviation for a single measurement of Z, which is upper bounded by 1; i.e., $\Delta Z \le \frac{1-(-1)}{2} = 1$

$\therefore$

The standard deviation in our estimate $\sum_1 z_i/m$ is at most $1/\sqrt{m}$.

$$s.d \le \frac{\text{range}}{2} \iff \sigma \le \frac{a_{max} - a_{min}}{2} = r/2$$

Proof.

$\sum_{i=1}^{n}(x_i-a)^2$ is minimized when $a = \bar{x}$

i.e., $$\sum_{i=1}^{n}(x_i-\bar{x})^2 \le \sum_{i=1}^{n}(x_i-a)^2$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 \le \sum_{i=1}^{n}(x_i-a_c)^2$$ where a_c: centre of the range.

$$\le n\left(r/2\right)^2 = nr^2/4$$

$$\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} \le r^2/4$$

$$\therefore \sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}} \le r/2$$

In a similar way, we can estimate the quantities $tr(X\rho)$ and $tr(Y\rho)$ with a high degree of confidence in the limit of a large sample size, and thus obtain a good estimate for ρ.

Similar to the single qubit case, an arbitrary density matrix on n qubits can be expanded as,

$$\boxed{\rho = \sum_{\vec{v}} \frac{tr(\sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n} \rho)\, \sigma_{v_1} \otimes \sigma_{v_2} \otimes \cdots \otimes \sigma_{v_n}}{2^n}}$$

where the sum is over vectors $\vec{v} = (v_1, \ldots, v_n)$ with entries v_i chosen from the set $0, 1, 2, 3$.

By performing measurements of observables which are products of Pauli matrices, we can estimate each term in this sum, and thus obtain an estimate for ρ.

Q. Stack
7/11/2022

Proof.

Consider the density matrix on 2 qubits,

We need to prove, $\rho = \sum_{i,j} \frac{tr(\sigma_i \otimes \sigma_j \rho)\, \sigma_i \otimes \sigma_j}{4}$

Lemma: Let V and W be normed spaces. If $\{v_i\} \in V$ and $\{w_j\} \in W$ are linearly independent in V and W respectively, then $\{v_i \otimes w_j\}$ is linearly independent in the algebraic tensor product $V \otimes W$.

QCO

Since we can define a vector space $\mathbb{C}^{16}$ isometric to the matrix space $M_{4\times4}(\mathbb{C})$, lemma can be used to prove that $\sigma_i \otimes \sigma_j$ form an orthogonal basis for the 2 qubit density matrix.

ie., $\rho = \sum_{i,j} c_{ij}\, \sigma_i \otimes \sigma_j$

$$\sigma_m \otimes \sigma_n \rho = \sum_{i,j} c_{ij}\, \sigma_m \sigma_i \otimes \sigma_n \sigma_j$$

$$tr\left(\sigma_m \otimes \sigma_n \rho\right) = tr\left(\sum_{ij} c_{ij}\, \sigma_m \sigma_i \otimes \sigma_n \sigma_j\right)$$

$$= \sum_{ij} c_{ij}\ tr\left(\sigma_m \sigma_i \otimes \sigma_n \sigma_j\right)$$

$$= \sum_{i,j} c_{ij}\ tr\left(\sigma_m \sigma_i\right) tr\left(\sigma_n \sigma_j\right)$$

$$\sigma_a \sigma_b = \delta_{ab} I + i \varepsilon_{abc} \sigma_c$$

$$\therefore\ tr\left(\sigma_a \sigma_b\right) = 2\delta_{ab}$$

$$= 4 \sum_{i,j} c_{ij}\, \delta_{mi}\, \delta_{nj}$$

$$= 4\, c_{mn}$$

$$\longrightarrow \quad c_{mn} = \frac{tr\left(\sigma_m \otimes \sigma_n \rho\right)}{4}$$

$$\therefore \quad \rho = \sum_{i,j} c_{ij}\, \sigma_i \otimes \sigma_j$$

$$= \frac{1}{4} \sum_{i,j} tr(\sigma_i \otimes \sigma_j\, \rho)\, \sigma_i \otimes \sigma_j$$

How can we use the quantum state tomography to do quantum process tomography?

Suppose the state space of the system has d dimensions, eg: $d=2$ for a single qubit.

We chose d^2 pure quantum states $|\psi_1\rangle, \ldots, |\psi_{d^2}\rangle$ chosen so that the corresponding density matrices $|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_{d^2}\rangle\langle\psi_{d^2}|$ from a basis set for the space of matrices.

