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sooraj ss1729@gmail.com +91-9400635788

I N D E X

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Sl. No.	Date	Title	Page No.	Teacher Sign/ Remarks
		QUANTUM COMPUTATION		
		& QUANTUM INFORMATION		
		- Nielsen & Chuang		

© Axiomatic approach to quantum operations

Until now,
the main motivation for our study of quantum operations has been that they provide an elegant way of studying systems which are interacting with the environment.

Now,
we write down physically motivated axioms which we expect quantum operations to obey.

- This viewpoint is more abstract than our earlier approach, based on explicit system-environment models, but it is also extremely powerful because of that abstraction.

Method :

First, we are going to forget everything we have learned about quantum operations, and start over by defining quantum operations according to a set of axioms, which we will justify on physical grounds.

2^{nd}, we'll prove that a map $\mathcal{E}$ satisfies these axioms iff it has an operator-sum representation, thus providing the missing link b/w the abstract axiomatic formulation, and the earlier discussion.

Define a quantum operation $\mathcal{E}$ as a map from the set of density operators of the i/p space Q_1 to the set of density operators for the output space Q_2, with the following 3 axiomatic properties:

(For notational simplicity in the proofs we take $Q_1 = Q_2 = Q$)

① $tr[\mathcal{E}(\rho)]$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the initial state.

$\therefore \quad 0 \le tr[\mathcal{E}(\rho)] \le 1$ for any state ρ.

② $\mathcal{E}$ is a convex-linear map on the set of density matrices, ie.,
for probabilities $\{p_i\}$,

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$$

③ $\mathcal{E}$ is a completely +ve map.

ie.,

if $\mathcal{E}$ maps density operators of system Q_1 to density operators of system Q_2, then $\mathcal{E}(A)$ must be +ve for any positive operator A.

Furthermore,

if we introduce an extra system R of arbitrary dimensionality, it must be true that $(I \otimes \mathcal{E})(A)$ is +ve for any positive operator on the combined system RQ_1, where I denotes the identity map on system R.

1st property

To cope with the case of measurements, it turns out that it is extremely convenient to make the convention that $\mathcal{E}$ does not necessarily preserve the trace property of density matrices, that $tr(\rho) = 1$.

Rather, we make the convention that $\mathcal{E}$ is to be defined in such a way that $tr[\mathcal{E}(\rho)]$ is equal to the probability of the measurement outcome described by $\mathcal{E}$ occurring.

Ex:-
Suppose that we are doing a projective measurement in the computational basis of a single qubit. Then 2 quantum operations are used to describe this process, defined by $\mathcal{E}_0(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\mathcal{E}_1(\rho) = |1\rangle\langle 1|\rho|1\rangle\langle 1|$.

The probabilities of the respective outcomes are correctly given by $tr[\mathcal{E}_0(\rho)] = tr[|0\rangle\langle 0|\rho|0\rangle\langle 0|]$ and $tr[\mathcal{E}_1(\rho)] = tr[|1\rangle\langle 1|\rho|1\rangle\langle 1|]$.

With this convention the correctly normalized final quantum state is therefore,

$$\frac{\mathcal{E}(\rho)}{tr(\mathcal{E}(\rho))} \iff \rho_m = \frac{M_m \rho M_m^\dagger}{tr(M_m \rho M_m^\dagger)}$$

In the case where the process is deterministic, ie., no measurement is taking place, this reduces to the requirement that $tr[\mathcal{E}(\rho)] = 1 = tr(\rho)$, for all ρ.

$\Longrightarrow$ the quantum operation is a trace-preserving quantum operation, since on its own $\mathcal{E}$ provides a complete description of the quantum process.

If there is a ρ such that $tr[\mathcal{E}(\rho)] < 1$, then the quantum operation is non-trace preserving, since on its own $\mathcal{E}$ does not provide a complete description of the processes that may occur in the system. (ie., other measurement outcomes may occur, with some probability).

A physical quantum operation is one that satisfies the requirement that probabilities never exceed 1, ie., $tr[\mathcal{E}(\rho)] \leq 1$.

2nd property

Suppose the input ρ to the quantum operation is obtained by randomly selecting the state from an ensemble $\{p_i, \rho_i\}$ of quantum states, ie., $\rho = \sum_i p_i \rho_i$.

Then we would expect that the resulting state, $\frac{\mathcal{E}(\rho)}{tr[\mathcal{E}(\rho)]} = \frac{\mathcal{E}(\rho)}{p(\rho)}$ corresponds to a random selection from the ensemble $\{p(i|\mathcal{E}), \mathcal{E}(\rho_i)/tr[\mathcal{E}(\rho_i)]\}$ where $p(i|\mathcal{E})$ is the probability that the state prepared was ρ_i, given that the process represented by $\mathcal{E}$ occurred.

Thus, we demand that,

$$\mathcal{E}(\rho) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{tr[\mathcal{E}(\rho_i)]} \Leftarrow \frac{\mathcal{E}(\rho)}{p(\mathcal{E})} = \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{tr[\mathcal{E}(\rho_i)]}$$

where,

$p(\mathcal{E}) = tr[\mathcal{E}(\rho)]$ is the probability that the process described by $\mathcal{E}$ occurs on input of ρ.

By Bayes' rule,

$$p(i|\mathcal{E}) = p(\mathcal{E}|i)\,\frac{p_i}{P(\mathcal{E})} = \mathrm{tr}\left[\mathcal{E}(\rho_i)\right]\frac{p_i}{P(\mathcal{E})}$$

$\therefore$

$$\mathcal{E}(\rho) = \mathcal{E}\left(\sum_i p_i \rho_i\right) = p(\mathcal{E}) \sum_i p(i|\mathcal{E}) \frac{\mathcal{E}(\rho_i)}{\mathrm{tr}\left[\mathcal{E}(\rho_i)\right]}$$

$$= \cancel{p(\mathcal{E})} \sum_i \mathrm{tr}\cancel{\left[\mathcal{E}(\rho_i)\right]} \frac{p_i}{\cancel{p(\mathcal{E})}} \times \frac{\mathcal{E}(\rho_i)}{\cancel{\mathrm{tr}\left[\mathcal{E}(\rho_i)\right]}}$$

$$= \sum_i p_i\, \mathcal{E}(\rho_i)$$

3rd property

Not only must $\mathcal{E}(\rho)$ be a valid density matrix (up to normalization) so long as ρ is valid, but furthermore,

if $\rho = \rho_{RQ}$ is the density matrix of some joint system of R and Q, if $\mathcal{E}$ acts only on Q, then $\mathcal{E}(\rho_{RQ})$ must still result in a valid density matrix (up to normalization) of the joint system.

Formally, suppose we introduce a second (finite dimensional) system R. Let I denote the identity map on system R. Then the map $I \otimes \mathcal{E}$ must take positive operators to positive operators.

Ex:- Transpose operation on a single qubit

This map transposes the density operator in the computational basis :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

This map preserves positivity of a single qubit.

However, suppose that qubit is part of a two qubit system initially in the entangled state,

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle}{\sqrt{2}}$$

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\Big(|0\rangle\langle 0|\otimes|0\rangle\langle 0| + |0\rangle\langle 1|\otimes|0\rangle\langle 1| + |1\rangle\langle 0|\otimes|1\rangle + |1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\otimes\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\otimes\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\otimes\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$(T\otimes I)\left(|\phi^+\rangle\langle\phi^+|\right) = \frac{1}{2}\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\1&0\end{bmatrix}\otimes\begin{bmatrix}0&1\\0&0\end{bmatrix} + \begin{bmatrix}0&1\\0&0\end{bmatrix}\otimes\begin{bmatrix}0&0\\1&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

which is the density operator of the system after the dynamics has been applied.

This operator has eigenvalues $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$.

$\Longrightarrow$ not a valid density operator.

$\therefore$ the transpose operation is an example of a positive map which is not completely positive.

ie., it preserves the positivity of operators on the principal system, but does not continue to preserve positivity when applied to systems which contain the principal system as a subsystem.

⇒ These 3 axioms are sufficient to define quantum operations, are equivalent to the system-environment models and the definition in terms of an operator-sum representation.

Theorem: The map $\mathcal{E}$ satisfies axioms A_1, A_2 and A_3 if and only if

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger,$$

for some set of operators $\{E_i\}$ which map the i/p Hilbert space to the o/p Hilbert space, and $\sum_i E_i^\dagger E_i \leq I$.

