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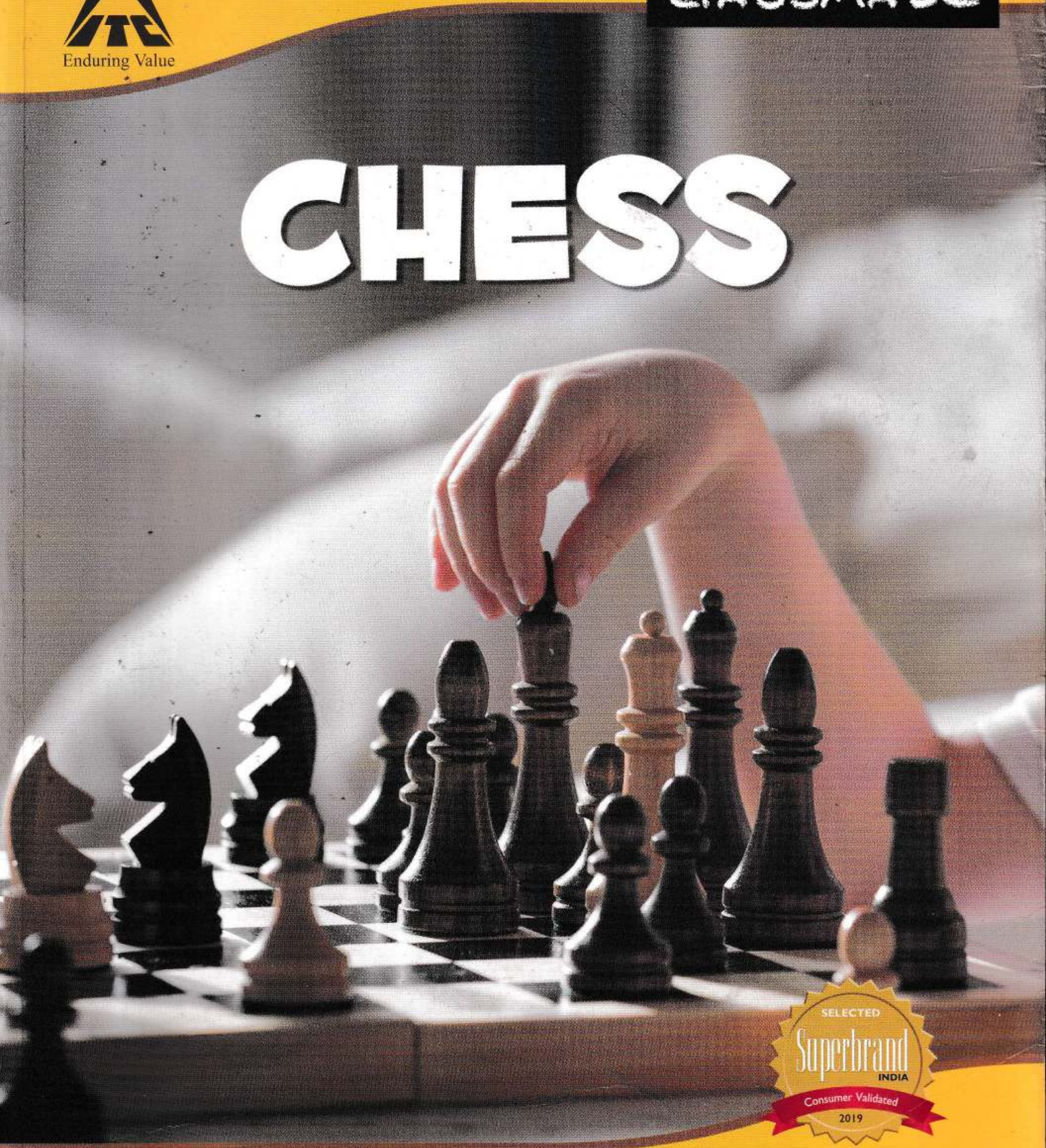

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		– Nielsen & Chuang		

QUANTUM INFORMATION

8

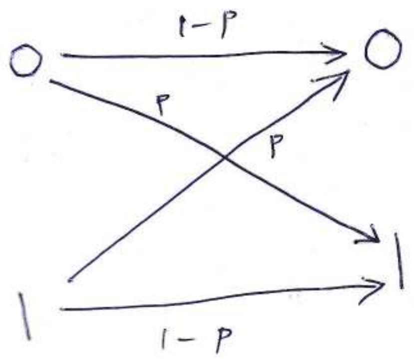
QUANTUM NOISE & QUANTUM OPERATIONS

Classical noise & Markov processes

Imagine,

a bit being stored on a hard disk drive attached to an ordinary classical computer. The bit starts out in the state 0 or 1, but after a long time it becomes likely that stray magnetic fields will cause the bit to become scrambled, possibly flipping its state.

We can model this by a probability p for the bit to flip, and a probability $1-p$ for the bit to remain the same.



* After a long time a bit on a hard disk drive may flip with probability p.

Suppose,

P_0 and P_1 are the initial probabilities that the bit is in the states 0 and 1, respectively. Let q_0 and q_1 be the corresp. probabilities after the noise has occurred.

Let X be the initial state of the bit, and Y be the final state of the bit.

Law of total probability

$$P(Y=y) = \sum_x P(Y=y, X=x)$$
$$= \sum_x P(Y=y / X=x) \, P(X=x)$$

The conditional probabilities $P(Y=y | X=x)$ are called transition probabilities, since they summarize the changes that may occur in the system.

Writing these equations out explicitly for the bit on a hard disk,

$$q_0 = (1-p)\, p_0 + p\, p_1$$
$$q_1 = p\, p_0 + (1-p)\, p_1$$

$$\Longleftrightarrow \quad \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$$

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Slightly more complicated example of noice in a classical system

Imagine,

we are trying to Build a classical circuit to perform some computational task and unfortunately we have been given faulty components to build the circuit.

The circuit consists of a single input bit X, to which are applied 2 consecutive (faulty) NOT gates, producing an intermediate bit Y, and a final bit Z.

It seems reasonable to assume that whether the 2nd NOT gate works correctly is independent of whether the 1st NOT gate worked correctly.

This assumption that the consecutive noise processes acts independently is a physically reasonable assumption to make in many situations. It results in a stochastic process $X \rightarrow Y \rightarrow Z$ of a special type known as a Markov process.

Physically, this assumption of Markovicity corresponds to assuming that the environment causing the noise in the 1st NOT gate acts independently of the environment causing the noise in the 2nd NOT gate.

Noise in classical systems can be described using the theory of stochastic processes. Often in the analysis of multistage processes it is a good assumption to use Markov processes.

For a single stage process the o/p probabilities $\vec{q}$ are related to the i/p probabilities $\vec{p}$ by the equation,

$$\vec{q} = E\vec{p}$$

where E is a matrix of transition probabilities which we shall refer to as the evolution matrix.

$\therefore$

The final state of the system is linearly related to the starting state.

This feature of linearity is echoed in the description of quantum noise, with density matrices replacing probability distributions.

What properties must the evolution matrix E possess ?

$\vec{p}$ is a valid probability distribution $\Longrightarrow$ $E\vec{p}$ must also be a valid probability distribution.

This condition turns out to be equivalent to 2 conditions on E.

1. All the entries of E must be non-negative (positivity requirement).

   - if the entries of E weren't all non-negative then it would be possible to obtain -ve probabilities in $E\vec{p}$.

② All the columns of E must sum to one.
(completeness requirement).

$$E\vec{p} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

$$= \vec{e}_1 p_1 + \vec{e}_2 p_2 + \cdots + \vec{e}_n p_n$$

ILA②

$$= \begin{bmatrix} e_{11}p_1 + e_{12}p_2 + \cdots + e_{1n}p_n \\ e_{21}p_1 + e_{22}p_2 + \cdots + e_{2n}p_n \\ \vdots \\ e_{n1}p_1 + e_{n2}p_2 + \cdots + e_{nn}p_n \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = q$$

$$q_1 + q_2 + \cdots + q_n = 1$$

$$\Longrightarrow p_1(e_{11} + e_{21} + \cdots + e_{n1}) + p_2(e_{12} + e_{22} + \cdots + e_{n2}) + \cdots$$
$$\cdots + p_n(e_{1n} + e_{2n} + \cdots + e_{nn}) = 1$$

$p_1 + p_2 + \cdots + p_n = 1$ and $e_{ij} \geq 0$ for $i,j \in \{1, 2, \ldots, n\}$

$$\Longrightarrow e_{1i} + e_{2i} + \cdots + e_{ni} = 1$$

For example;

the 1st column didn't sum to one.

Letting $\vec{P}$ contain a one in the 1st entry and zeros everywhere else.

$\Rightarrow E\vec{P}$ would not be a valid probability distribution in this case.

The key features of classical noise are:

There is a linear relationship b/w i/p and o/p probabilities, described by a transition matrix with non-negative entries (positivity) and columns summing to one (completeness). Classical noise processes involving multiple stages are described as Markov processes, provided the noise is caused by independent environments.