How to choose such a set needs to be explained!

For each state $|\psi_j\rangle$ we prepare the quantum system in that state and then subject it to the process $\mathcal{E}$ which we wish to characterize.

After the process has run to completion we use quantum state tomography to determine the state $\mathcal{E}(|\psi_j\rangle\langle\psi_j|)$ output from the process.

From a purist's point of view we are now done, since in principle the quantum operation $\mathcal{E}$ is now determined by a linear extension of $\mathcal{E}$ to all states.

In practice, we would like to have a way of determining a useful representation of $\mathcal{E}$ from experimentally available data.

Our goal is to determine a set of operation elements $\{E_i\}$ for $\mathcal{E}$,

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$$

However, experimental results involve numbers, not operators, which are a theoretical concept.

To determine the E_i from measurable parameters, it is convenient to consider an equivalent description of $\mathcal{E}$ using a fixed set of operators $\tilde{E}_i$, which form a basis for the set of operators on the state space, so that

$$E_i = \sum_m e_{im} \tilde{E}_m$$

for some set of complex numbers e_{im}

$$\begin{aligned}
\therefore \quad \mathcal{E}(\rho) &= \sum_i E_i \rho E_i^\dagger \\
&= \sum_i \sum_m e_{im} \tilde{E}_m \rho \sum_n e_{in}^* \tilde{E}_n^\dagger \\
&= \sum_{m,n} \tilde{E}_m \rho \tilde{E}_n^\dagger \sum_i e_{im} e_{in}^* \\
&= \sum_{m,n} \tilde{E}_m \rho \tilde{E}_n^\dagger \chi_{mn}
\end{aligned}$$

and $$\left.\begin{aligned} \chi_{mn}^* &= \Big(\sum_i e_{im} e_{in}^*\Big)^* = \sum_i e_{in} e_{im}^* = \chi_{nm} \\ \chi_{mm} &= \sum_i |e_{im}|^2 \geq 0 \end{aligned}\right\}$$

$$\boxed{\mathcal{E}(\rho) = \sum_{m,n} \tilde{E}_m \rho \tilde{E}_n^{\dagger} \chi_{mn}}$$ $$\boxed{\chi = \sum_{m,n} \chi_{mn} |m\rangle\langle n|}$$

where $\boxed{\chi_{mn} \equiv \sum_i e_{im} e_{in}^*}$ are the entries of a matrix which is positive Hermitian by definition. This expression, known as the chi matrix representation, shows that $\mathcal{E}$ can be completely described by a complex number matrix, χ, once the set of operators E_i has been fixed.

χ is a hermitian $d^2 \times d^2$ matrix.

$\therefore$ χ must have d^2 diagonal real (independent) terms, and the # of terms below the diagonal is $\frac{(d^2-1)d^2}{2}$ that contains a total of $2 \times \frac{(d^2-1)d^2}{2} = d^2(d^2-1) = d^4 - d^2$ real (independent) terms.

$\therefore$

The total # of independent real parameters considering only that χ is hermitian, is $(d^4-d^2)+d^2 = d^4$

Q. Stark
9/11/2022
13/11/2022

$\mathcal{E}$ is a trace-preserving quantum operation
(ρ remains Hermitian with trace 1 under $\mathcal{E}$)

$$\Rightarrow \sum_i E_i^\dagger E_i = I \quad , \text{ where } E_m = \sum e_{im} \tilde{E}_m$$

$$\sum_i E_i^\dagger E_i = \sum_i \sum_m e_{im}^* \tilde{E}_m^\dagger \sum_n e_{in} \tilde{E}_m$$

$$= \sum_{m,n} \tilde{E}_m^\dagger \tilde{E}_n \sum_i e_{im}^* e_{in}$$

$$= \sum_{m,n} \tilde{E}_m^\dagger \tilde{E}_n \chi_{mn}$$

$$\Longrightarrow \sum_{m,n} \tilde{E}_m^\dagger \tilde{E}_n \chi_{mn} = I$$

where $\tilde{E}_n \cdot \tilde{E}_m^{\dagger}$, $\sum_{m,n} \chi_{mn} \tilde{E}_m^{\dagger} \tilde{E}_n$, I are all $d \times d$ matrices with d^2 terms.

