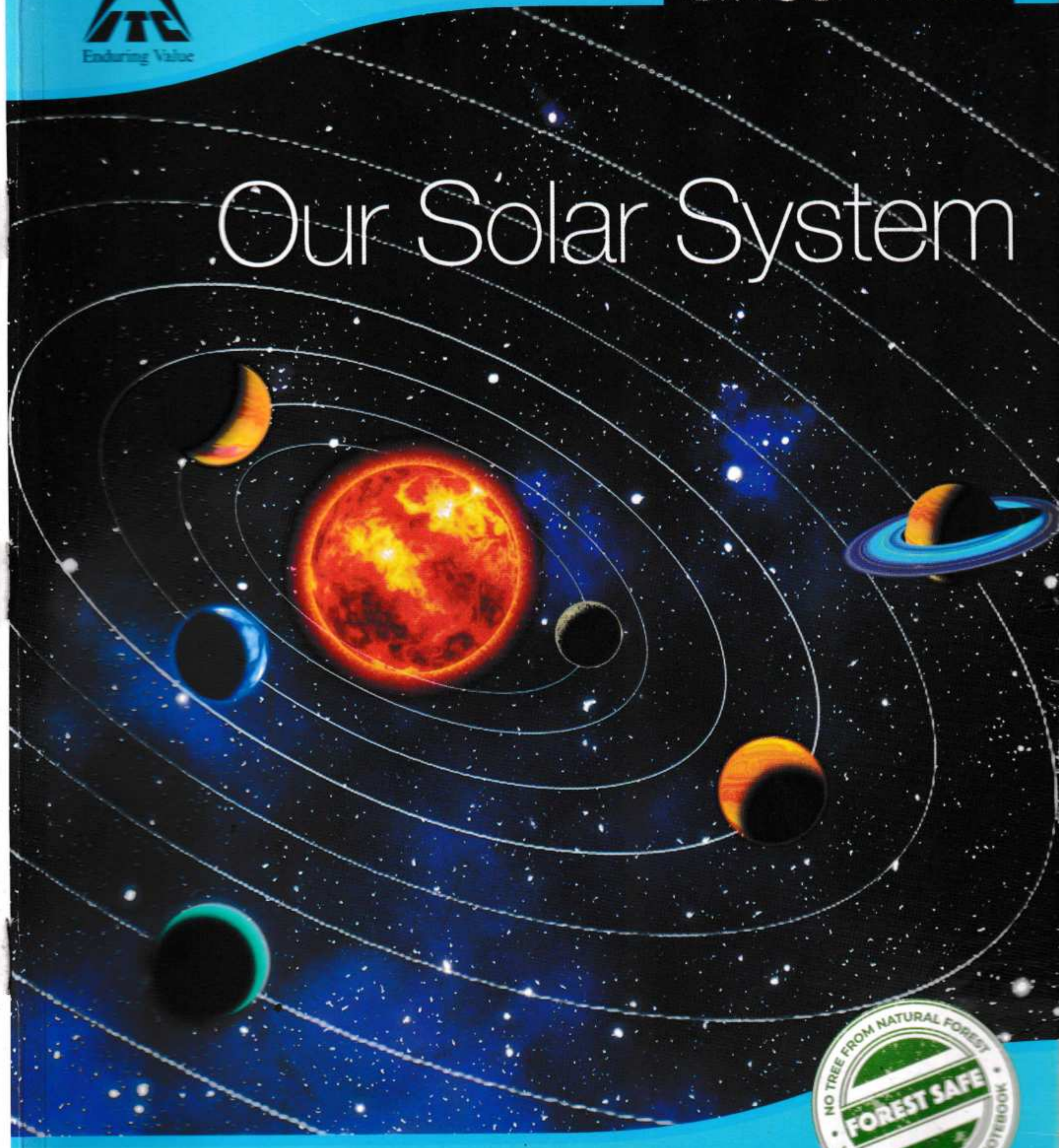
classmate

Enduring Value

Our Solar System



NO TREE FROM NATURAL FOREST WAS CUT TO MAKE THIS NOTEBOOK

FOREST SAFE

soorajss1729@gmail.com

+91-94006357[illegible]

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11.2

NAME: SOORAJ.S. STD.: SEC.: ROLL NO.: SUB.:

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		QUANTUM COMPUTATION		
		& QUANTUM INFORMATION		
		– Nielsen & Chuang.		