Proof

Part 1

Suppose $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$

$\Rightarrow \mathcal{E}$ is obviously linear

$\therefore$ To check that $\mathcal{E}$ is a quantum operation, we need only to prove that it is completely positive.

Let 'A' be any positive operator acting on the state space of an extended system, RQ, and let $|\psi\rangle$ be some state of RQ.

Defining $|\phi_i\rangle \equiv \left(I_R \otimes E_i^\dagger\right)|\psi\rangle$,

$$\langle\psi|\left(I_R \otimes E_i\right) A \left(I_R \otimes E_i^\dagger\right)|\psi\rangle = \langle\phi_i|A|\phi_i\rangle \geq 0$$

By the positivity of the operator 'A'.

$$\Rightarrow \langle\psi|(I\otimes\mathcal{E})(A)|\psi\rangle = \langle\psi|\sum_i \left(I_R\otimes E_i\right)A\left(I_R\otimes E_i^\dagger\right)|\psi\rangle$$

$$= \sum_i \langle\psi|\left(I_R\otimes E_i\right)A\left(I_R\otimes E_i^\dagger\right)|\psi\rangle$$

$$= \sum_i \langle\phi_i|A|\phi_i\rangle \geq 0.$$

$\therefore$ For any operator 'A', the operator $(I\otimes\mathcal{E})(A)$ is also positive.

$\sum_i E_i^\dagger E_i \leq I \Longrightarrow$ the probabilities are less than or equal to 1.

Part 2

Suppose that $\mathcal{E}$ satisfies axioms A_1, A_2 and A_3.

Aim: to find an operator-sum representation for $\mathcal{E}$.

Suppose, we introduce a system, R, with the same dimension as the original quantum system, Q.

Let $|i_R\rangle$ and $|i_Q\rangle$ be orthonormal bases for R and Q.

It will be convenient to use the same index, i for these 2 bases, and this can certainly be done as R and Q have the same dimensionality.

Define a joint state $|\alpha\rangle$ of RQ by,

$$|\alpha\rangle \equiv \sum_i |i_R\rangle|i_Q\rangle .$$

$$|\alpha\rangle\langle\alpha| = \left(\sum_i |i_R\rangle \otimes |i_Q\rangle\right)\left(\sum_j \langle j_R| \otimes \langle j_Q|\right)$$

$$= \sum_{i,j} |i_R\rangle\langle j_R| \otimes |i_Q\rangle\langle j_Q|$$

$$\rho_Q = tr_R\left(|\alpha\rangle\langle\alpha|\right)$$

$$= \sum_k \left(\langle k_R| \otimes I\right) |\alpha\rangle\langle\alpha| \left(|k_R\rangle \otimes I\right)$$

$$= \sum_k \left(\langle k_R| \otimes I\right)\left[\sum_{i,j} |i_R\rangle\langle j_R| \otimes |i_Q\rangle\langle j_Q|\right]\left(|k_Q\rangle \otimes I\right)$$

$$= \sum_k \sum_{i,j} \langle k_R|i_R\rangle\langle j_R|k_Q\rangle \otimes |i_Q\rangle\langle j_Q|$$

$$= \sum_k |k_Q\rangle\langle k_Q| = I_Q$$

∴

The state $|\alpha\rangle = \sum_i |i_R\rangle |i_Q\rangle$ is, upto a normalization factor, a maximally entangled state of the systems R and Q.

Define an operator σ on the state space of RQ by,

$$\sigma \equiv \left(I_R \otimes \mathcal{E}\right)\left(|\alpha\rangle\langle\alpha|\right)$$ – The Choi matrix

We may think of this as the result of applying the quantum operation $\mathcal{E}$ to one half of a maximally entangled state of the system RQ.

It is a ~~truly~~ remarkable fact, which we'll now demonstrate, that the operator σ completely specifies the quantum operation $\mathcal{E}$.

i.e., to know how $\mathcal{E}$ acts on an arbitrary state of Q, it is sufficient to know how it acts on a single maximally entangled state of Q with another system. !

Recover $\mathcal{E}$ from σ

Let $|\psi\rangle = \sum_j \psi_j |j_Q\rangle$ be any state of system Q.

Define a corresponding state $|\tilde{\psi}\rangle$ of system R by the equation,

$$|\tilde{\psi}\rangle = \sum_j \psi_j^* |j_R\rangle$$

$$\left(\langle\tilde{\psi}|\otimes I_Q\right)\sigma\left(|\tilde{\psi}\rangle\otimes I_Q\right) =$$

$$= \left(\langle\tilde{\psi}|\otimes I_Q\right)\left[\left(I_R\otimes\mathcal{E}\right)\left(|\alpha\rangle\langle\alpha|\right)\right]\left(|\tilde{\psi}\rangle\otimes I_Q\right)$$

$$= \left(\langle\tilde{\psi}|\otimes I_Q\right)\left[\left(I_R\otimes\mathcal{E}\right)\sum_{i,j}\left(|i_R\rangle\otimes|i_Q\rangle\right)\left(\langle j_R|\otimes\langle j_Q|\right)\right]\left(|\tilde{\psi}\rangle\otimes I_Q\right)$$

$$= \left(\langle\tilde{\psi}|\otimes I_Q\right)\left[\left(I_R\otimes\mathcal{E}\right)\sum_{i,j}|i_R\rangle\langle j_R|\otimes|i_Q\rangle\langle j_Q|\right]\left(|\tilde{\psi}\rangle\otimes I_Q\right)$$

$$= \left(\langle\tilde{\psi}|\otimes I_Q\right)\left[\sum_{i,j}|i_R\rangle\langle j_R|\otimes\mathcal{E}\left(|i_Q\rangle\langle j_Q|\right)\right]\left(|\tilde{\psi}\rangle\otimes I_Q\right)$$

$$= \left(\sum_k \psi_k \langle k_R| \otimes I_Q\right)\left[\sum_{i,j} |i_R\rangle\langle j_R| \otimes \mathcal{E}(|i_Q\rangle\langle j_Q|)\right]\left(\sum_l \psi_l^* |l_R\rangle \otimes I_Q\right)$$

$$= \sum_{k,l}\sum_{i,j} \psi_k \psi_l^* \langle k_R|i_R\rangle\langle j_R|l_R\rangle \otimes \mathcal{E}(|i_Q\rangle\langle j_Q|)$$

$$= \sum_{k,l}\sum_{i,j} \psi_k \psi_l^* \, \delta_{k,i}\, \delta_{j,l} \otimes \mathcal{E}(|i_Q\rangle\langle j_Q|)$$

$$= \sum_{k,l} \psi_k \psi_l^* \, \mathcal{E}(|k_Q\rangle\langle l_Q|)$$

$$= \mathcal{E}(|\psi\rangle\langle\psi|)$$

$$\therefore \left(\langle\tilde{\psi}| \otimes I_Q\right)\sigma\left(|\tilde{\psi}\rangle \otimes I_Q\right) = \mathcal{E}(|\psi\rangle\langle\psi|) \quad \text{— } (6.59)$$

Let $\sigma = \sum_i |s_i\rangle\langle s_i|$ be some decomposition of σ, where the vectors $|s_i\rangle$ need not be normalized., since

Axiom 3 $\Rightarrow$ $\sigma = (I_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|)$ is a positive operator (positive semidefinite $\Rightarrow$ σ is Hermitian & have non-negative eigenvalues.

$\sigma = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$, $\sqrt{\lambda_i}|\lambda_i\rangle = |s_i\rangle$

Define a map,

$$E_i\left(|\psi\rangle\right) \equiv \left(\langle\tilde{\psi}| \otimes I_Q\right)|s_i\rangle$$

which is a linear operator on the state space of Q.

$$\sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger = \sum_i \left(\langle\tilde{\psi}|\otimes I_Q\right)|s_i\rangle\langle s_i|\left(|\tilde{\psi}\rangle \otimes I_Q\right)$$

$$= \left(\langle\tilde{\psi}|\otimes I_Q\right)\sum_i |s_i\rangle\langle s_i| \left(|\tilde{\psi}\rangle \otimes I_Q\right)$$

$$= \left(\langle\tilde{\psi}|\otimes I_Q\right)\sigma\left(|\tilde{\psi}\rangle \otimes I_Q\right)$$

$$= \mathcal{E}\left(|\psi\rangle\langle\psi|\right)$$

$\therefore$

$$\mathcal{E}\left(|\psi\rangle\langle\psi|\right) = \sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger$$

for all pure states $|\psi\rangle$ of Q.