$\therefore$

The equation $\sum_{m,n} \tilde{E}_m^{\dagger} \tilde{E}_n \chi_{mn} = I$ obtains d^2 number of linear equations with d^2 unknown χ_{mn}, which put d^2 constraints on the terms of χ. (d^2 linear equations involving d^4 real independent parameters, which introduce d^2 additional constraints).

$\therefore$

χ will contain a total of $d^4 - d^2$ independent real parameters.

Alternative method

The Choi matrix is given by,

$$\sigma = (I_R \otimes \mathcal{E}) |\alpha\rangle\langle\alpha|$$

QC (13)

where $|\alpha\rangle = \sum_i |i_R\rangle \otimes |i_Q\rangle$ is, upto a normalization factor, a maximally entangled state of the systems R and Q.

$$|\alpha\rangle\langle\alpha| = \left(\sum_i |i\rangle \otimes |i\rangle\right)\left(\sum_j |j\rangle \otimes |j\rangle\right)$$

$$= \sum_{i,j} \left(|i\rangle \otimes |i\rangle\right)\left(|j\rangle \otimes |j\rangle\right)$$

$$= \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|$$

$$\sigma = (I_R \otimes \mathcal{E})|\alpha\rangle\langle\alpha|$$

$$= \sum_{m,n} \chi_{mn} (I \otimes \tilde{E}_m)|\alpha\rangle\langle\alpha|(I \otimes \tilde{E}_n^\dagger)$$

$$= \sum_{m,n} \chi_{mn} |\tilde{E}_m\rangle\langle\tilde{E}_n|$$

where $|\tilde{E}_m\rangle = I \otimes \tilde{E}_m |\alpha\rangle$

$$\Rightarrow \boxed{\begin{aligned} \chi_{mn} &= \langle\tilde{E}_m|\sigma|\tilde{E}_n\rangle \\ \sigma &= \sum_{m,n} \chi_{mn} |\tilde{E}_m\rangle\langle\tilde{E}_n| \end{aligned}}$$

$$\sigma = (I_R \otimes \mathcal{E})|\alpha\rangle\langle\alpha|$$

$$= \sum_m (I \otimes E_m)|\alpha\rangle\langle\alpha|(I \otimes E_m^\dagger)$$

$$= \sum_m (I \otimes E_m)\left(\sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|\right)(I \otimes E_m^\dagger)$$

$$= \sum_{i,j} |i\rangle\langle j| \otimes \sum_m E_m |i\rangle\langle j| E_m^\dagger$$

$\therefore$

The $(i,j)^{th}$ block of the Choi matrix σ is, $\sum_m E_m |i\rangle\langle j| E_m^\dagger$

The $(k,l)^{th}$ term of the $(i,j)^{th}$ block of the Choi matrix is,

$$\boxed{\begin{aligned}\sigma_{ij,kl} &= \langle k| \left(\sum_m E_m |i\rangle\langle j| E_m^\dagger \right) |l\rangle \\ &= \sum_m \langle k| E_m |i\rangle\langle j| E_m^\dagger |l\rangle\end{aligned}}$$

Now,

$$\chi_{mn} = \langle \tilde{E}_m | \sigma | \tilde{E}_n \rangle \longleftrightarrow \sigma = \sum_{mn} \chi_{mn} |\tilde{E}_m\rangle\langle \tilde{E}_n|$$

$\therefore$ For $\sigma = \chi$, we need $|\tilde{E}_m\rangle = |m\rangle$ where $m : 0 \rightarrow d^2-1$.

$$|\tilde{E}_m\rangle = (I \otimes \tilde{E}_m)|\alpha\rangle$$
$$= (I \otimes \tilde{E}_m)\sum_i |i\rangle \otimes |i\rangle$$
$$= \sum_i |i\rangle \otimes \tilde{E}_m|i\rangle$$

where $\tilde{E}_m|i\rangle$ is the i^{th} column of $\tilde{E}_m$.

$\Rightarrow |i\rangle \otimes \tilde{E}_m|i\rangle$ is a column vector with the i^{th} column of $\tilde{E}_m$ as its i^{th} block.

$\Rightarrow |\tilde{E}_m\rangle = \sum_i |i\rangle \otimes \tilde{E}_m|i\rangle$ is the column vector with the columns of $\tilde{E}_m$ stacked on top of each other in order.

We can divide $|m\rangle$ into d blocks of dimension d such that $m = qd + t$, where $q, t : 0 \rightarrow d-1$.