Von Neumann entropy

The Shannon entropy measures the uncertainty associated with a classical distribution.

Quantum states are described in a similar fashion, with density operators replacing probability distribution.

Generalizing the definition of the Shannon entropy to quantum states, $\Rightarrow$

* Von Neumann defined the entropy of a quantum state ρ by the formula,

$$\boxed{\begin{aligned} S(\rho) &\equiv -\mathrm{tr}(\rho \log \rho) \\ &= -\sum_x \lambda_x \log \lambda_x \end{aligned}}$$

where λ_x: eigenvalues of ρ and $\log \rho = \log_2 \rho$.

$0 \log 0 \equiv 0$, as for the Shannon entropy.

$$\boxed{A = VDV^{-1} \Rightarrow \log A = V(\log D)V^{-1}}$$

Ex:-

The completely mixed density operator in a d-dimensional space, I/d has entropy $\log d$.

$\rho = I/d$.

$$S(\rho) = S(I/d) = -\sum_x \lambda_x \log \lambda_x$$

$$= -\sum_{x=1}^{d} \frac{1}{d} \log \frac{1}{d}$$

$$= \sum_{x=1}^{d} \frac{1}{d} \log d = \underline{\log d}.$$

Logarithm of a matrix

- A logarithm of a matrix is another matrix such that the matrix exponential of the matrix equals the original matrix.

Ex. 11.11 Calculate $S(\rho)$ for

① $\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Ans: $S(\rho) \equiv -tr(\rho \log \rho)$

$$= -\sum_x \lambda_x \log \lambda_x$$

$$= -1 \log 1 - 0 \log 0 = 0$$

② $\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Ans: $|\rho| = 0$ & $tr(\rho) = 1 = \lambda_1 + \lambda_2$

$\lambda = 0, 1$

$S(\rho) = 0$

③ $\rho = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

Ans: $tr(\rho) = \lambda_1 + \lambda_2 = 1$ & $|\rho| = \frac{1}{9} = \lambda_1 \lambda_2$

$\Delta = 81 - 36 = 45$

$\lambda_1 + \frac{1}{9\lambda_1} = \frac{9\lambda_1^2 + 1}{9\lambda_1} = 1 \Rightarrow 9\lambda_1^2 - 9\lambda_1 + 1 = 0$

$$\left.\begin{array}{l} \lambda_1 = \frac{9 \pm 3\sqrt{5}}{2 \times 9} = \frac{3 \pm \sqrt{5}}{6} \end{array}\right\} \begin{array}{l} \lambda_1 = \frac{3 + \sqrt{5}}{6} = \frac{1}{2} + \frac{\sqrt{5}}{6} \\ \lambda_2 = \frac{3 - \sqrt{5}}{6} = \frac{1}{2} - \frac{\sqrt{5}}{6} \end{array}$$