Quantum Operations

The quantum operations formalism is a general tool for describing the evolution of quantum systems in a wide variety of circumstances, including stochastic changes to quantum states, much as Markov processes describe stochastic changes to classical states.

Just as a classical state is described by a vector of probabilities, we shall describe quantum states in terms of the density operator (density matrix) ρ.

Similar to how classical states transform as described by $\vec{q} = E\vec{p}$, quantum states transform as,

$$\boxed{\rho' = \mathcal{E}(\rho)}$$

The map $\mathcal{E}$ in this equation is a quantum operation.

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Two simple examples of quantum operations are: unitary transformations

$$\mathcal{E}(\rho) = U\rho U^{\dagger}$$

Measurements

$$\mathcal{E}_m(\rho) = M_m \rho M_m^{\dagger}$$

The quantum operation captures the dynamical change to a state which occurs as a result of some physical process.

ρ is the initial state before the process, and $\mathcal{E}(\rho)$ is the final state after the process occurs, possibly upto some normalization factor.

* Pure states evolve under unitary transforms as, $|\psi\rangle \longrightarrow U|\psi\rangle$

Equivalently,

$$\rho \longrightarrow \mathcal{E}(\rho) \equiv U\rho U^{\dagger} \quad \text{for } \rho = |\psi\rangle\langle\psi|$$

* A quantum measurement with outcomes labeled by m is described by a set of measurement operators M_m such that $\sum_m M_m^{\dagger} M_m = I$.

Let the state of the system immediately before the measurement be ρ.
For $\mathcal{E}_m(\rho) = M_m \rho M_m^{\dagger}$, the state of the system immediately after the measurement is,

QC(2)

$$\boxed{\rho_m = \frac{\mathcal{E}_m(\rho)}{tr(\mathcal{E}_m(\rho))} = \frac{M_m \rho M_m^{\dagger}}{tr(M_m \rho M_m^{\dagger})}}$$

The probability of obtaining this measurement result is,

$$\boxed{p(m) = tr(\mathcal{E}(\rho)) = tr(M_m \rho M_m^{\dagger})}$$

We'll develop a general theory of quantum operations incorporating unitary evolution, measurement, and even more general process!

We shall develop 3 separate ways of understanding quantum operations, all of which turn out to be equivalent.

System coupled to environment

- based on the idea of studying dynamics as the result of interaction b/w a system and an environment, much as classical noise was described; This method is concrete & easy to relate to the real world. Unfortunately, it suffers from the drawback of not being mathematically convenient.

Operator-sum representation

- equivalent to the 1st, overcomes the mathematical inconvenience by providing a powerful mathematical representation for quantum operations. This method is rather abstract, but is very useful for calculations and theoretical work.

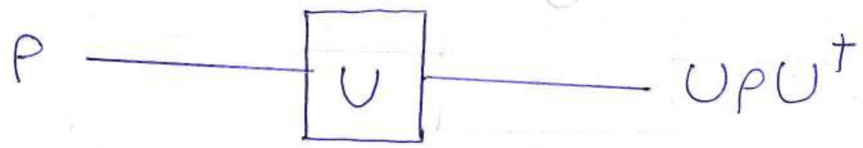
Physically motivated axioms

—

(A) Environment and quantum operations

The dynamics of a closed quantum system are described by a unitary transform.

Think of the unitary transform as a box into which the i/p state enters and from which the o/p exits.



* Model of closed quantum system.

Assume (for now) that the system-environment i/p state is a product state, $\rho \otimes \rho_{env}$.

After the box's transformation U the system no longer interacts with the environment, and thus we perform a partial trace over the environment to obtain the reduced state of the system alone:

$$\boxed{\mathcal{E}(\rho) = tr_{env}\left[U(\rho \otimes \rho_{env})U^{\dagger}\right]} \quad \text{— (8.6)}$$

If U does not involve any interaction with the environment, then $\mathcal{E}(\rho) = \tilde{U}\rho\tilde{U}^{\dagger}$, where $\tilde{U}$ is the part of U which acts on the system alone.

Assumption :

We assume that the system and the environment start in a product state. In general, this is not true. Quantum systems interact constantly with their environments, building up correlations.

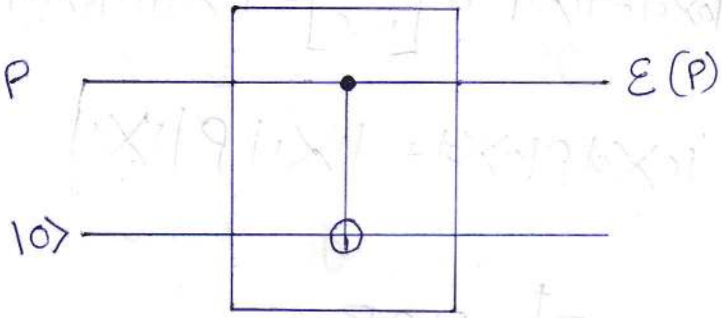
Another issue:

How can U be specified if the environment has nearly infinite degrees of freedom ?

It turns out, very interestingly, that in order for this model to properly describe any possible transformation $\rho \rightarrow \mathcal{E}(\rho)$, if the principal system has a Hilbert space of d-dimensions, then it suffices to model the environment as being in a Hilbert space of no more than d^2 dimensions.

It also turns out not to be necessary for the environment to start out in a mixed state; a pure state will do.

Ex:- $\mathcal{E}(\rho) = tr_{env}\left[U(\rho \otimes \rho_{env}) U^{\dagger}\right]$



* CNOT gate as an elementary example of a single qubit quantum operation.

Consider this quantum circuit, in which U is a controlled-NOT gate, with the principal system the control qubit, and the environment initially in the state $\rho_{env} = |0\rangle\langle 0|$ as the target qubit.

$$U_{CN} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$$

$$\mathcal{E}(\rho) = tr_{env}\left[\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X\right)\left(\rho \otimes |0\rangle\langle 0|\right)\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X^{\dagger}\right)\right]$$

$$= tr_{env}\left[|0\rangle\langle 0| \rho |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \rho |1\rangle\langle 1| \otimes X |0\rangle\langle 0| X^{\dagger} + |0\rangle\langle 0| \rho |1\rangle\langle 1| \otimes X^{\dagger} + |1\rangle\langle 1| \rho |0\rangle\langle 0| \otimes X\right]$$

$$= |0\rangle\langle 0| \rho |0\rangle\langle 0| \cdot tr(|0\rangle\langle 0|) + |1\rangle\langle 1| \rho |1\rangle\langle 1| \cdot tr(X |0\rangle\langle 0| X^{\dagger})$$
$$+ |0\rangle\langle 0| \rho |1\rangle\langle 1| \cdot tr(X^{\dagger}) + |1\rangle\langle 1| \rho |0\rangle\langle 0| \cdot tr(X)$$

$$= |0\rangle\langle 0|\rho|0\rangle\langle 0| \cdot \langle 0|0\rangle + |1\rangle\langle 1|\rho|1\rangle\langle 1| \cdot \langle 0|X^{\dagger}X|0\rangle$$

$$+ |0\rangle\langle 0|\rho|1\rangle\langle 1| \; tr\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + |1\rangle\langle 1|\rho|0\rangle\langle 0| \; tr\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1$$

where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are projection operators.

(effect of the CNOT gate on the principal system)

Intuitively, this dynamics occurs because the environment stays in the $|0\rangle$ state only when the system is $|0\rangle$; otherwise the environment is flipped to the state $|1\rangle$.

Q. Stack
8/9/2022

Intuition about CNOT gate is that the control qubit remains the same and only the target qubit flips ~~iff~~ the control qubit is $|1\rangle$?

This is untrue for quantum systems!

For the CNOT, if the i/p state is in the computational basis then it's true. But, generally a multi-qubit gate can't leave any i/p qubit the same for all i/ps.

Let $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$ and $|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)$.

$$U_{CNOT}|+\rangle\otimes|-\rangle = U_{CNOT}\left(\frac{(|0\rangle+|1\rangle)(|0\rangle-|1\rangle)}{2}\right)$$

$$= U_{CNOT}\left(\frac{|0\rangle|0\rangle-|0\rangle|1\rangle+|1\rangle|0\rangle-|1\rangle|1\rangle}{2}\right)$$

$$\left[\begin{array}{l} X|0\rangle=|1\rangle \\ X|1\rangle=|0\rangle \end{array}\right.$$

$$U_{CNOT} = |0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X$$

$$= \frac{|0\rangle\otimes|0\rangle - |0\rangle\otimes|1\rangle + |1\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle}{2}$$

$$= \frac{|0\rangle\otimes|0\rangle - |0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle}{2}$$

$$= |-\rangle\otimes|-\rangle$$

$$U_{CNOT}|+\rangle \otimes |-\rangle = |-\rangle \otimes |-\rangle$$

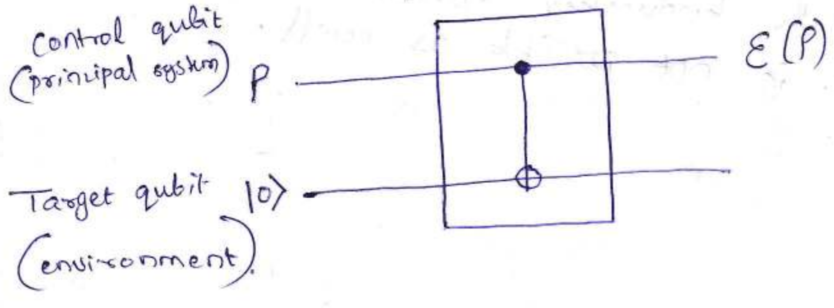
$$U_{CNOT}|-\rangle \otimes |-\rangle = |+\rangle \otimes |-\rangle$$

$\Longrightarrow$ In this basis, the CNOT gate does nothing to the target qubit, but flips the control qubit.