By convex-linearity it follows that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \quad \text{in general.}$$

The condition $\sum_i E_i E_i^\dagger \leq I$ follows immediately from axiom A_1 identifying the trace of $\mathcal{E}(\rho)$ with a probability.

ly

- Freedom in the operator-sum representation

Consider quantum operations $\mathcal{E}$ and $\mathcal{F}$ acting on a single qubit with the operator sum representations,

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{and} \quad \mathcal{F}(\rho) = \sum_k F_k \rho F_k^\dagger$$

where the operation elements for $\mathcal{E}$ and $\mathcal{F}$ are defined by,

$$E_1 = \frac{I}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \frac{Z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and,

$$F_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

These appears to be very different quantum operations.

$$F_1 = \frac{E_1 + E_2}{\sqrt{2}} \quad \text{and} \quad F_2 = \frac{E_1 - E_2}{\sqrt{2}}$$

$$\mathcal{F}(\rho) = \sum_k F_k \rho F_k^\dagger = F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger$$

$$= \frac{(E_1+E_2)\rho(E_1^\dagger + E_2^\dagger) + (E_1 - E_2)\rho(E_1^\dagger - E_2^\dagger)}{2}$$

$$= E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger$$

$$= \mathcal{E}(\rho)$$

$\Rightarrow$ the operation elements appearing in an operator-sum representation for a quantum operation are not unique.

Ex:

Suppose we flipped a fair coin, and depending on the outcome of the coin toss, applied either the unitary operator I or Z to the quantum system. This process corresp. to the 1st operator-sum representation for $\mathcal{E}$.

$$\Longrightarrow \mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

$$= \frac{1}{2} I \rho I + \frac{1}{2} Z \rho Z^\dagger$$

The 2nd operator-sum representation for $\mathcal{E}$ (labeled $\mathcal{F}$ above) corresponds to performing a projective measurement in the $\{|0\rangle, |1\rangle\}$ basis, with the outcome of the measurement unknown.

$$\Rightarrow \mathcal{F}(\rho) = \sum_k F_k \rho F_k^\dagger = |0\rangle\langle 0| \rho |0\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1|$$

⇒ These two apparently very different physical processes give rise to exactly the same dynamics for the principal system.

When do 2 sets of operation elements give rise to the same quantum operation ?

This is important for atleast 2 reasons

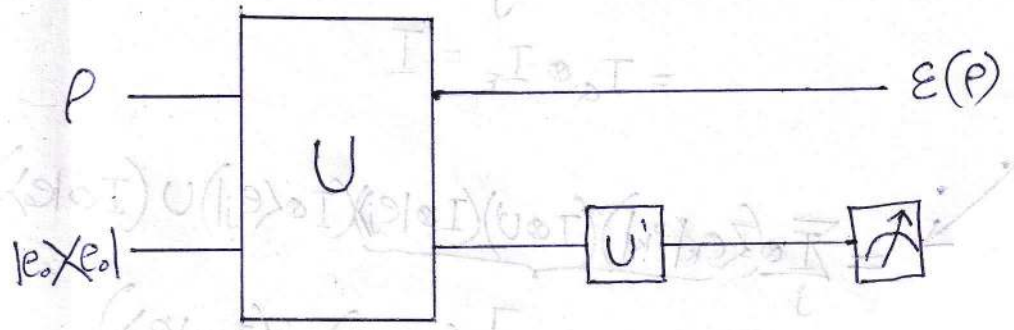
(1st) From a physical point of view, understanding the freedom in the representation gives us more insight into how different physical processes can give rise to the same system dynamics

(2nd) Understanding the freedom in operator-sum representation is crucial to a good understanding of quantum error-correction.

Consider a trace-preserving quantum operation $\mathcal{E}$ which describes the dynamics of some system.

The operation elements $E_k = \langle e_k|U|e_0\rangle = (I \otimes \langle e_k|)U(I \otimes |e_0\rangle)$ for $\mathcal{E}$ may be associated with an orthonormal basis $|e_k\rangle$ for the environment.

Suppose that, we supplement the interaction U with an additional unitary action U' on the environment alone, as shown in Fig. 8.7. This does not change the state of the principal system.



* Origin of the unitary freedom in the operator-sum representation.

What are the corresponding operation elements to this new process, $(I \otimes U')U$?

$$F_k = \langle e_k | (I \otimes U') U | e_0 \rangle$$

$$= \left(I \otimes \langle e_k|\right)(I \otimes U')U\left(I \otimes |e_0\rangle\right)$$

$$= \left(I \otimes \langle e_k|\right)(I \otimes U')\, I\, U\left(I \otimes |e_0\rangle\right)$$

$$= \left(I \otimes \langle e_k|\right)(I \otimes U')\left(\sum_j (I \otimes |e_j\rangle)(I \otimes \langle e_j|)\right)U\left(I \otimes |e_0\rangle\right)$$

where, $$\sum_j (I \otimes |e_j\rangle)(I \otimes \langle e_j|) = \sum_j I_Q \otimes |e_j\rangle\langle e_j|$$

$$= I_Q \otimes \sum_j |e_j\rangle\langle e_j|$$

$$= I_Q \otimes I_E = I$$

$$= \sum_j \underbrace{\left(I \otimes \langle e_k|\right)(I \otimes U')(I \otimes |e_j\rangle)}\left(I \otimes \langle e_j|\right)U\left(I \otimes |e_0\rangle\right)$$

$$= \sum_j \left[I \otimes \langle e_k|U'|e_j\rangle\right]\left(I \otimes \langle e_j|\right)U\left(I \otimes |e_0\rangle\right)$$

$$= \sum_j \langle e_k | U' | e_j \rangle \left(I \otimes \langle e_j | \right) U \left(I \otimes | e_0 \rangle \right)$$

$$= \sum_j U'_{kj} E_j$$

where $U'_{kj} = \langle e_k | U' | e_j \rangle$ are the matrix elements of U' w.r.t the basis $|e_k\rangle$.

Theorem (8.2): Unitary freedom in the operator-sum representation

Suppose $\{E_1, \ldots, E_m\}$ and $\{F_1, \ldots, F_n\}$ are operation elements giving rise to quantum operations $\mathcal{E}$ and $\mathcal{F}$, respectively. By appending zero operators to the shorter list of operation elements we may ensure that $m = n$.

Then,

$\mathcal{E} = \mathcal{F}$ if and only if there exists complex numbers u_{ij} such that $E_i = \sum_j u_{ij} F_j$, and $[u_{ij}]$ is an $m \times m$ matrix.

Theorem: (8.2) <u>Unitary freedom in the operator-sum representation</u>

Suppose $\{E_1, \dots, E_m\}$ and $\{F_1, \dots, F_n\}$ are operation elements giving rise to quantum operations $\mathcal{E}$ and $\mathcal{F}$, respectively. By appending zero operators to the shorter list of operation elements we may ensure that $m = n$.

Then,

$\mathcal{E} = \mathcal{F}$ if and only if there exists complex numbers u_{ij} such that $E_i = \sum_j u_{ij} F_j$, and $[u_{ij}]$ is an $m \times m$ matrix.

Proof

Theorem 2.6 : Unitary freedom in the ensemble for density matrices

QC②

Two sets of vectors $|\psi_i\rangle$ and $|\phi_j\rangle$ generate the same operator if and only if $|\psi_i\rangle = \sum_j u_{ij} |\phi_j\rangle$, where $[u_{ij}]$ is a unitary matrix of complex numbers, and we 'pad' whichever set of states $|\psi_i\rangle$ or $|\phi_j\rangle$ is smaller with additional states 0 so that the 2 sets have the same # of elements.

ie.,

$$|\psi_i\rangle = \sum_j u_{ij} |\phi_j\rangle \iff \sum_i |\psi_i\rangle\langle\psi_i| = \sum_j |\phi_j\rangle\langle\phi_j|$$

for some unitary matrix $[u_{ij}]$

Suppose $\{E_i\}$ and $\{F_j\}$ are 2 sets of operation elements for the same quantum operation,

$$\sum_i E_i \rho E_i^\dagger = \sum_j F_j \rho F_j^\dagger \quad \text{for all } \rho.$$

Define,

$$|e_i\rangle = \sum_k |k_R\rangle \otimes \left(E_i |k_Q\rangle\right)$$

$$|f_j\rangle = \sum_k |k_R\rangle \otimes \left(F_j |k_Q\rangle\right)$$

Define a joint state $|\alpha\rangle$ of RQ by,

$$|\alpha\rangle = \sum_k |k_R\rangle \otimes |k_Q\rangle$$

which is, upto a normalization factor, a maximally entangled state of the systems R and Q.