$\therefore$

For $|\tilde{E}_m\rangle = |m\rangle$ we need to choose $\tilde{E}_m = |t\rangle\langle q|$, such that

$$\sigma = \sum_{m,n} \chi_{mn} |\tilde{E}_m\rangle\langle \tilde{E}_n| = \sum_{m,n} \chi_{mn} |m\rangle\langle n| = \chi$$

$\therefore$

$$\chi_{ij,kl} = \sum_m \langle k|E_m|i\rangle\langle j|E_m^\dagger|l\rangle$$

$\mathcal{E}$ is a trace-preserving quantum operation

ie, $\sum_m E_m^\dagger E_m = I$

$$\Longrightarrow \sum_m \langle j|E_m^\dagger E_m|i\rangle = \delta_{ij}$$

$$\chi_{ij,kk} = \sum_m \langle k|E_m|i\rangle\langle j|E_m^\dagger|k\rangle$$

$$= \sum_m \langle j|E_m^\dagger|k\rangle\langle k|E_m|i\rangle$$

Summing over all k and using the completeness relation,

$$\sum_k \chi_{ij,kk} = \sum_k\sum_m \langle j|E_m^\dagger|k\rangle\langle k|E_m|i\rangle$$

$$= \sum_m \langle j|E_m^\dagger\left(\sum_k |k\rangle\langle k|\right)E_m|i\rangle$$

$$\sum_k X_{ij,kk} = \sum_m \langle j|E_m^\dagger E_m|i\rangle = \delta_{ij}$$

$$\therefore \quad \sum_{k=0}^{d-1} X_{ij,kk} = \delta_{ij}$$

$i,j : 0 \to d-1$

$\therefore$

Let ρ_j, $1 \le j \le d^2$ be a fixed, linearly independent basis for the space of $d \times d$ matrices

A convenient choice is the set of operators $|n\rangle\langle m|$.

Experimentally, $\mathcal{E}(|n\rangle\langle m|)$ may be obtained the o/p state by preparing the i/p states $|n\rangle, |m\rangle, |+\rangle = \frac{|n\rangle + |m\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|n\rangle + i|m\rangle}{\sqrt{2}}$, and forming linear combinations of $\mathcal{E}(|n\rangle\langle n|)$, $\mathcal{E}(|m\rangle\langle m|)$, $\mathcal{E}(|+\rangle\langle +|)$ and $\mathcal{E}(|-\rangle\langle -|)$, as :

$$\mathcal{E}(|n\rangle\langle m|) = \mathcal{E}(|+\rangle\langle +|) + i\,\mathcal{E}(|-\rangle\langle -|) - \frac{1+i}{2}\mathcal{E}(|n\rangle\langle n|) - \frac{1+i}{2}\mathcal{E}(|m\rangle\langle m|)$$

$$|+\rangle\langle+| = \frac{1}{2}\left(|n\rangle\langle n| + |m\rangle\langle m| + |n\rangle\langle m| + |m\rangle\langle n|\right)$$

$$i|-\rangle\langle-| = \frac{i}{2}\left(|n\rangle\langle n| + |m\rangle\langle m| - i|n\rangle\langle m| + i|m\rangle\langle n|\right)$$

$$= \frac{i}{2}\left(|n\rangle + i|m\rangle\right)\left(\langle n| - i\langle m|\right)$$

$$|+\rangle\langle+| + i|-\rangle\langle-| - \frac{1+i}{2}|n\rangle\langle n| - \frac{1+i}{2}|m\rangle\langle m| =$$

$$= \frac{1}{2}\left(\cancel{|n\rangle\langle n|} + \cancel{|m\rangle\langle m|} + \cancel{|n\rangle\langle m|} + \cancel{|m\rangle\langle n|}\right)$$

$$+ \frac{i}{2}\left(\cancel{|n\rangle\langle n|} + |m\rangle\langle m| - \cancel{i|n\rangle\langle m|} + \cancel{i|m\rangle\langle n|}\right)$$

$$-\frac{1}{2}\cancel{|n\rangle\langle n|} - \frac{i}{2}\cancel{|n\rangle\langle n|} - \frac{1}{2}\cancel{|m\rangle\langle m|} - \frac{i}{2}|m\rangle\langle m|$$

$$= \frac{1}{2}\cancel{|n\rangle\langle n|} + \frac{1}{2}\cancel{|m\rangle\langle m|} + \frac{1}{2}|n\rangle\langle m| + \cancel{\frac{1}{2}|m\rangle\langle n|}$$