We have given a principal system in initial state ρ and the environment being simplified to a system in an initial state $|0\rangle$.

This is a method to describe an open quantum system as a closed quantum system consisted of 2 sub-systems - the principal system & the environment.

The interaction b/w the principal system and the environment occurs only through the unitary operator U, which in the example is defined as a CNOT gate with the principal system as the control qubit and the environment as the target qubit.



CNOT gates can create correlation b/w qubits known as entanglement.

If $\rho = |0\rangle\langle 0|$ or $\rho = |1\rangle\langle 1|$ then there is no superposition in the control qubit and no entanglement is being created.

But, if the principal system is in some superposition state then the CNOT entangles the principal system and the environment, and now the overall state of the system is not separable.

i.e., One can't describe the state of the principal system alone nor the environment alone, and thus also describing the overall state with kronecker product of the two subsystems is not possible as well.

Ex:- $\rho = |+\rangle\langle+|$ and $\rho_{env} = |0\rangle\langle 0|$

$$U_{CNOT}|+\rangle\otimes|0\rangle = U_{CNOT}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes|0\rangle$$

$$= \left(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X\right)\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes|0\rangle$$

$$= \left(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X\right)\left(\frac{|0\rangle\otimes|0\rangle+|1\rangle\otimes|0\rangle}{\sqrt{2}}\right)$$

$$= \frac{|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle}{\sqrt{2}} \qquad \left[X|0\rangle = |1\rangle\right.$$

which the 1st Bell state (maximally entangled).

What we can do is obtain some information on the state of a subsystem by taking partial trace in order to form the reduced density matrix of a subsystem. The equation, $\mathcal{E}(\rho) = tr_{env}\left[U(\rho\otimes\rho_{env})U^{\dagger}\right]$ is exactly that.

i.e., we are obtaining the reduced density matrix of the principal system by taking the partial trace of the environment from the density matrix of the overall system after going through U (which is $U(\rho\otimes\rho_{env})U^{\dagger}$).

$\therefore$

If the control qubit is in superposition before the CNOT, then the state of the system after the CNOT will not be separable, and thus no other way to describe the state of the control qubit (or the target qubit) alone, but to use this method.

Note: $U_{CNOT}|+\rangle\otimes|-\rangle = |-\rangle\otimes|-\rangle$

$U_{CNOT}|-\rangle\otimes|-\rangle = |+\rangle\otimes|-\rangle$

It's a private case of CNOT that doesn't create entanglement. It happens because $|-\rangle$ is an eigenstate of the X gate. So by applying $U_{CNOT}|+\rangle\otimes|-\rangle$ the eigenvalue -1 is kicked back (it is the well known phase kick back phenomenon) towards the control qubit and the resulted state is the product state (not entangled state) $|-\rangle\otimes|-\rangle$.

We have described quantum operations as arising from the interaction of a principal system with an environment; however it is convenient to generalize the definition somewhat to allow different i/p and o/p spaces.

For example,

imagine that a single qubit, which we label 'A', is prepared in an unknown state ρ. A 3-level quantum system ('qutrit') labelled 'B' is prepared in some standard state $|0\rangle$ and then interacts with system 'A' via a unitary interaction U, causing the joint system to evolve into the state $U(\rho \otimes |0\rangle\langle 0|)U^\dagger$. We then discard system 'A', leaving system 'B' in some final state ρ'.

The quantum operation $\mathcal{E}$ describing this process is,

$$\mathcal{E}(\rho) = \rho' = tr_A\left(U(\rho \otimes |0\rangle\langle 0|)U^\dagger\right)$$

Here: $\mathcal{E}$ maps density operators of the i/p system, A, to density operators of the o/p system, B

Most of our discussion of quantum operations below is concerned with quantum operations on some system A, that is, they map density operators of system A to density operators of system A.

However,
it is occasionally useful in applications to allow a more general definition. Such a definition is provided by defining quantum operations as the class of maps which arise as a result of the following process:

Some initial system is prepared in an unknown quantum state ρ, and then brought into contact with other systems prepared in standard Basis, allowed to interact according to some unitary interaction, and then some part of the combined system is discarded, leaving just a final system in some state ρ'. The quantum operation $\mathcal{E}$ defining this process simply maps ρ to ρ'.

(B) Operator - sum representation

Quantum operations can be represented in an elegant form known as the operator - sum representation, which is essentially a re-statement of equation (8.6) explicitly in terms of operators on the principal system's Hilbert space alone.

There is no loss of generality is assuming that the environment starts in a pure state, since if it starts in a mixed state we are free to introduce an extra system purifying the environment. Although this extra system is 'fictitious', it makes no difference to the dynamics experienced by the principal system, and thus can be used as an intermediate step in calculations.

Let $|e_k\rangle$ be an orthonormal basis for the (finite dimensional) state space of the environment and let $\rho_{env} = |e_0\rangle\langle e_0|$ be the initial state of the environment.

Equation (8.6) can thus be rewritten as,

$$\mathcal{E}(\rho) = tr_{env}\left[U(\rho \otimes \rho_{env})U^\dagger\right]$$

$$= \sum_k \left(I \otimes \langle e_k|\right)\left(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right)\left(I \otimes |e_k\rangle\right)$$

$$\rho \otimes |e_0\rangle\langle e_0| = \underbrace{(\rho \otimes I)(I \otimes |e_0\rangle)}(I \otimes \langle e_0|)$$

$$= \left((\rho I) \otimes (I|e_0\rangle)\right)(I \otimes \langle e_0|)$$

$$= \left((I\rho) \otimes (|e_0\rangle \cdot 1)\right)(I \otimes \langle e_0|)$$

$$= (I \otimes |e_0\rangle)(\rho \otimes 1)(I \otimes \langle e_0|)$$

$$= (I \otimes |e_0\rangle)\rho(I \otimes \langle e_0|)$$

Considering the k^{th} term in the partial trace sum,

$$\mathcal{E}(\rho)=\sum_k I\otimes\langle e_k|\left(U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger\right) I\otimes|e_k\rangle =$$

$$= \sum_k \underbrace{(I\otimes\langle e_k|)U(I\otimes|e_0\rangle)}_{E_k}\,\rho\,\underbrace{(I\otimes\langle e_0|)U^\dagger(I\otimes|e_k\rangle)}_{E_k^\dagger}$$

$$= \sum_k E_k \rho E_k$$

Operator-sum representation of $\mathcal{E}$

$$\mathcal{E}(\rho) = \sum_k I \otimes \langle e_k | \left(U(\rho \otimes |e_0\rangle\langle e_0|) U^\dagger \right) I \otimes |e_k\rangle$$

$$= \sum_k \langle e_k | U \left[\rho \otimes |e_0\rangle\langle e_0| \right] U^\dagger | e_k \rangle$$

$$= \sum_k E_k \rho E_k^\dagger$$

— Eq^n. (8.10)

where,

$$E_k = (I \otimes \langle e_k|) U (I \otimes |e_0\rangle) = \langle e_k | U | e_0 \rangle$$

is an operator on the state space of the principal system.

The operators $\{E_k\}$ are known as operation elements for the quantum operation $\mathcal{E}$.

The operation elements satisfy an important constraint known as the completeness relation, analogous to the completeness relation for evolution matrices in the description of classical noise.

In the classical case, the completeness relation arose from the requirement that probability distributions be normalized to one.
In the quantum case the completeness relation arises from the analogous requirement that the trace of $\mathcal{E}(\rho)$ be equal to one,

$$
\begin{aligned}
1 &= tr\left(\mathcal{E}(\rho)\right) \\
&= tr\left(\sum_k E_k \rho E_k^{\dagger}\right) \\
&= tr\left(\sum_k E_k^{\dagger} E_k \rho\right) \\
&= tr\left(\left(\sum_k E_k^{\dagger} E_k\right)\rho\right)
\end{aligned}
$$

This relationship is true for all ρ, it follows that we must have $\boxed{\sum_k E_k^{\dagger} E_k = I}$

Proof

A matrix 'A' is positive semidefinite if and only if $v^{\dagger}Av \geq 0$ for all $v \neq 0$.

$\therefore$

For $A = E_k^{\dagger} E_k$,

$$v^{\dagger}Av = v^{\dagger}E_k^{\dagger}E_k v = \|E_k v\|^2 \geq 0$$

ILA(8) $\Longrightarrow E_k^{\dagger}E_k$ is positive semi-definite.

If A and B are positive semi definite so is $A+B$.