Define an operator σ on the state space of RQ by,

$$\sigma \equiv (I_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|)$$

which can be thought of as the result of applying the quantum operation $\mathcal{E}$ to one half of a maximally entangled state of the system RQ.

$$|\alpha\rangle\langle\alpha| = \left(\sum_k |k_R\rangle \otimes |k_Q\rangle\right)\left(\sum_l \langle l_R| \otimes \langle l_Q|\right)$$

$$= \sum_{k,l} |k_R\rangle\langle l_R| \otimes |k_Q\rangle\langle l_Q|$$

$$\sigma = (I_R \otimes \mathcal{E})(|\alpha\rangle\langle\alpha|)$$

$$= (I_R \otimes \mathcal{E})\left[\sum_{k,l} |k_R\rangle\langle l_R| \otimes |k_Q\rangle\langle l_Q|\right]$$

$$= \sum_{k,l} |k_R\rangle\langle l_R| \otimes \mathcal{E}\left(|k_Q\rangle\langle l_Q|\right)$$

$$\sum_i |e_i\rangle\langle e_i| = \sum_i \left[\sum_k |k_R\rangle \otimes \left(E_i|k_Q\rangle\right)\right]\left[\sum_l \langle l_R| \otimes \left(\langle l_Q|E_i^\dagger\right)\right]$$

$$= \sum_i \sum_{k,l} |k_R\rangle\langle l_R| \otimes E_i|k_Q\rangle\langle l_Q|E_i^\dagger$$

$$= \sum_{k,l} |k_R\rangle\langle l_R| \otimes \sum_i E_i|k_Q\rangle\langle l_Q|E_i^\dagger$$

$$= \sum_{k,l} |k_R\rangle\langle l_R| \otimes \mathcal{E}\left(|k_Q\rangle\langle l_Q|\right) = \sigma$$

$$\sum_j |f_j\rangle\langle f_j| = \sum_j \left[\sum_k |k_R\rangle \otimes \left(F_j|k_Q\rangle\right)\right]\left[\sum_l \langle l_R| \otimes \left(\langle l_Q|F_j^\dagger\right)\right]$$

$$= \sum_j \sum_{k,l} |k_R\rangle\langle l_R| \otimes F_j|k_Q\rangle\langle l_Q|F_j^\dagger$$

$$= \sum_{k,l} |k_R\rangle\langle l_R| \otimes \sum_j F_j|k_Q\rangle\langle l_Q|F_j^\dagger$$

$$= \sum_{k,l} |k_R\rangle\langle l_R| \otimes \mathcal{E}\left(|k_Q\rangle\langle l_Q|\right) = \sigma$$

since $\sum_i E_i \rho E_i^\dagger = \sum_j E_j \rho E_j^\dagger$ for all ρ.

$$\therefore \quad \sigma = \sum_i |e_i\rangle\langle e_i| = \sum_j |f_j\rangle\langle f_j|$$

Theorem 2.6 $\Longrightarrow$

There exists unitary $[u_{ij}]$ such that

$$|e_i\rangle = \sum_j u_{ij} |f_j\rangle \quad \text{— (8.77)}$$

Let $|\psi\rangle = \sum_j \psi_j |j_Q\rangle$ be any state of system Q. Define a corresponding state $|\tilde{\psi}\rangle$ of system R by the equation,

$$|\tilde{\psi}\rangle = \sum_j \psi_j^* |j_R\rangle$$

Define a map,

$$E_i|\psi\rangle \equiv \left(\langle\tilde{\psi}|\otimes I_Q\right)|e_i\rangle \quad \& \quad F_j|\psi\rangle \equiv \left(\langle\tilde{\psi}|\otimes I_Q\right)|f_j\rangle$$

Axiomatic approach $\Rightarrow$ $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger = \sum_j F_j \rho F_j^\dagger$

$\therefore$ For arbitrary $|\psi\rangle$, we have

$|\tilde{\psi}\rangle$

$$\begin{aligned} E_i|\psi\rangle &= \left(\langle\tilde{\psi}|\otimes I_Q\right)|e_i\rangle \\ &= \left(\langle\tilde{\psi}|\otimes I_Q\right)\left(\sum_j u_{ij}|f_j\rangle\right) \\ &= \sum_j u_{ij} \left(\langle\tilde{\psi}|\otimes I_Q\right)|f_j\rangle \\ &= \sum_j u_{ij} F_j|\psi\rangle \end{aligned}$$

$$\therefore \quad E_i = \sum_j u_{ij} F_j$$

Theorem 8.3: All quantum operations $\mathcal{E}$ on a system of Hilbert space dimension d can be generated by an operator-sum representation containing at most d^2 elements,

$$\mathcal{E}(\rho) = \sum_{k=1}^{M} E_k \rho E_k^\dagger$$

where, $1 \le M \le d^2$

PROOF

Ex: 8.10

Let $\{E_j\}$ be a set of operation elements for $\mathcal{E}$. Define a matrix $W_{jk} \equiv \mathrm{tr}(E_j^\dagger E_k)$. Show that the matrix W is Hermitian and of rank at most d^2, and thus there is unitary matrix u such that $uWu^\dagger$ is diagonal with at most d^2 non-zero entries. Use u to define a new set of at most d^2 non-zero operation elements $\{F_j\}$ for $\mathcal{E}$.

Ans: The Kraus operators are,

$$E_j = \left(I \otimes \langle e_j|\right) U \left(I \otimes |e_0\rangle\right)$$

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Given that the quantum operation $\mathcal{E}$ acts on a Hilbert space of dimension d, ie., $\rho \in \mathbb{C}^{d\times d}$ and $E_j \in \mathbb{C}^{d\times d}$.

$\therefore$

A maximum of d^2 of the E_j can be linearly independent.

$$W = \begin{bmatrix} tr(E_1^\dagger E_1) & tr(E_1^\dagger E_2) & \cdots & tr(E_1^\dagger E_M) \\ tr(E_2^\dagger E_1) & tr(E_2^\dagger E_2) & \cdots & tr(E_2^\dagger E_M) \\ \vdots & \vdots & \ddots & \vdots \\ tr(E_M^\dagger E_1) & tr(E_M^\dagger E_2) & \cdots & tr(E_M^\dagger E_M) \end{bmatrix}$$

$$W_{jk} = tr\left(E_j^\dagger E_k\right) = tr\left(\left(E_k^\dagger E_j\right)^\dagger\right) = tr\left(E_k^\dagger E_j\right)^* = W_{kj}^*$$

$\therefore$ W is hermitian.

Let the k^{th} column of W is,

$$|W_k\rangle = \sum_{j=1}^{M} W_{jk}|j\rangle = \sum_{j=1}^{M} tr\left(E_j^\dagger E_k\right)|j\rangle$$

Let E_j are arranged such that $E_1, \ldots, E_{d^2}$ are linearly independent, then for any $k > d^2$ we have,

$$E_k = \sum_{l=1}^{d^2} c_{kl} E_l \quad \text{where } c_{kl} \in \mathbb{C}$$

$$|w_k\rangle = \sum_{j=1}^{M} tr\left(E_j^\dagger E_k\right)|j\rangle$$

$$= \sum_{j=1}^{M} tr\left(E_j^\dagger \sum_{l=1}^{d^2} c_{kl} E_l\right)|j\rangle$$

$$= \sum_{j=1}^{M} tr\left(\sum_{l=1}^{d^2} c_{kl} E_j^\dagger E_l\right)|j\rangle$$

$$= \sum_{j=1}^{M} \sum_{l=1}^{d^2} c_{kl}\, tr\left(E_j^\dagger E_l\right)|j\rangle$$

$$= \sum_{l=1}^{d^2} c_{kl}\left[\sum_{j=1}^{M} tr\left(E_j^\dagger E_l\right)|j\rangle\right]$$

$$= \sum_{l=1}^{d^2} c_{kl}\,|w_l\rangle$$

$\Rightarrow$ For every $k > d^2$, the k^{th} column of W is a linear combination of the first d^2 linearly independent columns of W.

$\Rightarrow$ the # of linearly independent columns of W is therefore at most d^2.