$$+ \frac{i}{2}\cancel{|n\rangle\langle n|} + \cancel{\frac{i}{2}|m\rangle\langle m|} + \frac{1}{2}|n\rangle\langle m| - \cancel{\frac{1}{2}|m\rangle\langle n|}$$

$$-\frac{1}{2}\cancel{|n\rangle\langle n|} - \cancel{\frac{i}{2}|n\rangle\langle n|} - \frac{1}{2}\cancel{|m\rangle\langle m|} - \cancel{\frac{i}{2}|m\rangle\langle m|}$$

$$= |n\rangle\langle m|$$

$\therefore$

It is possible to determine $\mathcal{E}(\rho_j)$ by Quantum state tomography, for each ρ_j.

Each $\mathcal{E}(\rho_j)$ may be expressed as a linear combination of the basis states,

$$\mathcal{E}(\rho_j) = \sum_k \lambda_{jk} \rho_k \qquad \text{--- (8.155)}$$

Since $\mathcal{E}(\rho_j)$ is known from the state tomography, λ_{jk} can be determined by standard linear algebraic algorithms.

We may write,

$$\tilde{E}_m \rho_j \tilde{E}_n^{\dagger} = \sum_k \beta_{jk}^{mn} \rho_k \qquad (8.156)$$

where β_{jk}^{mn} are complex numbers which can be determined by standard algorithms from linear algebra given the $\tilde{E}_m$ operators and the ρ_j operators.

Combining 8.155, 8.156 and $\mathcal{E}(\rho) = \sum_{mn} \tilde{E}_m \rho \tilde{E}_n^{\dagger} \chi_{mn}$ we obtain,

$$\mathcal{E}(\rho_j) = \sum_k \lambda_{jk} \rho_k = \sum_k \sum_{mn} \chi_{mn} \beta_{jk}^{mn} \rho_k$$

From the linear independence of the ρ_k it follows that for each k,

From the linear independence of the ρ_k it follows that for each k,

$$\boxed{\sum_{m,n} \beta_{jk}^{mn} \chi_{mn} = \lambda_{jk}} \qquad \text{— (8.158)}$$

$\Rightarrow$ This relation is a necessary and sufficient condition for the matrix χ to give the correct quantum operation $\mathcal{E}$.

One may think of χ and λ as vectors, and β as a $d^2 \times d^2$ matrix, with columns indexed by mn, and rows by jk.

$jk, mn : 1 - d^2$

$mn, jk : 1 \to d^4$

$\beta_{d^4 \times d^4}$

$$\sum_{mn} \beta_{jk}^{mn} \chi_{mn} = \lambda_{jk}$$

$$\beta \vec{\chi} = \vec{\lambda}$$

A matrix $A^g \in \mathbb{R}^{n \times m}$ is a generalized inverse of a matrix $A \in \mathbb{R}^{m \times n}$ if $A A^g A = A$.

Let $\mathcal{K}$ be the generalized inverse for the matrix β, satisfying

$$\underbrace{\beta = \beta \mathcal{K} \beta}$$

and

$$\underbrace{A = BCD}$$

$$\beta_{jk}^{mn} = \sum_{st,\, xy} \beta_{jk}^{st} \mathcal{K}_{st}^{xy} \beta_{xy}^{mn}$$

$$[a]_{ij} = \sum_{k,l} b_{ik} c_{kl} d_{lj}$$

* Most computer packages for matrix multiplication are capable of finding such generalized inverses.

We can prove that χ defined by the equation, $\chi_{mn} = \sum_{jk} \mathcal{K}_{jk}^{mn} \lambda_{mn}$ satisfies the relation $\sum_{mn} \beta_{jk}^{mn} \chi_{mn} = \lambda_{jk}$

$$\sum_{mn} \beta^{mn}_{jk} \chi_{mn} = \lambda_{jk} \Longleftrightarrow \boxed{\beta \vec{\chi} = \vec{\lambda}}$$

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$$\chi_{mn} = \sum_{jk} \kappa^{mn}_{jk} \lambda_{mn} \Longleftrightarrow \vec{\chi} = \kappa \vec{\lambda}$$

* $Ax=b$ has a solution ⟺ $b \in \mathrm{Range}(A)$

The relation (8.158) which is $\sum_{mn} \beta^{mn}_{jk} \chi_{mn} = \lambda_{jk}$ is a necessary and sufficient condition for the matrix χ to give the correct quantum operation $\mathcal{E}$.