Proof

$v^{\dagger}Av \geq 0$ and $v^{\dagger}Bv \geq 0 \Longrightarrow v^{\dagger}(A+B)v \geq 0$

$\Longrightarrow A+B$ is positive semidefinite

$$\Longrightarrow X = \sum_k E_k^{\dagger}E_k \text{ is positive semi-definite.}$$

X, P are positive semidefinite.

$$\Rightarrow \quad X = UDU^{\dagger} = \sum_i \lambda_i |u_i\rangle\langle u_i|$$

Stark 1/9/2022

where λ_i are the non-negative eigenvalues of X, and $|u_i\rangle$ are the corresponding orthonormal eigenvectors.

Taking $P = |u_i\rangle\langle u_i|$,

$$1 = \mathrm{tr}\left[XP\right] = \mathrm{tr}\left[X|u_i\rangle\langle u_i|\right] = \lambda_i$$

Repeating for each eigenvector $\Longrightarrow \lambda_i = 1$ for all i

$$\therefore \quad X = UIU^{\dagger} = UU^{\dagger} = I$$

$$\Longrightarrow \sum_k E_k^{\dagger} E_k = I$$

→ $\sum_k E_k^\dagger E_k = I$ is satisfied by quantum operations which are trace-preserving. $\left(tr(\mathcal{E}(\rho)) = 1 \right)$

There are also non-trace-preserving quantum operations, for which $\sum_k E_k^\dagger E_k \leq I$, but they describe processes in which extra information about what occurred in the process is obtained by measurement.

The operator-sum representation is important because it gives us an intrinsic means of characterizing the dynamics of the principal system.

The operator-sum representation describes the dynamics of the principal system without having to explicitly consider properties of the environment; all that we need to know is bundled up into the operators E_k, which act on the principal system alone. This simplifies calculations and often provides considerable theoretical insight.

Furthermore, many different environmental interactions may give rise to the same dynamics on the principal system. If it is only the dynamics of the principal system which are of interest then it makes sense to choose a representation of the dynamics which does not include unimportant information about other systems.

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Ex: 8.3 Our derivation of the operator-sum representation implicitly assumed that the i/p and o/p spaces for the operation were the same.

Suppose a composite system AB initially in an unknown quantum state ρ is brought into contact with a composite system CD initially in some standard state $|0\rangle$, and the 2 systems interact according to a unitary interaction U. After the interaction we discard systems A and D, leaving a state ρ' of system BC. Show that the map $\mathcal{E}(\rho) = \rho'$ satisfies

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^{\dagger}$$

for some set of linear operators E_k from the state space of system AB to the state space of system BC, and such that $\sum_k E_k^{\dagger} E_k = 1$.

Ans: The system of interest is a composite system AB in the state ρ, and it gets in contact with an environment that is also composite CD initially in the state $|0\rangle$. The whole system evolves accordingly to U and then we discard A and D.

The overall state of the system after the interaction is $U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}$.

Let $|a_m\rangle, |b_n\rangle, |c_p\rangle, |d_q\rangle$ be orthonormal basis for the state space of the systems A, B, C, D, respectively.

Discarding systems A and D,
i.e., tracing out system A first and then tracing out ~~the~~ system D obtains.

$$\mathcal{E}(\rho) = tr_D\left[tr_A\left[U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}\right]\right]$$

$$= tr_D\left[\sum_m (\langle a_m| \otimes I \otimes I \otimes I)\left[U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}\right] \times (|a_m\rangle \otimes I \otimes I \otimes I)\right]$$

$$=\sum_q \left(I\otimes I\otimes I\otimes \langle d_q|\right)\left[\sum_m \left(\langle a_m|\otimes I\otimes I\otimes I\right)\left[U(\rho\otimes|0\rangle\langle 0|)U^\dagger\right] \times\left(|a_m\rangle\otimes I\otimes I\otimes I\right)\right]$$

$$\times\left(I\otimes I\otimes I\otimes |d_q\rangle\right).$$

$$=\sum_{m,q}\left(\langle a_m|\otimes I\otimes I\otimes \langle d_q|\right)\left[U(\rho\otimes|0\rangle\langle 0|)U^\dagger\right]\left(|a_m\rangle\otimes I\otimes I\otimes |d_q\rangle\right)$$

Expanding $\rho\otimes|0\rangle\langle 0|$ into the products,

$$\rho\otimes|0\rangle\langle 0| = \rho_{AB}\otimes|0_{CD}\rangle\langle 0_{CD}|$$

$$=\left(\rho_{AB}\otimes I_{CD}\right)\left(I_{AB}\otimes|0_{CD}\rangle\right)\left(I_{AB}\otimes\langle 0_{CD}|\right)$$

$$=\left(\rho_{AB}I_{AB}\otimes I_{CD}|0_{CD}\rangle\right)\left(I_{AB}\otimes\langle 0_{CD}|\right)$$

$$=\left(I_{AB}\rho_{AB}\otimes|0_{CD}\rangle\cdot 1\right)\left(I_{AB}\otimes\langle 0_{CD}|\right)$$

$$=\left(I_{AB}\otimes|0_{CD}\rangle\right)\left(\rho_{AB}\otimes 1\right)\left(I_{AB}\otimes\langle 0_{CD}|\right)$$

$$=\left(I_{AB}\otimes|0_{CD}\rangle\right)\rho_{AB}\left(I_{AB}\otimes\langle 0_{CD}|\right)$$

$$\rho \otimes |0\rangle\langle 0| = \left(I_{AB} \otimes |0_{CD}\rangle\right) \rho \left(I_{AB} \otimes \langle 0_{CD}|\right)$$

Substituting back into the expression for $\mathcal{E}(\rho)$,

$$\mathcal{E}(\rho) = \sum_{m,q} \left(\langle a_m| \otimes I \otimes I \otimes \langle d_q|\right) U \left(I_{AB} \otimes |0_{CD}\rangle\right) \rho \left(I_{AB} \otimes \langle 0_{CD}|\right) U^{\dagger} \left(|a_m\rangle \otimes I \otimes I \otimes |d_q\rangle\right)$$

$$= \sum_{m,q} E_{m,q} \rho E_{m,q}^{\dagger}$$

where, $E_{m,q} = \left(\langle a_m| \otimes I \otimes I \otimes \langle d_q|\right) U \left(I_{AB} \otimes |0_{CD}\rangle\right)$

$\boxed{\text{Part 1:}}$ $E_{m,q} = (\langle a_m| \otimes I \otimes I \otimes \langle d_q|)\, U\, (I_{AB} \otimes |0_{CD}\rangle)$ is a linear operator from the state space of AB to the state space of BC.

Proof

QC. Stark 13/9/2022

If $|a_m\rangle, |b_n\rangle, |c_p\rangle, |d_q\rangle$ be orthonormal basis for the state space of the systems $\mathbb{C}^i, \mathbb{C}^j, \mathbb{C}^k, \mathbb{C}^l$ respectively, then

$|a_m\rangle \otimes |b_n\rangle \otimes |c_p\rangle \otimes |d_q\rangle$ is an orthonormal basis for the combined system $\mathbb{C}^i \otimes \mathbb{C}^j \otimes \mathbb{C}^k \otimes \mathbb{C}^l = \mathbb{C}^{ijkl}$.

QC.2

Lemma: If V and W are vector spaces with basis $|v_i\rangle$ and $|w_j\rangle$ respectively. Then $|v_i\rangle \otimes_{outer} |w_j\rangle = |v_i\rangle\langle w_j|$ form a basis for the vector space $V \otimes_{outer} W$.

Therefore,

$$\left(|a_m\rangle \otimes |b_n\rangle \otimes |c_p\rangle \otimes |d_q\rangle\right)\left(\langle a_{m'}| \otimes \langle b_{n'}| \otimes \langle c_{p'}| \otimes \langle d_{q'}|\right) =$$

$$= |a_m\rangle\langle a_{m'}| \otimes |b_n\rangle\langle b_{n'}| \otimes |c_p\rangle\langle c_{p'}| \otimes |d_q\rangle\langle d_{q'}|$$

form a basis for $\mathbb{C}^{ijkl} \otimes_{outer} \mathbb{C}^{ijkl}$.