$\Rightarrow$ $\text{rank}(W) \leq d^2$

And since W is a hermitian matrix, there exists a decomposition $uWu^\dagger$ which is a diagonal matrix with at most d^2 non-zero diagonal entries.

Using the unitary u that diagonalizes W to define new operators, $F_i = \sum_j u_{ij}^* E_j$

$$F_i^\dagger F_i = \left(\sum_j u_{ij} E_j^\dagger\right)\left(\sum_k E_k u_{ik}^*\right)$$

$$= \sum_{j,k} u_{ij} E_j^\dagger E_k u_{ik}^*$$

$$tr\left(F_i^\dagger F_i\right) = tr\left(\sum_{j,k} u_{ij} E_j^\dagger E_k u_{ik}^*\right)$$

$$= \sum_{j,k} u_{ij}\, tr\left(E_j^\dagger E_k\right) u_{ik}^*$$

$$= \sum_{j,k} u_{ij} W_{jk} u_{ik}^* = \left(uWu^\dagger\right)_{ii}$$

$\therefore$ For all but d^2 of the F_i, we have

$tr(F_i^\dagger F_i) = 0$

$F_i^\dagger F_i$ is positive semi-definite

$\lambda_i \geq 0$

ILA 8

$tr(F_i^\dagger F_i) = 0 \implies$ all $\lambda_i = 0$

$\therefore$ $F_i = 0$ for all but d^2 of the F_i

$\Rightarrow$ Using the unitary u, we have defined a new set of at most d^2 non-zero operation elements $\{F_j\}$ for $\mathcal{E}$.

Theorem 8.2 $\Rightarrow$ F_i describe the same quantum operation as the original E_i.

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Ex: 8.11 Suppose $\mathcal{E}$ is a quantum operation mapping a d-dimensional i/p space to a d'-dimensional o/p space. Show that $\mathcal{E}$ can be described using a set of at most dd' operation elements $\{E_k\}$.

Examples of quantum noise & quantum operations

Trace & partial trace

One of the main uses of the quantum operations formalism is to describe the effects of measurement. Quantum operations can be used to describe both the probability of getting a particular outcome from a measurement on a quantum system, and also the state change in the system effected by the measurement.

The simplest operation related to measurement is the trace map, $\rho \to tr(\rho)$, which can be shown to be a quantum operation.

Let H_Q be any i/p Hilbert space, spanned by an orthonormal basis $|1\rangle, \ldots, |d\rangle$, and let H'_Q be a one-dimensional o/p space, spanned by the state $|0\rangle$.

Define,

$$\mathcal{E}(\rho) \equiv \sum_{i=1}^{d} |0\rangle\langle i|\rho|i\rangle\langle 0| = |0\rangle\left[\sum_{i=1}^{d} \langle i|\rho|i\rangle\right]\langle 0|$$
$$= tr(\rho)|0\rangle\langle 0|$$

So $\mathcal{E}$ is a quantum operation by Theorem 8.1.

$\mathcal{E}(\rho) = tr(\rho)|0\rangle\langle 0|$, so that, upto an unimportant $|0\rangle\langle 0|$ multiplier, this quantum operation is identical to the trace function.

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The partial trace is a quantum operation.

Suppose we have a joint system QR, and wish to trace out system R. Let $|j\rangle$ be a basis for system R.

Define a linear operator $E_i : H_{QR} \rightarrow H_Q$ by,

$$E_i \left[\sum_j \lambda_j |q_j\rangle \otimes |j\rangle \right] \equiv \lambda_i |q_i\rangle$$

where $\lambda_i \in \mathbb{C}$ and $|q_j\rangle$ are arbitrary states of system Q.

$$\therefore \quad E_i = I \otimes \langle i|$$

Define $\mathcal{E}$ to be the quantum operation with operation elements $\{E_i\}$, i.e.,

$$\mathcal{E}(\sigma) = \sum_i E_i \sigma E_i^\dagger$$

where σ is a density matrix on QR.

By Theorem 8.1, this is a quantum operation from system QR to system Q.

Let ρ be a density matrix on Q.
Then,

$$\mathcal{E}(\rho \otimes |j\rangle\langle j'|) = \sum_i E_i (\rho \otimes |j\rangle\langle j'|) E_i^\dagger$$

$$= \sum_i (I \otimes \langle i|)(\rho \otimes |j\rangle\langle j'|)(I \otimes |i\rangle)$$

$$= \sum_i \rho \otimes \langle i|j\rangle\langle j'|i\rangle$$

$$= \sum_i \rho \otimes \delta_{i,j}\, \delta_{j',i}$$

$$= \rho \otimes \delta_{j,j'}$$

$$= \rho\, \delta_{j,j'} = tr_R (\rho \otimes |j\rangle\langle j'|)$$

where ρ is any Hermitian operator on the state space of system Q, and $|j\rangle$ and $|j'\rangle$ are members of the orthonormal basis for system R.

By linearity of $\mathcal{E}$ and tr_R

$$\Longrightarrow \mathcal{E} = tr_R$$

Geometric picture of single qubit quantum operation

The state of a single qubit can always be written in the Bloch representation,

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}, \quad \|r\| \le 1$$

where, $\vec{r}$ is a 3-component real vector, which is known as the Bloch vector for the state ρ.

Proof

$$S^{\dagger} = S \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \quad ; \quad a, b, c, d \in \mathbb{C}$$

$$a^* = a \;\; \& \;\; d^* = d \Longrightarrow a, d \in \mathbb{R}$$

$$c^* = b \;\; \& \;\; b^* = c \Longrightarrow b = v + iw, \; c = v - iw$$

$\begin{bmatrix} u & v+iw \\ v-iw & z \end{bmatrix}$ is 2×2 hermitian

$$\sigma_0 = I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \sigma_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$\rho = n_0\sigma_0 + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$$

$$= \begin{bmatrix}n_0+n_3 & n_1-in_2\\ n_1+in_2 & n_0-n_3\end{bmatrix} \text{ is hermitian}$$

$\therefore$ The Pauli matrices along with the identity matrix form an orthonormal basis for the complex 2D Hilbert space.

$\Rightarrow$ An arbitrary density matrix for a mixed state qubit is written as real linear combination of $I, \sigma_1, \sigma_2, \sigma_3$.

$$tr(\rho) = 1 \Longrightarrow n_0 = 1/2$$

$$\det(\rho - \lambda I) = \begin{vmatrix}n_0+n_3-\lambda & n_1-in_2\\ n_1+in_2 & n_0-n_3-\lambda\end{vmatrix}$$

$$= n_0^2 - n_3^2 - 2n_0\lambda + \lambda^2 - n_1^2 - n_2^2$$

$$= \lambda^2 - 2n_0\lambda + n_0^2 - (n_1^2+n_2^2+n_3^2) = 0$$

$$\lambda = n_0 \pm \sqrt{n_1^2+n_2^2+n_3^2} = \frac{1}{2}\left[1 \pm \sqrt{(2n_1)^2+(2n_2)^2+(2n_3)^2}\right]$$

ρ is +ve semidefinite $\Longrightarrow \lambda \geq 0$

$$\sqrt{(2n_1)^2+(2n_2)^2+(2n_3)^2} \le 1$$

$$\sqrt{n_1^2+n_2^2+n_3^2} \le 1/2$$

$$r = (2n_1, 2n_2, 2n_3) = (2r_x, 2r_y, 2r_z)$$

$\therefore$

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$

$$= \frac{1}{2}\begin{bmatrix} 1+n_3 & n_1 - in_2 \\ n_1 + in_2 & 1-n_3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}$$

* The Pauli matrices, along with the identity matrix I, form an orthonormal basis for the Hilbert space of all 2×2 complex matrices; $M_{2\times 2}(\mathbb{C})$.

ie.,

Any 2×2 matrix can be written as,

$$A = cI + \sum_k a_k \sigma_k$$

, where c is a complex #, and $a = (a_1, a_2, a_3)$ is a 3-component complex vector.

$\therefore$

The operators that generate the operator-sum representation for $\mathcal{E}$ can be written in the form:

$$E_i = \alpha_i I + \sum_{k=1}^{3} a_{ik} \sigma_k$$

* The Pauli matrices, along with the identity matrix I, form an orthonormal basis for the Hilbert space of all 2×2 complex matrices; $M_{2\times 2}(\mathbb{C})$.

ie.,

Any 2×2 matrix can be written as,

$$A = cI + \sum_k a_k \sigma_k$$

where c is a complex #, and $a = (a_1, a_2, a_3)$ is a 3-component complex vector.