$\therefore$

There exists at least one solution to the equation $\beta \vec{\chi} = \vec{\lambda}$, which we shall call χ', because we defined β to make this system satisfied. (there is such a solution).

What might be less obvious is still in general an entire space of solutions, whenever $\mathcal{B}$ has non-trivial kernel.

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$\therefore$

The difficulty in verifying that X defined by $X_{mn} = \sum_{jk} \mathcal{B}^{mn}_{jk} \lambda_{mn}$ satisfies $\sum_{mn} \mathcal{B}^{mn}_{jk} X_{mn} = \lambda_{jk}$ is that, in general, X is not entirely determined by the equation $\sum_{mn} \mathcal{B}^{mn}_{jk} X_{mn} = \lambda_{jk}$

Premultiplying the definition of $\vec{\chi}$ by β gives,

$$\vec{\chi} = \kappa\vec{\lambda} \longrightarrow \beta\vec{\chi} = \beta\kappa\vec{\lambda}$$

$$= \beta\underbrace{\kappa\beta}\vec{\chi}'$$

$$= \beta\vec{\chi}'$$

$$= \vec{\lambda}$$

$\therefore$ χ defined by $\vec{\chi} = \kappa\vec{\lambda}$ satisfies the equation $\beta\vec{\chi} = \vec{\lambda}$.

Having determined χ one immediately obtains the operator-sum representation for $\mathcal{E}$ as follows:

Let the unitary matrix $U^\dagger$ diagonalize χ,

$$\chi = UDU^\dagger$$

$$x_{mn} = \sum_{xy} U_{mx} D_{xy} U^{\dagger}_{yn}$$

$$= \sum_{xy} U_{mx} d_x \delta_{xy} U^*_{ny}$$

$$= \sum_i U_{mi} d_i U^*_{ni}$$

$$= \sum_i e_{im} e^*_{in}$$

$$\Rightarrow e_{im} = \sqrt{d_i}\, U_{mi}$$

$$E_i = \sum_m e_{im} \tilde{E}_m$$

$$\Rightarrow E_i = \sum_m \sqrt{d_i}\, U_{mi} \tilde{E}_m$$

$$= \sqrt{d_i} \sum_m U_{mi} \tilde{E}_m$$

are operation elements for $\mathcal{E}$.

Summary

λ is experimentally determined using state tomography, which in turn determines χ via the equation $\vec{\chi} = \kappa \vec{\lambda}$, which gives us a complete description of $\mathcal{E}$, including a set of operation elements E_i.

- The process of quantum process tomography is analogous to the system identification step performed in classical control theory, and plays a similar role in understanding and controlling noisy quantum systems.

Ex:- Process tomography for a single qubit

Let's choose,

$\tilde{E}_0 = I$

$\tilde{E}_1 = X$

$\tilde{E}_2 = -iY$

$\tilde{E}_3 = Z$

- There are $d^4 - d^2 = 2^4 - 2^2 = 12$ parameters, specified by χ, which determine an arbitrary single qubit quantum operation $\mathcal{E}$.

- These parameters may be measured using 4 sets of experiments.

Suppose the i/p states $|0\rangle, |1\rangle, |+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle+i|1\rangle}{\sqrt{2}}$ are prepared, and the 4 matrices,

$$\rho_1' = \mathcal{E}(|0\rangle\langle 0|)$$

$$\rho_4' = \mathcal{E}(|1\rangle\langle 1|)$$

$$\rho_2' = \mathcal{E}(|+\rangle\langle +|) - i\,\mathcal{E}(|-\rangle\langle -|) - \frac{(1-i)}{2}(\rho_1' + \rho_4')$$

$$\rho_3' = \mathcal{E}(|+\rangle\langle +|) + i\,\mathcal{E}(|-\rangle\langle -|) - \frac{(1+i)}{2}(\rho_1' + \rho_4')$$

are determined by state tomography.

These correspond to $\rho_j' = \mathcal{E}(\rho_j)$, where

$$\rho_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho_4 = X\rho_1 X = |1\rangle\langle 1|$$

$\boxed{X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

$\boxed{X = |0\rangle\langle 1| + |1\rangle\langle 0|}$

$$\rho_3 = |0\rangle\langle 1| = \rho_1 X$$

$$\rho_2 = \rho_3' = |1\rangle\langle 0| = X\rho_1$$

We may determine β_1 and similarly ρ_j' determines λ.