$S(\rho) = \frac{-1}{\ln 2}(-0.1188 - 0.3812) \approx 0.55$

$$\lambda_1 = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{1}{\sqrt{3}} e^{i\pi/6}$$

$$\lambda_2 = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{1}{\sqrt{3}} e^{-i\pi/6}$$

$$\ln\lambda_1 = \ln\frac{1}{\sqrt{3}} + i\frac{\pi}{6} \approx -0.54931 + i\pi/6 \Rightarrow \log_2\lambda_1 = \frac{\ln\lambda_1}{\ln 2}$$

$$\ln\lambda_2 = \ln\frac{1}{\sqrt{3}} - i\pi/6 \approx -0.54931 - i\pi/6 \Rightarrow$$

$$S(\rho) = \sum_x \lambda_x \log_2\lambda_x = \frac{-1}{\ln 2}\left[\lambda\ln\lambda + \bar{\lambda}\ln\bar{\lambda}\right]$$

$$= \frac{-1}{\ln 2}\left[2\,\mathrm{Re}\{\lambda\ln\lambda\}\right]$$

$$= \frac{-2}{\ln 2}\left[\frac{1}{2}(-0.54931) + \frac{1}{2\sqrt{3}}\cdot\frac{\pi}{6}\right]$$

$$= \frac{1}{\ln 2}\left(0.54931 + 0.3022998\right)$$

$$= \frac{0.851606}{\ln 2} =$$

Complex Logarithm

A complex logarithm is a generalization of the natural logarithm to non zero complex numbers.

A complex logarithm of a non zero complex number z, defined to be any complex number w for which $e^{w} = z$, which is denoted by $\log z$.

If z is given is polar form as $z = re^{i\theta}$, where r and θ are real numbers with $r > 0$, then $\ln r + i\theta$ is one logarithm of z, and all complex logarithms of z are exactly the numbers of the form $\ln r + i(\theta + 2\pi k)$ for $k \in \mathbb{Z}$.

These logarithms are equally spaced along a vertical line in the complex plane.

Consider any non-zero complex number z and we would like to solve for w in the equation $e^{w} = z$.

If $\theta = \operatorname{Arg}(z)$ with $-\pi < \theta \leq \pi$ then z and w can be written as follows,

$z = re^{i\theta}$ and $w = u+iv$

$$e^{w} = z \implies e^{u}e^{iv} = re^{i\theta}$$

$\therefore$ $e^{u} = r$ and $\cos v = \cos\theta$ & $\sin v = \sin\theta$

$v = 2n\pi \pm \theta$ & $v = n\pi + (-1)^{n}\theta$

$v = 2n\pi \pm \theta$ & $v = 2n\pi + \theta$ (or) $v = (2n+1)\pi - \theta$

$$e^{u} = r \quad \text{and} \quad \underbrace{}_{v = 2n\pi + \theta,\ n \in \mathbb{Z}}$$

$u = \ln r$

$$w = u+iv \implies w = \ln r + i(2n\pi + \theta) \quad , n \in \mathbb{Z}$$

$\therefore$ The equation $e^{w} = z$ is satisfied iff w has one of the values,

$$w = \ln r + i(2n\pi + \theta) \quad , n \in \mathbb{Z}$$

The (multivalued) logarithmic function of a nonzero complex variable $z = re^{i\theta}$ is defined by,

$$\boxed{\log z = \ln r + i(\theta + 2n\pi)} \quad , \; n \in \mathbb{Z}$$

Ex: 11.12 Comparison of quantum & classical entropies.

Suppose $\rho = p|0\rangle\langle 0| + (1-p)\frac{(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)}{2}$.

Evaluate $S(\rho)$. Compare the value of $S(\rho)$ to $H(p, 1-p)$.

Ans: $\rho = p|0\rangle\langle 0| + \frac{(1-p)}{2}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right]$

$$= \frac{1+p}{2}|0\rangle\langle 0| + \frac{1-p}{2}\left(|0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= \begin{bmatrix} \frac{1+p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & \frac{1-p}{2} \end{bmatrix}$$

$$|\rho| = \frac{1}{4}\left[\cancel{1}-p^2-\cancel{1}-p^2+2p\right] = \frac{2p-2p^2}{4} = \frac{p(1-2p)}{2} = \lambda_1\lambda_2$$