$\therefore$

The operators that generate the operator-sum representation for $\mathcal{E}$ can be written in the form:

$$E_i = \alpha_i I + \sum_{k=1}^{3} a_{ik} \sigma_k$$

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$ — State of a single qubit in Bloch representation

Lemma 1: ~~The pauli matrices~~, along with the identity matrix I, form an orthonormal basis for the Hilbert space of all 2×2 complex matrices, $M_{2\times 2}(\mathbb{C})$,

i.e., $$M_{2\times 2} = aI + \sum_{k=1}^{3} a_k \sigma_k$$

Lemma 2: The pauli matrices, along with the identity matrix I, form an orthogonal basis of the Hilbert space of 2×2 Hermitian matrices, $\mathcal{H}_2$ over $\mathbb{R}$. (all coeff. being real numbers).

From the lemma 1, in the most general case, let the quantum operation $\mathcal{E}$ do the following to the four basis matrices we have for the Bloch sphere, as

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$$\mathcal{E}(I) = aI + \sum_i c_i \sigma_i \qquad \text{(Lemma 2)}$$

$$\mathcal{E}(\sigma_j) = b_j I + \sum_k M_{kj} \sigma_k \quad ; \; b_j, M_{kj} \in \mathbb{R}$$

$$\mathcal{E}(\rho) = \mathcal{E}\left(\frac{I + \vec{r}\cdot\vec{\sigma}}{2}\right) = \mathcal{E}\left(\frac{I + \sum_j r_j \sigma_j}{2}\right)$$

$$= \frac{1}{2}\left[\mathcal{E}(I) + \sum_j r_j \mathcal{E}(\sigma_j)\right]$$

$$= \frac{1}{2}\left[aI + \sum_i c_i \sigma_i + \sum_j r_j \left(b_j I + \sum_k M_{kj} \sigma_k\right)\right]$$

$$= \frac{1}{2}\left[aI + \sum_i c_i \sigma_i + \sum_j r_j b_j I + \sum_{j,k} M_{kj} r_j \sigma_k\right]$$

$$= \frac{1}{2}\left[aI + \sum_i c_i \sigma_i + \sum_j r_j b_j I + \sum_k \left(\sum_j M_{kj} r_j\right) \sigma_k\right]$$

$$= \frac{1}{2}\left[aI + \sum_i c_i \sigma_i + \sum_j r_j b_j I + \sum_k (M\vec{r})_k \sigma_k\right]$$

$$= \frac{1}{2}\left[aI + \vec{c}\cdot\vec{\sigma} + \vec{r}\cdot\vec{b}\,I + M\vec{r}\cdot\vec{\sigma}\right]$$

$$= \frac{1}{2}\left[aI + \vec{r}\cdot\vec{b} + M\vec{r}\cdot\vec{\sigma} + \vec{c}\cdot\vec{\sigma}\right]$$

$$= \frac{1}{2}\left[aI + \vec{r}\cdot\vec{b} + \left(M\vec{r} + \vec{c}\right)\cdot\vec{\sigma}\right]$$

Assuming that the quantum operation $\mathcal{E}$ is trace preserving, i.e., $tr(\mathcal{E}(\rho)) = 1$

$$tr(\mathcal{E}(\rho)) = 1 \Rightarrow tr\left[\frac{aI + \sum_i c_i\sigma_i + \sum_j r_j b_j\, I + \sum_{j,k} r_j \sigma_k M_{kj}}{2}\right] =$$

$$1 = \frac{1}{2}\left[a\, tr(I) + \sum_i c_i\, tr(\sigma_i) + \sum_j r_j b_j\, tr(I) + \sum_{j,k} r_j\, tr(\sigma_k) M_{kj}\right]$$

$$1 = \frac{1}{2}\left[a\cdot 2 + \sum_i c_i \cdot 0 + \sum_j r_j b_j \cdot 2 + \sum_{j,k} r_j \cdot 0 \cdot M_{kj}\right]$$

$$1 = \frac{1}{\cancel{2}}\left[\cancel{2}a + \cancel{2}\sum_j r_j b_j\right]$$

$$\Longrightarrow a + \sum_j r_j b_j = 1$$

The trace-preserving condition holds irrespective of $\vec{r}$, which implies $a=1$ and $\vec{b}=0$.

Then,

$$\mathcal{E}(\rho) = \frac{1}{2}\left[I + (M\vec{r}+\vec{c})\cdot\vec{\sigma}\right]$$

So, we see that the most general transformation of the Bloch sphere is an affine transformation.

$\therefore$

In the Bloch sphere representation, an arbitrary trace-preserving quantum operation is equivalent to a map of the form,

$$\boxed{\vec{r} \xrightarrow{\quad\mathcal{E}\quad} \vec{r}\,' = M\vec{r}+\vec{c}}$$

where M is a 3×3 real matrix, and $\vec{c}$ is a constant vector.

This is an affine map, mapping the Bloch sphere into itself.

Consider the polar decomposition of the matrix M, $M = U|M|$, where U is unitary.

$$A = U\Sigma V^T = (UV^T)(V\Sigma V^T) = QS$$

M is real $\Rightarrow$ $|M|$ is a symmetric matrix

$\Rightarrow$ U is real & an orthogonal matrix.

$$U^T U = I$$

* An orthogonal matrix, is a real square matrix whose columns and rows are orthonormal vectors.

*

$$\det(U) = \pm 1$$

$$\det(kA) = \det(kI_n A) = \det(kI_n)\det(A)$$
$$= k^n \det(A)$$

$\Rightarrow$ A 3×3 orthogonal matrix with -ve determinant can be converted to a pure rotation by factoring out a $-I$.

$$\therefore \quad M = OS \qquad —(8.93)$$

where,

O is a real orthogonal matrix with determinant 1, representing a proper rotation, and S is a real symmetric matrix.

$\Rightarrow$ $\vec{r} \xrightarrow{\mathcal{E}} \vec{r}' = M\vec{r} + \vec{c}$ is just a deformation of the Bloch sphere along principal axes determined by S, followed by a proper rotation due to O, followed by a displacement due to $\vec{c}$.

(A) Bit flip & phase flip channels

The bit flip channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1-p$.

It has operation elements,

$$E_0 = \sqrt{p}\, I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \sqrt{1-p}\, X = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger =$$

$$= \frac{p\rho}{2} + \frac{(1-p)}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1+z & x-iy\\ x+iy & 1-z\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$= \frac{p}{2}\begin{bmatrix}1+z & x-iy\\ x+iy & 1-z\end{bmatrix} + \frac{1-p}{2}\begin{bmatrix}1-z & x+iy\\ x-iy & 1+z\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1+z(2p-1) & x-iy(2p-1)\\ x+iy(2p-1) & 1-z(2p-1)\end{bmatrix}$$

$$\sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \frac{1}{2}\left[I + x\sigma_x + (2p-1)y\sigma_y + (2p-1)z\sigma_z\right]$$

$$= \frac{I + \vec{r}\,'\cdot\vec{\sigma}}{2}$$

$$\vec{r} = (x, y, z) \xrightarrow{\ \mathcal{E}\ } \vec{r}\,' = (x, (2p-1)y, (2p-1)z)$$

$\Rightarrow$ The states on the $\hat{x}$ axis are left alone, while the $\hat{y}\hat{z}$-plane is uniformly contracted by a factor of $1-2p$.

$$\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2} = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

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$$\rho^2 = \frac{1}{4}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+z^2+2z+x^2+y^2 & x-iy+\cancel{xz}-iyz + x-iy-\cancel{xz}+iz \\ x+iy+\cancel{xz}+iyz + x+iy-\cancel{xz}-iyz & x^2+y^2+1-2z+z^2 \end{bmatrix}$$

$$tr(\rho^2) = \frac{1}{4}\left[2+2x^2+2y^2+2z^2\right] = \underline{\frac{1}{2}\left[1+|r|^2\right]}$$

for a single qubit., where $|r|$ is the norm of the Bloch vector.

$$\vec{r} = (x,y,z) \xrightarrow{\mathcal{E}} \vec{r}' = (x, (2p-1)y, (2p-1)z)$$

$\therefore$ The contraction of the Bloch sphere, due to the effect of the Bit flip channel, cannot increase the norm of the Bloch vector.