$\therefore$ $U \in \mathbb{C}^{ijkl} \otimes_{outer} \mathbb{C}^{ijkl}$ can be written as a linear combination of this basis

i.e.,

$$U = \sum_{m',n',p',q'} \upsilon_{m',n',p',q'} \left(|a_{m'}\rangle \otimes |b_{n'}\rangle \otimes |c_{p'}\rangle \otimes |d_{q'}\rangle\right)\left(\langle a_{m''}| \otimes \langle b_{n''}| \otimes \langle c_{p''}| \otimes \langle d_{q''}|\right)$$

$$= \sum_{m',n',p',q'} \upsilon_{m',n',p',q'} |a_{m'}\rangle\langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle\langle c_{p''}| \otimes |d_{q'}\rangle\langle d_{q''}|$$

And $|0_{CD}\rangle$ being a standard state of the system CD can be represented in terms of the orthonormal basis vectors as,

$$|0_{CD}\rangle = \sum_{p',q'} \eta_{p',q'} |c_{p'}\rangle \otimes |d_{q'}\rangle$$

Substituting back into the expression for $E_{m,q}$ obtains,

$$E_{m,q} = \left(\langle a_m| \otimes I \otimes I \otimes \langle d_q|\right) U \left(I_{AB} \otimes |\phi_{CD}\rangle\right)$$

$$= \left(\langle a_m| \otimes I_B \otimes I_C \otimes \langle d_q|\right) U \sum_{p''',q'''} \eta_{p''',q'''} \left(I_A \otimes I_B \otimes |c_{p'''}\rangle \otimes |d_{q'''}\rangle\right)$$

$$= \left(\langle a_m| \otimes I_B \otimes I_C \otimes \langle d_q|\right) \left[\sum_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} \upsilon_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} |a_{m'}\rangle\langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle\langle c_{p''}| \otimes |d_{q'}\rangle\langle d_{q''}|\right]$$

$$\times \sum_{p''',q'''} \eta_{p''',q'''} \left(I_A \otimes I_B \otimes |c_{p'''}\rangle \otimes |d_{q'''}\rangle\right)$$

$$= \left(\langle a_m| \otimes I_B \otimes I_C \otimes \langle d_q|\right) \left[\sum_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} \sum_{p''',q'''} \eta_{p''',q'''} \upsilon_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} |a_{m'}\rangle\langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle\langle c_{p''}| \otimes |d_{q'}\rangle\langle d_{q''}|\right]$$

$$\times \left(I_A \otimes I_B \otimes |c_{p'''}\rangle \otimes |d_{q'''}\rangle\right)$$

$$= \sum_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} \sum_{p''',q'''} \eta_{p''',q'''} \upsilon_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} \langle a_m|a_{m'}\rangle\langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle\langle c_{p''}|c_{p'''}\rangle \otimes \langle d_q|d_{q'}\rangle\langle d_{q''}|d_{q'''}\rangle$$

$$= \sum_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} \sum_{p''',q'''} \eta_{p''',q'''} \upsilon_{\substack{m',n',p',q' \\ m'',n'',p'',q''}} \delta_{m,m'} \langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle \delta_{p'',p'''} \otimes \delta_{q,q'} \delta_{q'',q'''}$$

$$= \sum_{\substack{n',p',q \\ m'',n'',p'',q''}} \eta_{p'',q''}\, \upsilon_{\substack{m,n',p',q \\ m'',n'',p''}} \langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle \otimes 1$$

$$= \sum_{\substack{n',p',q \\ m'',n'',p'',q''}} \eta_{p',q}\, \upsilon_{m,n',p',q} \left(1\cdot \langle a_{m''}| \otimes |b_{n'}\rangle\langle b_{n''}| \otimes |c_{p'}\rangle \cdot 1 \otimes 1\cdot 1\right)$$

$$= \sum_{\substack{n',p',q \\ m'',n'',p'',q''}} \eta_{p',q}\, \upsilon_{m,n',p',q} \left(1 \otimes |b_{n'}\rangle \otimes |c_{p'}\rangle \otimes 1\right)\left(\langle a_{m''}| \otimes \langle b_{n''}| \otimes 1 \otimes 1\right)$$

$$= \sum_{\substack{n',p',q \\ m'',n'',p'',q''}} K_{m,n',p',q} \left(|b_{n'}\rangle \otimes |c_{p'}\rangle\right)\left(\langle a_{m''}| \otimes \langle b_{n''}|\right)$$

$\therefore$ $E_{m,q}$ is a sum of operators from the system AB to BC.

Part 2:

$$tr(\mathcal{E}(\rho)) = tr\left(\sum_{m,q} E_{m,q}\rho E^\dagger_{m,q}\right)$$

$$= tr\left(\sum_{m,q} E^\dagger_{m,q} E_{m,q}\rho\right)$$

$$= tr\left(\left(\sum_{m,q} E^\dagger_{m,q} E_{m,q}\right)\rho\right)$$

where ρ, $M = \sum_{m,q} E^\dagger_{m,q} E_{m,q}$ are positive semidefinite linear operators, and thus can be written as

$$M = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$$

where λ_j are the non-negative eigenvalues and $|\lambda_j\rangle$ are the corresponding eigenvectors of the positive semidefinite ~~linear~~ linear operator

$M = \sum_{m,q} E^\dagger_{m,q} E_{m,q}$.

Since $tr\left(\left(\sum_{m,q} E^{\dagger}_{m,q} E_{m,q}\right)\rho\right) = tr(M\rho) = 1$ is true for all ρ, let's consider $\rho = |\lambda_k\rangle\langle\lambda_k|$ then

$$1 = tr(M\rho) = tr\left(\left(\sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|\right)|\lambda_k\rangle\langle\lambda_k|\right) = \lambda_k$$

$$\Longrightarrow \lambda_k = 1$$

Repeating this for all k, we obtain that all the eigenvalues of $M = \sum_j E^{\dagger}_{m,q} E_{m,q}$ are 1,

and therefore,

$$M = \sum_{m,q} E^{\dagger}_{m,q} E_{m,q} = \sum_j |\lambda_j\rangle\langle\lambda_j| = I$$

Ex: 8.4 (Measurement) Suppose we have a single qubit principal system, interacting with a single qubit environment through the transform

$$U = P_0 \otimes I + P_1 \otimes X$$

where X is the Pauli matrix (acting on the environment), and $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$ are projectors (acting on the system). Give the quantum operation for this process, in the operator-sum representation, assuming the environment starts in the state $|0\rangle$.

Ans: $U = P_0 \otimes I + P_1 \otimes X = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$

$$\mathcal{E}(\rho) = \sum_k (I \otimes \langle e_k|)(U(\rho \otimes |e_0\rangle\langle e_0|) U^\dagger)(I \otimes |e_k\rangle)$$

$$= \sum_k E_k \rho E_k^\dagger$$

where, $E_k = (I \otimes \langle e_k|) U (I \otimes |e_0\rangle)$

$$E_0 = \left(I \otimes \langle 0|\right) U \left(I \otimes |0\rangle\right)$$

$$= \left(I \otimes \langle 0|\right)\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X\right)\left(I \otimes |0\rangle\right)$$

$$= |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \langle 0|X|0\rangle \qquad \left[\begin{array}{l} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{array}\right.$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| \otimes \langle 0|1\rangle$$

$$= |0\rangle\langle 0|$$

Similarly,

$$E_1 = \left(I \otimes \langle 1|\right)\left(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X\right)\left(I \otimes |0\rangle\right)$$

Check previous section

$$= |0\rangle\langle 0| \otimes \langle 1|0\rangle + |1\rangle\langle 1| \otimes \langle 1|X|0\rangle$$

$$= 0 + |1\rangle\langle 1| \otimes \langle 1|1\rangle = |1\rangle\langle 1|$$

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

$$= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

$$= P_0 \rho P_0 + P_1 \rho P_1$$

Ex. 8.5

Ans:

Ex. 8.5 (Spin flips)

Just as in Ex: 8.4, but now let

$$U = \frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X$$

Give the quantum operation for this process, in the operator-sum representation.

Ans: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$ $\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ $\begin{cases} Y|0\rangle = i|1\rangle \\ Y|1\rangle = -i|0\rangle \end{cases}$

$$E_0 = (I \otimes \langle 0|)\, U\, (I \otimes |0\rangle)$$

$$= (I \otimes \langle 0|)\left(\frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X\right)(I \otimes |0\rangle)$$

$$= \frac{X}{\sqrt{2}} \otimes \langle 0|0\rangle + \frac{Y}{\sqrt{2}} \otimes \langle 0|X|0\rangle$$

$$= \frac{X}{\sqrt{2}} \otimes 1 + \frac{Y}{\sqrt{2}} \otimes \langle 0|1\rangle$$

$$= \frac{X}{\sqrt{2}}$$

$$E_1 = (I \otimes \langle 1|)\left(\frac{X}{\sqrt{2}} \otimes I + \frac{Y}{\sqrt{2}} \otimes X\right)(I \otimes |0\rangle)$$

$$= \frac{X}{\sqrt{2}} \otimes \langle 1|0\rangle + \frac{Y}{\sqrt{2}} \otimes \langle 1|X|0\rangle$$

$$= 0 + \frac{Y}{\sqrt{2}} \otimes \langle 1|1\rangle$$

$$= \frac{Y}{\sqrt{2}}$$

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \frac{X}{\sqrt{2}} \rho \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}} \rho \frac{Y}{\sqrt{2}}$$

$$= \frac{1}{2} X \rho X + \frac{1}{2} Y \rho Y$$

Ex: 8.6 Composition of quantum operations

Physical interpretation of the operator-sum representation

Imagine that,

a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary transformation U has been applied.

The principle of implicit measurement $\Rightarrow$ Such a measurement affects only the state of the environment & does not change the state of the principal system.