$tr(\rho) = 1 = \lambda_1+\lambda_2 \implies \lambda_2 = 1-\lambda_1$

$\lambda_1(1-\lambda_1) = \lambda_1 - \lambda_1^2 = \frac{p(1-2p)}{2} \implies \lambda_1^2 - \lambda_1 + \frac{p(1-2p)}{2} = 0$

$\Delta = 1 - \frac{^2\cancel{4}(p-2p^2)}{\cancel{2}} = 1-2p+4p^2 = (1-2p)^2$

$\lambda = \frac{1 \pm (1-2p)}{2} = \frac{2-2p}{2}$ (or) $\frac{2p}{2}$

$\lambda_1 = p$ and $\lambda_2 = 1-p$

$S(\rho) = -p\log p - (1-p)\log(1-p) = H(p, 1-p)$

Quantum relative entropy

In quantum information theory, quantum relative entropy is a measure of distinguishability b/w 2 quantum states.

* Suppose ρ and σ are density operators. The relative entropy of ρ to σ is defined by,

$$\boxed{\begin{aligned} S(\rho\|\sigma) &\equiv \mathrm{tr}(\rho\log\rho) - \mathrm{tr}(\rho\log\sigma) \\ &= -\mathrm{tr}(\rho\log\sigma) - S(\rho) \\ &= \mathrm{tr}\,\rho(\log\rho - \log\sigma) \end{aligned}}$$

- The relative entropy is defined to be $+\infty$ if the kernel of σ (the vector space spanned by the eigenvectors of σ with eigenvalue 0) has non-trivial intersection with the support of ρ (the vector space spanned by the eigenvectors of ρ with nonzero eigenvalue), and is finite otherwise.

$$S(\rho\|\sigma) = \begin{cases} +\infty & ; \quad \mathrm{supp}(\rho) \cap \ker(\sigma) \neq \{0\} \\ \text{finite} & ; \quad \text{otherwise} \end{cases}$$

- The support of a matrix M is the orthogonal complement of its kernel, ie, $\mathrm{supp}(M) = \ker(M)^{\perp}$

Proof

$$S(\rho\|\sigma) = tr(\rho\log\rho) - tr(\rho\log\sigma)$$
$$= -S(\rho) - tr(\rho\log\sigma)$$

where $S(\rho)$ is the von Neumann entropy of ρ such that $S(\rho) = -tr(\rho\log\rho) = -\sum_i p_i \log p_i$

Using the convention $-p\log 0 = +\infty$, and $\rho = \sum_i p_i |i\rangle\langle i|$ such that $\log\rho = \sum_i \log p_i |i\rangle\langle i|$, and $\sigma = \sum_j q_j |j\rangle\langle j|$ such that $\log\sigma = \sum_j \log q_j |j\rangle\langle j|$.

$$\boxed{A = VDV^{-1} \Rightarrow \log A = V(\log D)V^{-1}}$$

$$S(\rho\|\sigma) = tr(\rho\log\rho) - tr(\rho\log\sigma)$$
$$= \sum_i p_i \log p_i - \sum_i \langle i|\rho\log\sigma|i\rangle$$
$$= \sum_i p_i \log p_i - \sum_i p_i \langle i|\log\sigma|i\rangle$$

where we used $\langle i|\rho = \langle i|p_i$

$$= \sum_i p_i \log p_i - \sum_i p_i \langle i|\left(\sum_j \log q_j |j\rangle\langle j|\right)|i\rangle$$
$$= \sum_i p_i \log p_i - \sum_{i,j} p_i \log q_j \langle i|j\rangle\langle j|i\rangle$$
$$= \sum_i p_i \log p_i - \sum_{i,j} |\langle i|j\rangle|^2 \times p_i \log q_j$$

$S(\rho\|\sigma) = +\infty$ if there exists $|\langle i|j\rangle|^2 p_i > 0$ and $q_j = 0$ for some i, j.