$\Rightarrow tr(\rho^2)$ can only ever decrease for the Bit flip channel.

* A state ρ is pure iff $tr(\rho^2) = 1$.

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The phase flip channel has operation elements.

$$E_0 = \sqrt{p}\, I = \sqrt{p}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$E_1 = \sqrt{1-p}\; Z = \sqrt{1-p}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

where $Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ -\beta\end{bmatrix}$ (or) $\alpha|0\rangle + \beta|1\rangle \xrightarrow{Z} \alpha|0\rangle - \beta|1\rangle$

$$\mathcal{E}(\rho) = E_0 \rho E_0^{\dagger} + E_1 \rho E_1^{\dagger}$$

$$= \frac{p\rho}{2} + \frac{1-p}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1+z & x-iy\\ x+iy & 1-z\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \frac{p}{2}\begin{bmatrix}1+z & x-iy\\ x+iy & 1-z\end{bmatrix} + \frac{1-p}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1+z & -x+iy\\ x+iy & -1+z\end{bmatrix}$$

$$= \frac{p}{2}\begin{bmatrix}1+z & x-iy\\ x+iy & 1-z\end{bmatrix} + \frac{1-p}{2}\begin{bmatrix}1+z & -x+iy\\ -x-iy & 1-z\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1+z & x(2p-1) - iy(2p-1)\\ x(2p-1) + iy(2p-1) & 1-z\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1+z & (x-iy)(2p-1)\\ (x+iy)(2p-1) & 1-z\end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\left[I + x(2p-1)X + y(2p-1)Y + zZ \right]$$

$$= \frac{1}{2}\left[I + \vec{r}\,' \cdot \vec{\sigma} \right]$$

$\therefore$

$$\vec{r} = (x, y, z) \xrightarrow{\mathcal{E}} \vec{r}\,' = \left(x(2p-1), y(2p-1), z \right)$$

$\Rightarrow$ The effect of the phase flip channel on the Bloch sphere is that, the states on the $\hat{z}$ axis are left alone, while the $\hat{x}\hat{y}$-plane is uniformly contracted by a factor of $1-2p$.

Special case: Consider the quantum operation which arises when we choose $p = 1/2$.

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1+z & 0 \\ 0 & 1-z \end{bmatrix}$$

$$= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

$$= P_0 \rho P_0 + P_1 \rho P_1$$

This corresponds to a measurement of the qubit in the $|0\rangle, |1\rangle$ basis, with the result of the measurement unknown.

The corresponding map on the Bloch sphere is given by,

$$\vec{r} = (r_x, r_y, r_z) \longrightarrow \vec{r}' = (0, 0, r_z)$$

$\therefore$ The Bloch vector is projected along the $\hat{z}$-axis, and the x and y components of the Bloch vector are lost.

The bit-phase flip channel has operation elements,

$$E_0 = \sqrt{p}\, I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_1 = \sqrt{1-p}\, Y = \sqrt{1-p}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

⇒ This is a combination of a phase flip and a bit flip, since $Y = iXZ$.

$$iXZ = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

$$= \frac{p}{2}\rho + \frac{1-p}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \frac{p}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} + \frac{1-p}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} ix+y & -i-iz \\ i-iz & -ix+y \end{bmatrix}$$

$$= \frac{p}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} + \frac{1-p}{2}\begin{bmatrix} 1-z & -x-iy \\ -x+iy & 1+z \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1(1-z)+p(2z) & -x-iy+p(2x) \\ -x+iy+p(2x) & 1+z+p(-2z) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+z(2p-1) & x(2p-1)-iy \\ x(2p-1)+iy & 1-z(2p-1) \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\left[I + x(2p-1)X + yY + z(2p-1)Z\right]$$

$\therefore$

$$\vec{r} = (x, y, z) \xrightarrow{\mathcal{E}} \vec{r}' = ((2p-1)x,\ y,\ (2p-1)z)$$

$\Rightarrow$ The effect of the bit-phase flip channel on the Bloch sphere is that, the states on the $\hat{y}$-axis are left alone, while the $\hat{x}\hat{z}$-plane is uniformly contracted by a factor of $1-2p$.

Ex: 8.15 Suppose a projective measurement is performed on a single qubit in the basis $|+\rangle, |-\rangle$, where $|\pm\rangle \equiv \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$. In the event that we are ignorant of the result of the measurement, the density matrix evolves according to the equation,

$$\rho \longrightarrow \mathcal{E}(\rho) = |+\rangle\langle+|\rho|+\rangle\langle+| + |-\rangle\langle-|\rho|-\rangle\langle-|$$

Illustrate this transformation on the Bloch sphere.

Ans: $|+\rangle\langle+| = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$$|-\rangle\langle-| = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$\mathcal{E}(\rho) = \frac{1}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1+z & x-iy\\x+iy & 1-z\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1+z & x-iy\\x+iy & 1-z\end{bmatrix} \times \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

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$$= \frac{1}{4}\begin{bmatrix}1+z+x+iy & 1-z+x-iy\\1+z+x+iy & 1-z+x-iy\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1+z-x-iy & -1+z+x-iy\\-1-z+x+iy & 1-z-x+iy\end{bmatrix} \times \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}2+2x & 2+2x\\2+2x & 2+2x\end{bmatrix} + \frac{1}{4}\begin{bmatrix}2-2x & -2+2x\\-2+2x & 2-2x\end{bmatrix}$$

$$\mathcal{E}(\rho) = \frac{1}{2}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathcal{E}(\rho) = \frac{I + \vec{r}'\cdot\vec{\sigma}}{2}$$

$$= \frac{1}{2}\left[I + xX\right]$$

$$\vec{r} = (x, y, z) \xrightarrow{\mathcal{E}} \vec{r}' = (x, 0, 0)$$

Ex: 8.16 The graphical method for understanding single qubit quantum operations was derived for trace-preserving quantum operations. Find an explicit example of a non-trace preserving quantum operation which cannot be described as a deformation of the Bloch sphere, followed by a rotation and a displacement.

Ans:

(B) ## 1) Depolarizing channel

The depolarizing channel is an important type of quantum noise. Imagine we take a single qubit, and with probability p that qubit is depolarized, ie., it is replaced by the completely mixed state, $I/2$. With probability $1-p$ the qubit is left untouched.

The state of the quantum system after this noise is,

$$\mathcal{E}(\rho) = p\frac{I}{2} + (1-p)\rho$$

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{bmatrix} 1+Z & x-iy \\ x+iy & 1-Z \end{bmatrix}$$

$$\mathcal{E}(\rho) = p\frac{I}{2} + (1-p)\rho$$

$$= \frac{p}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{(1-p)}{2}\begin{bmatrix} 1+Z & x-iy \\ x+iy & 1-Z \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+z(1-p) & (x-iy)(1-p) \\ (x+iy)(1-p) & 1-z(1-p) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+z(1-p) & x(1-p) - iy(1-p) \\ x(1-p) + iy(1-p) & 1-z(1-p) \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

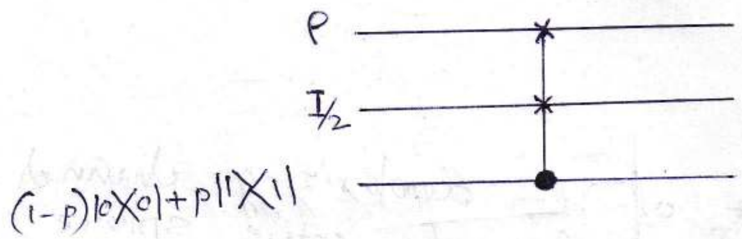
$$= \frac{1}{2}\left[I + x(1-p)X + y(1-p)Y + z(1-p)Z\right]$$

$\therefore$

$$\vec{r} = (x, y, z) \xrightarrow{\mathcal{E}} \vec{r}\,' = \big(x(1-p),\, y(1-p),\, z(1-p)\big)$$

$\Rightarrow$ The effect of the depolarizing channel on the Bloch sphere, is that the entire sphere contracts uniformly as a function of p.

* Circuit implementation of the depolarizing channel.



The top line of the circuit is the input to the depolarizing channel, while the bottom two lines are an 'environment' to simulate the channel. We have used an environment with a mixed state inputs.

The idea is that, the 3^{rd} qubit, initially a mixture of the state $|0\rangle$ with probability $1-p$ and state $|1\rangle$ with probability p acts as a control for whether or not the completely mixed state $I/2$ stored in the 2^{nd} qubit is swapped into the 1^{st} qubit.