Let ρ_k be the state of the principal system given that outcome k occurs;

$\therefore$

$$\rho_k \propto tr_E\left[|e_k\rangle\langle e_k|\left(U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger\right)|e_k\rangle\langle e_k|\right]$$

$$= \sum_j \langle e_j|e_k\rangle\langle e_k|\left[U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger\right]|e_k\rangle\langle e_k|e_j\rangle$$

$$= \langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger|e_k\rangle$$

$$= E_k\rho E_k^\dagger$$

$|e_k\rangle\langle e_k| = (I\otimes|e_k\rangle)(I\otimes\langle e_k|)$

$|e_j\rangle = I\otimes|e_j\rangle$

Normalizing ρ_k,

$$\rho_k = \frac{E_k\rho E_k^\dagger}{tr(E_k\rho E_k^\dagger)}$$

QC②

The probability of outcome k is given by,

$$p(k) = tr\left[|e_k\rangle\langle e_k|\left(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right)|e_k\rangle\langle e_k|\right]$$

$$= tr\left[\langle e_k|\left(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right)|e_k\rangle\right]$$

$$= tr\left(E_k \rho E_k^\dagger\right)$$

$\therefore$

$$\mathcal{E}(\rho) = \sum_k p(k)\,\rho_k$$

$$= \sum_k E_k \rho E_k^\dagger$$

This gives us a beautiful physical interpretation of what is going on in a quantum operation with operation elements $\{E_k\}$.

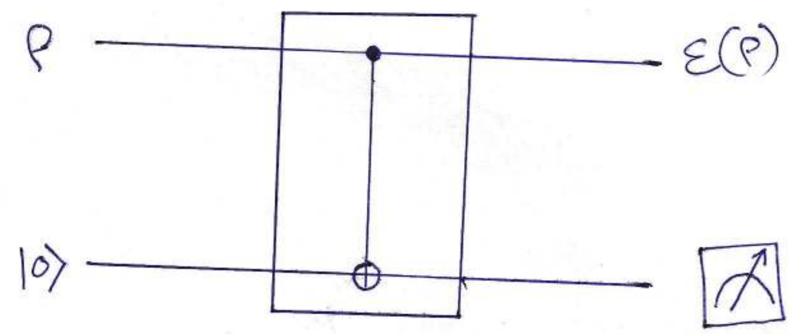
$\Rightarrow$ The action of the quantum operation $\mathcal{E}(\rho)$ is equivalent to taking the state ρ and randomly replacing it by $\dfrac{E_k \rho E_k^\dagger}{tr(E_k \rho E_k^\dagger)}$, with probability $tr(E_k \rho E_k^\dagger)$.

$$\begin{tikzcd}
 & \dfrac{E_1 \rho E_1^\dagger}{tr(E_1 \rho E_1^\dagger)} \\
\rho \arrow[ur, "p(1)"] \arrow[dr, "p(n)"'] & \vdots \\
 & \dfrac{E_n \rho E_n^\dagger}{tr(E_n \rho E_n^\dagger)}
\end{tikzcd}$$

$$\Rightarrow \mathcal{E}(\rho) = \sum_k p(k)\, \rho_k$$

Ex:-

* CNOT gate as an elementary model of single qubit measurement



Suppose we choose the states $|e_k\rangle = |0_E\rangle$ and $|1_E\rangle$

E : state of the environment

P : state of the principal system

This can be interpreted as doing a measurement in the computational basis of the environment qubit. Doing such a measurement does not change the state of the principal system.

Measurements & the operator-sum representation

Given a description of an open quantum system, how do we determine an operator-sum representation for its dynamics?

We found the answer.

Given the unitary system-environment transformation operation U, and basis of states $|e_k\rangle$ for the environment, the operation elements are:

$$E_k = \left(I \otimes \langle e_k|\right) U \left(I \otimes |e_0\rangle\right) = \langle e_k|U|e_0\rangle$$

It is possible to extend this result even further by allowing the possibility that a measurement is performed on the combined system-environment after the unitary interaction, allowing the acquisition of information about the quantum state.

This physical possibility is naturally connected to,

Non-trace-preserving quantum operations

- maps $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ such that $\sum_k E_k^\dagger E_k \le I$.

ILA②

* $A \leq B \Rightarrow B - A$ is positive semi-definite

ILA(2) $\rightarrow$

$$\sum_k E_k^\dagger E_k \leq I \Rightarrow I - \sum_k E_k^\dagger E_k \text{ is positive semi definite}$$

Non-trace-preserving quantum operation $\Rightarrow tr(\mathcal{E}(\rho)) < 1$

$$tr(\mathcal{E}(\rho)) = tr\left(\sum_k E_k \rho E_k^\dagger\right) = tr\left(\sum_k E_k^\dagger E_k \rho\right)$$

$$= tr\left[\left(\sum_k E_k^\dagger E_k\right)\rho\right] < 1 \quad \text{for all } \rho$$

$X = \sum_k E_k^\dagger E_k$ and ρ are both positive semidefinite

$$X = UDU^\dagger = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

where λ_i are the non-negative eigenvalues of X and $|\lambda_i\rangle$ are the corresponding eigenvectors.

Taking $\rho = |\lambda_j\rangle\langle\lambda_j|$,

$$\mathrm{tr}\left[X\,|\lambda_j\rangle\langle\lambda_j|\right] = \mathrm{tr}\left[\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|\lambda_j\rangle\langle\lambda_j|\right] \leq 1$$

$$\lambda_j < 1$$

Repeating for each j,

$$\Rightarrow \lambda_i \leq 1 \quad \text{for all } i$$

$$\Rightarrow 1 - \lambda_i > 0 \quad \text{for all } i$$

$$\Rightarrow I - X = I - \sum_k E_k^\dagger E_k \quad \text{is positive definite}$$

$$\Rightarrow X = \sum_k E_k^\dagger E_k < I$$

$\therefore$

$$\sum_k E_k^\dagger E_k < I \Longrightarrow \lambda_i < 1 \quad \text{for all } i$$

Suppose, the principal system is initially in a state ρ:

Q : the principal system

E : an environment system adjoined to Q

Suppose that Q and E are initially independent systems, and that E starts in some standard state, σ.

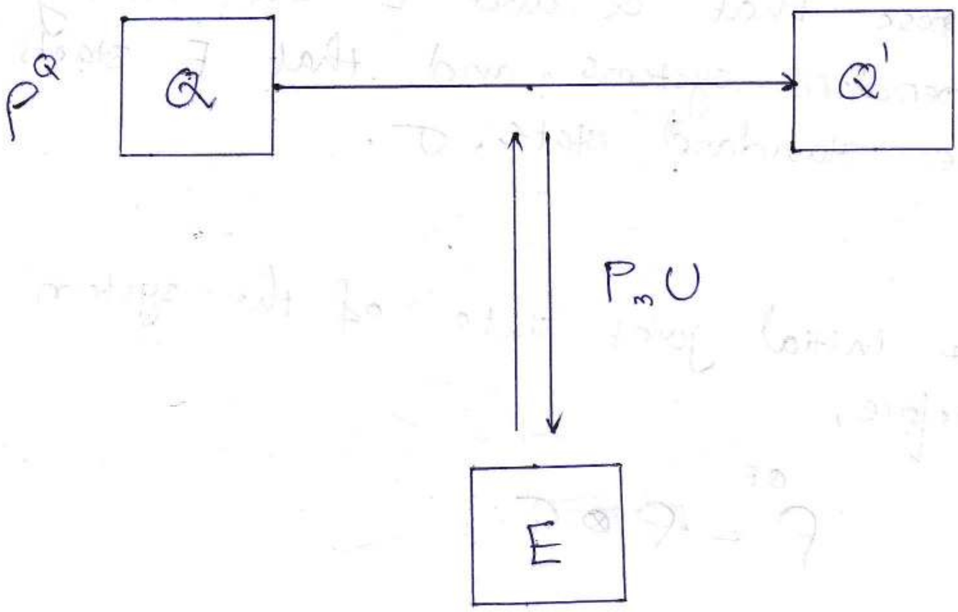
The initial joint state of the system is therefore,

$$\rho^{QE} = \rho \otimes \sigma$$

Suppose that the systems interact according to some unitary interaction U. After the unitary interaction a projective measurement is performed on the joint system, described by projectors P_m.

The case where no measurement is made corresponds to the special case where there is only a single measurement outcome, $m=0$, which corresponds to the projector $P_0 = I$.

* Environmental model for a quantum operation



Our aim is to determine the final state of Q as a function of the initial state, ρ.

The final state of QE is given by,

$$\frac{P_m U(\rho\otimes\sigma)U^\dagger P_m}{tr\left(P_m U(\rho\otimes\sigma)U^\dagger P_m\right)}$$

given that measurement outcome m occurred.

Tracing out E, the final state of Q alone is,

$$\frac{tr_E\left[P_m U(\rho\otimes\sigma)U^\dagger P_m\right]}{tr\left[P_m U(\rho\otimes\sigma)U^\dagger P_m\right]}$$

This representation of the final state involves the initial state σ of the environment, the interaction U and the measurement operators P_m.

Define a map,

$$\mathcal{E}_m(\rho) = \mathrm{tr}_E\left[P_m U(\rho \otimes \sigma)U^\dagger P_m\right]$$

$\therefore$ The final state of Q alone is, $\dfrac{\mathcal{E}_m(\rho)}{\mathrm{tr}(\mathcal{E}_m(\rho))}$

$\mathrm{tr}(\mathcal{E}_m(\rho))$: the probability of outcome m of the measurement occurring.