$\boxed{\sigma|j\rangle = q_j|j\rangle}$

$q_j = 0$ corresponds to the state $|j\rangle \in \ker(\sigma)$

$$|\langle i|j\rangle|^2 p_i > 0 \Rightarrow p_i > 0 \text{ and } |\langle i|j\rangle| > 0$$
$$\Rightarrow p_i \neq 0 \text{ and } |\langle i|j\rangle| \neq 0$$

$p_i \neq 0$ corresponds to the state $|i\rangle \in \mathrm{supp}(\rho)$

$|\langle i|j\rangle| \neq 0$ for some $|i\rangle \in \mathrm{supp}(\rho)$ and $|j\rangle \in \ker(\sigma)$.

$S(\rho\|\sigma) = +\infty$ iff $|\langle i|j\rangle| \neq 0$ for some $|i\rangle \in \mathrm{supp}(\rho)$ and $|j\rangle \in \ker(\sigma)$

$\Rightarrow \mathrm{supp}(\rho) \cap \ker(\sigma) \neq \{0\}$

* Klein's inequality

The quantum relative entropy is non-negative

$$\boxed{S(\rho\|\sigma) \geq 0}$$

with equality iff $\rho = \sigma$.

Proof

Let $\rho = \sum_i p_i |i\rangle\langle i|$ and $\sigma = \sum_j q_j |j\rangle\langle j|$ be orthonormal decompositions for ρ and σ.

$$S(\rho\|\sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma)$$
$$= \sum_i p_i \log p_i - \sum_i \langle i|\rho \log \sigma|i\rangle$$
$$= \sum_i p_i \log p_i - \sum_i p_i \langle i|\log \sigma|i\rangle$$
$$= \sum_i p_i \log p_i - \sum_i p_i \langle i|\Big(\sum_j \log q_j |j\rangle\langle j|\Big)|i\rangle$$
$$= \sum_i p_i \Big(\log p_i - \sum_j \langle i|j\rangle\langle j|i\rangle \log q_j\Big)$$
$$= \sum_i p_i \Big(\log p_i - \sum_j P_{ij} \log q_j\Big)$$

where $P_{ij} = \langle i|j\rangle\langle j|i\rangle = |\langle i|j\rangle|^2 \geq 0$

$$P_{ij} = \langle i|j\rangle\langle j|i\rangle = |\langle i|j\rangle|^2 \geq 0$$

$$\sum_i P_{ij} = \sum_i \langle i|j\rangle\langle j|i\rangle = \sum_i \langle j|i\rangle\langle i|j\rangle$$

$$= \langle j|\left(\sum_i |i\rangle\langle i|\right)|j\rangle = \langle j|j\rangle = 1$$

$$\sum_j P_{ij} = 1$$

For the matrix $P = [P_{ij}]$ this is known as double stochasticity.

QC : 11.1 Binary entropy

$\log(\cdot)$ is a strictly concave function.

$$\sum_j P_{ij} \log q_j \leq \log\left(\sum_j P_{ij} q_j\right)$$

with equality iff there exists a value of j for which $P_{ij} = 1$

$$\sum_j P_{ij} \log q_j \le \log r_i \quad , \text{ where } \quad r_i \equiv \sum_j P_{ij} q_j$$

$$-\sum_j P_{ij} \log q_j \ge -\log r_i$$

$$\log p_i - \sum_j P_{ij} \log q_j \ge \log p_i - \log r_i = \log p_i / r_i$$

$$\sum_i p_i \left(\log p_i - \sum_j P_{ij} \log q_j \right) \ge \sum_i p_i \log \frac{p_i}{r_i}$$

$$\therefore \quad S(\rho \| \sigma) \ge \sum_i p_i \log \frac{p_i}{r_i}$$

with equality iff for each i there exists a value of j such that $P_{ij} = 1$.

ie, iff $P = [P_{ij}]$ is a permutation matrix.