$\mathcal{E}(\rho) = p\frac{I}{2} + (1-p)\rho$ is not in the operator sum representation.

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

$$\rho + X\rho X + Y\rho Y + Z\rho Z = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1-z & x+iy \\ x-iy & 1+z \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1-z & -x-iy \\ -x+iy & 1+z \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1+z & -x+iy \\ -x-iy & 1-z \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 2I$$

$\therefore$

For arbitrary ρ,

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

2 pages forward $\xrightarrow[\text{Proof}]{}$

Substituting for $I/2$,

$$\mathcal{E}(\rho) = p\frac{I}{2} + (1-p)\rho$$

$$= p\left[\frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}\right] + (1-p)\rho$$

$$= \rho\left(\frac{p}{4} + 1 - p\right) + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

$$= \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

$\Rightarrow$ The depolarizing channel has operation elements, $\{E_0, E_1, E_2, E_4\} = \left\{\sqrt{1-3p/4}\, I, \sqrt{p}\, X/2, \sqrt{p}\, Y/2, \sqrt{p}\, Z/2\right\}$.

It is frequently convenient to parametrize the depolarizing channel in different ways, such as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

which has the interpretation that the state ρ is left alone with probability $1-p$, and the operators X, Y and Z applied each with probability $p/3$.

Ex: 8.17 Verify $\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$

Define $\mathcal{E}(A) = \frac{A + XAX + YAY + ZAZ}{4}$ and

show that $\mathcal{E}(I) = I$; $\mathcal{E}(X) = \mathcal{E}(Y) = \mathcal{E}(Z) = 0$

Now use Bloch sphere representation for single qubit density matrices to verify the equation.

Ans: $\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right) = \frac{I}{2} + \frac{1}{2}\left(xX + yY + zZ\right)$

$$\mathcal{E}(\rho) = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

$$= \mathcal{E}(I/2) + \frac{x}{2}\mathcal{E}(X) + \frac{y}{2}\mathcal{E}(Y) + \frac{z}{2}\mathcal{E}(Z)$$

$$= \frac{I}{2}$$

* The depolarizing channel can be generalized to quantum systems of dimensions more than two. For a d-dimensional quantum system the depolarizing channel again replaces the quantum system with the completely mixed state I/d with probability p, and leaves the state untouched otherwise.

The corresp. quantum operation is,

$$\mathcal{E}(\rho) = p\frac{I}{d} + (1-p)\rho$$

Ex: 8.18 For $k \geq 1$, show that $tr(\rho^k)$ is never increased by the action of the depolarizing channel.

Ans:

Ex: 8.19 Find an operator-sum representation for a generalized depolarizing channel acting on a d-dimensional Hilbert space.

Ans:

© Amplitude damping.

An important application of quantum operations is the description of energy dissipation — effects due to loss of energy from a quantum system.

What are the dynamics of an atom which is spontaneously emitting a photon ?

How does a spin system at high temperature approach equilibrium with its environment ?

What is the state of a photon in an interferometer or cavity when it is subject to scattering and attenuation ?

Each of these processes has its own unique features, but the general behavior of all of them is well characterized by a quantum operation known as amplitude damping.

* The amplitude damping channel gradually maps a qubit to the $|0\rangle$ state. It is a useful representation of a qubit for which the $|0\rangle$ state is the ground state and the $|1\rangle$ state is an excited state. The qubit can go from the excited state to the ground state, called relaxation; the inverse is far less probable and is not modelled here.

The operation elements for amplitude damping are,

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} = \frac{1+\sqrt{1-\gamma}}{2} I + \frac{1-\sqrt{1-\gamma}}{2} Z$$

$$= |0\rangle\langle 0| + \sqrt{1-\gamma}\, |1\rangle\langle 1|$$

$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} = \sqrt{\gamma}\left(X + iY\right)$$

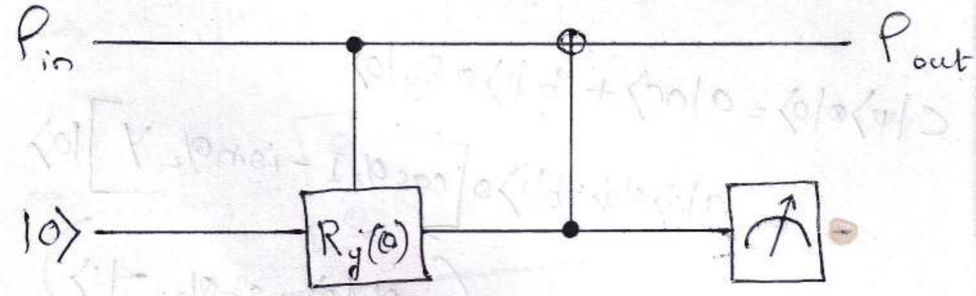
$$= \sqrt{\gamma}\, |0\rangle\langle 1|$$

- $\gamma = \sin^2\theta$ can be thought of as the probability of losing a photon.

The E_1 operation changes a $|1\rangle$ state into a $|0\rangle$ state, corresp. to the physical process of losing a quantum of energy to the environment.

E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; physically this happens because a quantum of energy was not lost to the environment

Ex: 8.20 Circuit model for amplitude damping



Show that the circuit models the amplitude damping quantum operation, with $\sin^2(\theta/2) = \gamma$

Ans: $E_k = (I \otimes \langle k|) U (I \otimes |0\rangle) = \langle k_E | U | 0_E \rangle$

$\rho_{in} = |\psi\rangle\langle\psi|$ where $|\psi\rangle = a|0\rangle + b|1\rangle$

$U = SC$ where

$C = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_y(\theta)$

$S = I \otimes |0\rangle\langle 0| + U_{CNOT} \otimes |1\rangle\langle 1|$

$R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \cos\theta/2 - i\sin\theta/2 \, Y$

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Rightarrow iY = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$iY \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}$

$$|\psi\rangle \otimes |0\rangle = \left(a|0\rangle + b|1\rangle\right) \otimes |0\rangle$$

$$= a|0\rangle \otimes |0\rangle + b|1\rangle \otimes |0\rangle$$

$$C|\psi\rangle \otimes |0\rangle = a|00\rangle + b|1\rangle \otimes R_y|0\rangle$$

$$= a|0\rangle \otimes |0\rangle + b|1\rangle \otimes \left[\cos\theta/_2 \, I - i\sin\theta/_2 \, Y\right]|0\rangle$$

$$= a|00\rangle + b|1\rangle \otimes \left(\cos\theta/_2 |0\rangle - \sin\theta/_2 \cdot -|1\rangle\right)$$

$$= a|00\rangle + b\cos\theta/_2 |10\rangle + b\sin\theta/_2 |11\rangle$$

$$U|\psi\rangle \otimes |0\rangle = SC|\psi\rangle \otimes |0\rangle$$

$$= \left(I \otimes |0\rangle\langle 0| + U_{CNOT} \otimes |1\rangle\langle 1|\right)\left(a|00\rangle + b\cos\theta/_2 |10\rangle + b\sin\theta/_2 |11\rangle\right)$$

$$= a|00\rangle + b\cos\theta/_2 |10\rangle + b\sin\theta/_2 |01\rangle$$

$$\left(I \otimes \langle 0|\right) U\left(|\psi\rangle \otimes |0\rangle\right) = a|0\rangle + b\cos\theta/_2 |1\rangle$$

$$\longrightarrow \quad E_0|\psi\rangle = E_0 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b\cos\theta/_2 \end{bmatrix}$$

$$\Rightarrow \quad E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta/_2 \end{bmatrix}$$

$$\left(I \otimes \langle 1|\right) U \left(|\psi\rangle \otimes |0\rangle\right) = b \sin\theta/_2 |0\rangle$$

$$\Rightarrow \quad E_1|\psi\rangle = E_1 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \sin\theta/_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad E_1 = \begin{bmatrix} 0 & \sin\theta/_2 \\ 0 & 0 \end{bmatrix}$$

This is called an amplitude damping channel as it reduces the amplitude of a $|1\rangle$ state.

$\sin\theta/_2 |11\rangle)$

* ρ_{in} is the quantum state being transmitted through the amplitude damping channel; and the environmental qubit is initially $|0\rangle$. Based on the contributions of $|1\rangle$ in the superposition $|\psi\rangle = a|0\rangle + b|1\rangle$ the environment qubit gets rotated about the Y-axis.

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