Let $\sigma = \sum_j q_j |j\rangle\langle j|$ be an ensemble decomposition for σ and introduce an orthonormal basis $|e_k\rangle$ for the system E

(Environment)

$$\mathcal{E}_m(\rho) = \sum_k \mathrm{tr}_E\left[|e_k\rangle\langle e_k| P_m U(\rho\otimes\sigma)U^\dagger P_m |e_k\rangle\langle e_k|\right]$$

Check physical interpretation of operator sum representation

$$= \sum_k \mathrm{tr}_E\left[|e_k\rangle\langle e_k| P_m U\left(\rho\otimes\sum_j q_j |j\rangle\langle j|\right)U^\dagger P_m |e_k\rangle\langle e_k|\right]$$

$$= \sum_{j,k} q_j\, \mathrm{tr}_E\left[|e_k\rangle\langle e_k| P_m U\left(\rho\otimes|j\rangle\langle j|\right)U^\dagger P_m |e_k\rangle\langle e_k|\right]$$

$$= \sum_{j,k} q_j \left(\sum_i \langle e_i|e_k\rangle\langle e_k| P_m U\left(\rho\otimes|j\rangle\langle j|\right)U^\dagger P_m |e_k\rangle\langle e_k|e_i\rangle\right)$$

$$= \sum_{j,k} q_j \langle e_k| P_m U\left(\rho\otimes|j\rangle\langle j|\right)U^\dagger P_m |e_k\rangle$$

$$= \sum_{j,k} E_{jk}\,\rho\, E_{jk}^\dagger$$

where, $E_{jk} \equiv \sqrt{q_j}\,\langle e_k|P_m U|j\rangle$ — (8.35)

This equation is a generalization of Eq^n. (8.10), and gives an explicit means for calculating the operators appearing in an operator-sum representation for $\mathcal{E}_m$, given that the initial state σ of E is known, and the dynamics b/w Q and E are known.

The quantum operations $\mathcal{E}_m$ can be thought of as defining a kind of measurement process generalizing the description of measurements given in Chapter 2.

Ex: 8·7 Suppose that instead of doing a projective measurement on the combined principal system and environment we had performed a general measurement described by measurement operators $\{M_m\}$. Find operator-sum representations for the corresp. quantum operations $\mathcal{E}_m$ on the principal system, and show that the respective measurement probabilities are $tr\left[\mathcal{E}(\rho)\right]$.

Ans:

- System-environment models for any operator-sum representation

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Interacting quantum systems give rise in a natural way to an operator-sum representation for quantum operations.

What about the converse problem ?

Given a set of operators $\{E_k\}$ is there some reasonable model environmental system and dynamics which give rise to a quantum operation with those operation elements ?

Reasonable ⇒ the dynamics must be either a unitary evolution or a projective measurement.

- For any trace-preserving or non-trace-preserving quantum operation, $\mathcal{E}$, with operation elements $\{E_k\}$, there exists a model environment, E, starting in a pure state $|e_0\rangle$, and model dynamics specified by a unitary operator U and projector P onto E such that,

$$\mathcal{E}(\rho) = \mathrm{tr}_E\left(P\,U\left(\rho \otimes |e_0\rangle\langle e_0|\right)U^\dagger P\right)$$

Proof: Let, $\mathcal{E}$ is a trace-preserving quantum operation, with operator-sum representation generated by operation elements $\{E_k\}$ satisfying the completeness relation $\sum_k E_k^\dagger E_k = I$.

We are only attempting to find an appropriate unitary operator U to model the dynamics.

Let $|e_k\rangle$ be an orthonormal basis set for E, in one-to-one correspondance with the index k for the operators E_k.

By definition, E has such a basis; we are trying to find a model environment giving rise to a dynamics described by the operation elements $\{E_k\}$.

Define an operator U which has the following action on the states of the form $|\psi\rangle|e_0\rangle$,

$$U|\psi\rangle|e_0\rangle \equiv \sum_k E_k|\psi\rangle|e_k\rangle$$

where $|e_0\rangle$ is just some standard state of the model environment.

For arbitrary states $|\psi\rangle$ and $|\phi\rangle$ of the principal system,

$$\langle\psi|\langle e_0|U^\dagger U|\phi\rangle|e_0\rangle = \left(\sum_{k'} E_{k'}|\psi\rangle|e_{k'}\rangle\right)^\dagger\left(\sum_k E_k|\phi\rangle|e_k\rangle\right)$$

$$= \left(\sum_{k'}\langle\psi|E_{k'}^\dagger\langle e_{k'}|\right)\left(\sum_k E_k|\phi\rangle|e_k\rangle\right)$$

$$= \sum_{k',k}\left(\langle\psi|E_{k'}^\dagger\otimes\langle e_{k'}|\right)\left(E_k|\phi\rangle\otimes|e_k\rangle\right)$$

$$= \sum_{k',k}\langle\psi|E_{k'}^\dagger E_k|\phi\rangle\otimes\langle e_{k'}|e_k\rangle$$

$$= \sum_{k',k}\langle\psi|E_{k'}^\dagger E_k|\phi\rangle\cdot\delta_{k'k}$$

$$= \sum_k\langle\psi|E_k^\dagger E_k|\phi\rangle$$

$$= \langle\psi|\phi\rangle$$

Completeness relation
$$\sum_k E_k^\dagger E_k = I$$

the

$|e_k\rangle\Big)$

on

$$\Big(\langle\psi|\langle e_0|\Big)\Big(|\phi\rangle|e_0\rangle\Big) = \Big(\langle\psi|\otimes\langle e_0|\Big)\Big(|\phi\rangle\otimes|e_0\rangle\Big)$$
$$= \langle\psi|\phi\rangle \otimes \langle e_0|e_0\rangle$$
$$= \langle\psi|\phi\rangle.$$

$\Rightarrow$ U preserves inner products of the form $|\psi\rangle|e_0\rangle$.

Let $\{|\psi_i\rangle\}$ is an orthonormal basis of the principal system. Such that $\langle\psi_i|\psi_j\rangle = \delta_{ij}$.

$\Rightarrow$ $\{|\psi_i\rangle|e_0\rangle\}$ is a set of orthonormal vectors, and form a subspace of the combined system QE.

$$\Big(\langle\psi_i|\langle e_0|U^\dagger\Big)\Big(U\,|\psi_j\rangle|e_0\rangle\Big) = \Big(\langle\psi_i|\langle e_0|\Big)\Big(|\psi_j\rangle|e_0\rangle\Big)$$

for all $|\psi_i\rangle, |\psi_j\rangle$

$\Rightarrow$ U is a unitary operator in the subspace spanned by $\{|\psi_i\rangle|e_0\rangle\}$. of the combined system QE

$\boxed{\text{Ex: 2.67}}$ Suppose V is a Hilbert space with a subspace W, and let $U: W \to V$ is a linear operator which preserves inner products.

i.e.,

for any $|w_i\rangle$ and $|w_j\rangle$ in W,

$$\langle w_1|U^\dagger U|w_1\rangle = \langle w_1|w_2\rangle$$

There exist a unitary operator $U': V \to V$ which extends U, ie.,

$U'|w\rangle = U|w\rangle \quad \forall \; |w\rangle \in W$, but U' is defined on the entire space V.

Proof

Let W is an m-dimensional subspace of the n-dimensional vector space V.

Let $\{|w_1\rangle, \ldots, |w_m\rangle\}$ be the orthonormal basis of W, and it can be extended to an orthonormal basis of V by adding some vectors $\{|w_{m+1}\rangle, \ldots, |w_n\rangle\}$.

If $U|w_i\rangle = |v_i\rangle$, then we have

$$\langle v_i|v_j\rangle = \langle w_i|U^\dagger U|w_j\rangle = \langle w_i|w_j\rangle = \delta_{ij}$$

$\Longrightarrow$ $U: W \to V$ maps the orthonormal basis of W to an orthonormal set in V, which is $\{|v_1\rangle, \ldots, |v_m\rangle\}$

$$\therefore \quad U = \sum_{i=1}^{m} |v_i\rangle\langle w_i|$$

$\{|v_1\rangle, \ldots, |v_m\rangle\}$ are an orthonormal set, hence it can be extended to an orthonormal basis of V by adding some orthonormal vectors $\{|v_{m+1}\rangle, \ldots, |v_n\rangle\}$.

ducts.

of

We can write down the unitary extension U' by setting,

$$U'|w_j\rangle = |v_j\rangle \quad \text{for} \quad j = m+1, \cdots, n$$

$$U' = \sum_{j=1}^{n} |w_j\rangle\langle v_j|$$

$\Rightarrow$ The operator U can be extended to a unitary operator acting on the entire state space of the joint system.

$$tr_E\left[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right] =$$

$$= \sum_k I \otimes \langle e_k| \left(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right) I \otimes |e_k\rangle$$

$$= \sum_k \underbrace{(I \otimes \langle e_k|)U(I \otimes |e_0\rangle)}\, \rho\, \underbrace{(I \otimes \langle e_0|)U^\dagger(I \otimes |e_k\rangle)}$$

$$(I \otimes \langle e_k|)U(I \otimes |e_0\rangle) = (I \otimes \langle e_k|)\left(\sum_j E_j I \otimes |e_j\rangle\right)$$

$$= \sum_j (I \otimes \langle e_k|)(E_j \otimes |e_j\rangle)$$

$$= \sum_j E_j \otimes \langle e_k|e_j\rangle \qquad \left[(A \otimes B)(C \otimes D) = AC \otimes BD\right.$$

$$= E_k.$$

$\therefore$

$$tr_E\left(U\left(\rho\otimes|e_0\rangle\langle e_0|\right)U^\dagger\right)=\sum_k E_k\rho E_k^\dagger$$

So this model provides a realization of the quantum operation $\mathcal{E}$ with operation elements $\{E_k\}$.

Mocking up a quantum operation

Given a trace-preserving quantum operation expressed in the operator-sum representation,

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger .$$

Construct a physical model.

From eq. (8.10) we want U to satisfy

$$E_k = \left(I \otimes \langle e_k|\right) U \left(I \otimes |e_0\rangle\right)$$

$$= \langle e_k | U | e_0 \rangle$$

where U is some unitary operator, and $|e_k\rangle$ are orthonormal basis vectors for the environment system.

Such a U is conveniently represented as the block matrix,

$$U = \begin{bmatrix} [E_1] & \cdot & \cdot & \cdot \\ [E_2] & \cdot & \cdot & \cdot \\ [E_3] & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

in the basis $|e_k\rangle$.

The operation elements E_k only determine the 1st block column of this matrix (unlike elsewhere, here it is convenient to have the 1st label of the states be the environment, and the 2nd, the principal system).

The entries of the rest of the matrix can be chosen such that U is unitary.

* The unitary matrix U can be implemented by a quantum circuit.

Proof

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$$E_k = \left(I \otimes \langle e_k|\right) U \left(I \otimes |e_0\rangle\right)$$

$$= \langle e_k|U|e_0\rangle$$

where $|e_k\rangle$ are the orthonormal basis vectors for the environment.

E_k are the Kraus operators

Given a matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \langle 0|A|0\rangle & \langle 0|A|1\rangle \\ \langle 1|A|0\rangle & \langle 1|A|1\rangle \end{bmatrix}$$

$$= \sum_{i,j} a_{ij} |i\rangle\langle j|$$

The matrix A in the orthonormal basis $\{|v_i\rangle\}$ is,

$$B = [A]_v = P_v A P_v^{-1} = P_v A P_v^{\dagger}$$

where, $P_v = V^{-1} = V^{\dagger} = \begin{bmatrix} |v_1\rangle & |v_2\rangle \end{bmatrix}^{\dagger}$

$$= \begin{bmatrix} \langle v_1| \\ \langle v_2| \end{bmatrix}$$

$$B = [A]_v = \begin{bmatrix} \langle v_1| \\ \langle v_2| \end{bmatrix} A \begin{bmatrix} |v_1\rangle & |v_2\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle v_1|A \\ \langle v_2|A \end{bmatrix} \begin{bmatrix} |v_1\rangle & |v_2\rangle \end{bmatrix} = \begin{bmatrix} \langle v_1|A|v_1\rangle & \langle v_1|A|v_2\rangle \\ \langle v_2|A|v_1\rangle & \langle v_2|A|v_2\rangle \end{bmatrix}$$

$$A = P_v^{\dagger} B P_v = \begin{bmatrix} |v_1\rangle & |v_2\rangle \end{bmatrix} \begin{bmatrix} \langle v_1|A|v_1\rangle & \langle v_1|A|v_2\rangle \\ \langle v_2|A|v_1\rangle & \langle v_2|A|v_2\rangle \end{bmatrix} \begin{bmatrix} \langle v_1| \\ \langle v_2| \end{bmatrix}$$

$$= \begin{bmatrix} |v_1\rangle\langle v_1|A|v_1\rangle + |v_2\rangle\langle v_2|A|v_1\rangle & |v_1\rangle\langle v_1|A|v_2\rangle + |v_2\rangle\langle v_2|A|v_2\rangle \end{bmatrix} \begin{bmatrix} \langle v_1| \\ \langle v_2| \end{bmatrix}$$

$$= \langle v_1|A|v_1\rangle |v_1\rangle\langle v_1| + \langle v_2|A|v_1\rangle |v_2\rangle\langle v_1| + \langle v_1|A|v_2\rangle |v_1\rangle\langle v_2| + \langle v_2|A|v_2\rangle |v_2\rangle\langle v_2|$$

$$= \sum_{i,j} \langle v_i|A|v_j\rangle |v_i\rangle\langle v_j|$$

$\therefore$

$$A = \sum_{i,j} a_{ij} |i\rangle\langle j|$$

$$= \sum_{i,j} b_{ij} |v_i\rangle\langle v_j|$$

where $a_{ij} = \langle i|A|j\rangle$ and $b_{ij} = \langle v_i|A|v_j\rangle$

$$B = [A]_v = P_v A P_v^\dagger$$

$$= \sum_{i,j} b_{ij} |i\rangle\langle j| = \sum_{i,j} \langle v_i|A|v_j\rangle |i\rangle\langle j|$$

$$\begin{bmatrix} \langle v_1|A|v_1\rangle & \langle v_2|A|v_1\rangle \\ \langle v_2|A|v_1\rangle & \langle v_2|A|v_2\rangle \end{bmatrix}$$

$\therefore$ $(i,j)^{th}$ term of A is the Basis $\{|v_i\rangle\}$ is, $b_{ij} = \langle v_i|A|v_j\rangle$.

$E_k = (I \otimes \langle e_k|)\, U\, (I \otimes |e_0\rangle)$ is an operator acting on the 1st system (principal system Q).

If we interchange the order of systems i.e., the 1st system is the environment E and 2nd is the principal system Q then,

Decomposing $(\langle e_k|\otimes I)U(|e_0\rangle\otimes I)$ and U in terms of the orthonormal basis vectors of the respective systems as,

$$U = \sum_{m,m',n,n'} \mu_{m,m',n,n'} \left(|e_m\rangle\otimes|P_n\rangle\right)\left(\langle e_{m'}|\otimes\langle P_{n'}|\right)$$

$$= \sum_{m,m',n,n'} \mu_{m,m',n,n'} |e_m\rangle\langle e_{m'}|\otimes|P_n\rangle\langle P_{n'}|$$

$$= \sum_{m,m'} |e_m\rangle\langle e_{m'}|\otimes\sum_{n,n'} \mu_{m,m',n,n'}|P_n\rangle\langle P_{n'}|$$

$$\left(\langle e_k|\otimes I\right)U\left(|e_0\rangle\otimes I\right) =$$

$$= \left(\langle e_k|\otimes I\right)\left[\sum_{m,m',n,n'} \mu_{m,m',n,n'} |e_m\rangle\langle e_{m'}|\otimes|P_n\rangle\langle P_{n'}|\right]\left(|e_0\rangle\otimes I\right)$$

$$= \sum_{m,m',n,n'} \mu_{m,m',n,n'} \langle e_k|e_m\rangle\langle e_{m'}|e_0\rangle \otimes |P_n\rangle\langle P_{n'}|$$

$$= \sum_{m,m',n,n'} \mu_{m,m',n,n'} \delta_{k,m}\delta_{m',0} \otimes |P_n\rangle\langle P_{n'}|$$

$$= \sum_{n,n'} \mu_{k,0,n,n'} |P_n\rangle\langle P_{n'}|$$

If we define the change of basis matrix $P = P_E \otimes P_P$ then (by extending the ideas in lemma) the operators in the new basis are:

$$\Big(\big(\langle e_k| \otimes I\big) U \big(|e_0\rangle \otimes I\big)\Big)' = \sum_{n,n'} \mu_{k,0,n,n'} |n\rangle\langle n'|$$

$$U' = \sum_{m,m',n,n'} \mu_{m,m',n,n'} |m\rangle\langle m'| \otimes |n\rangle\langle n'|$$

$$= \sum_{m,m'} |m\rangle\langle m'| \otimes \sum_{n,n'} \mu_{m,m',n,n'} |n\rangle\langle n'|$$

$$= \sum_{m,m'} |m\rangle\langle m'| \otimes \sigma_{m,m'}$$

where $\{|n\rangle\}$ and $\{|m\rangle\}$ are the standard basis such that $|m\rangle\langle m'|$ is the matrix with 1 as its (m,m') entry, and

$\sigma_{m,m'} = \sum_{n,n'} \mu_{m,m',n,n'} |n\rangle\langle n'|$ is the $(m,m')^{th}$ block entry of U'.

$\therefore$

The $(k,0)^{th}$ block entry of U' is,

$$\sigma_{k,0} = \sum_{n,n'} \mu_{k,0,n,n'} |n\rangle\langle n'|$$

$$= \Big(\big(\langle e_k| \otimes I\big) U \big(|e_0\rangle \otimes I\big)\Big)'$$

Ex: 8.8 Non-trace-preserving quantum operations