$$\sum_i r_i = \sum_i \sum_j P_{ij} q_j = \sum_j \left(\sum_i P_{ij} \right) q_j = \sum_j q_j = 1$$

$\therefore$ $\sum_i p_i \log p_i / r_i$ has the form of the classical relative entropy.

for

From the non-negativity of the classical relative entropy,

$$S(\rho\|\sigma) \geq 0$$

with equality iff $p_i = r_i$ for all i, and $P = [P_{ij}]$ is a permutation

Note: By relabeling the eigenstates of σ if necessary, we can assume that P_{ij} is the identity matrix,
and that ρ and σ ~~are~~ are diagonal in the same basis.

$\therefore$ $p_i = r_i \Rightarrow$ the corresp. eigenvalues of ρ & σ are identical & thus..

$$S(\rho\|\sigma) = 0 \text{ iff } \rho = \sigma$$

From the non-negativity of the classical relative entropy,

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Note: By relabeling the eigenstates of σ if necessary, we can assume that P_{ij} is the identity matrix, and that ρ and σ are diagonal in the same basis.

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$$s(\rho\|\sigma) = 0 \text{ iff } \rho = \sigma$$

Continuity of the entropy

Suppose we vary ρ by a small amount. How much does $S(\rho)$ change?

* Fannes' inequality

Suppose ρ and σ are density matrices such that the trace distance b/w them satisfies $2T(\rho,\sigma) \leq 1/e$. Then

$$\boxed{|S(\rho) - S(\sigma)| \leq 2T(\rho,\sigma) \log d + \eta(2T(\rho,\sigma))}$$

where d : dimension of the Hilbert space

$\eta(x) = -x \log x$

Removing the restriction that $2T(\rho,\sigma) \leq 1/e$,

$$\boxed{|S(\rho) - S(\sigma)| \leq 2T(\rho,\sigma) \log d + 1/e}$$

Proof.

Let $r_1 \geq r_2 \geq \cdots \geq r_d$ be the eigenvalues of ρ and $s_1 \geq s_2 \geq \cdots \geq s_d$ be the eigenvalues of σ, in descending order.

Let $H = \rho - \sigma$, which is hermitian since Both ρ & σ are hermitian (+ve semi definite).

$$H = W D_H W^{\dagger}$$

Let D_Q be the diagonal matrix which has all the +ve eigenvalues of H and D_R contains the negation of all the -ve eigenvalues of H. such that $W D_Q W^{\dagger} = Q$ and $W D_R W^{\dagger} = R$.

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$$\therefore \quad D_H = D_Q - D_R$$

$$H = W D_H W^{\dagger} = W (D_Q - D_R) W^{\dagger} = W D_Q W^{\dagger} - W D_R W^{\dagger} = Q - R$$

$\Rightarrow$ D_Q & D_R, and therefore Q & R have orthogonal supports, since Q only supports the +ve eigenspace and R only supports the -ve eigenspace.

$$\mathbb{P}_R \mathbb{P}_Q = \mathbb{P}_Q \mathbb{P}_R = 0 \implies RQ = QR = 0$$

$\therefore$

$\rho - \sigma = Q - R$, where Q & R are +ve operators with orthogonal support.

$$T(\rho, \sigma) = \frac{1}{2} \operatorname{tr} |\rho - \sigma| = \frac{1}{2} \operatorname{tr} \sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)}$$

$$\begin{aligned}
(\rho - \sigma)^\dagger (\rho - \sigma) &= (Q - R)^\dagger (Q - R) = (Q^\dagger - R^\dagger)(Q - R) \\
&= Q^\dagger Q + R^\dagger R - Q^\dagger R - R^\dagger Q \\
&= Q^\dagger Q + R^\dagger R - QR - RQ \\
&= Q^\dagger Q + R^\dagger R \quad \text{since } QR = RQ = 0 \\
&= Q^\dagger Q + R^\dagger R + QR + RQ \\
&= Q^2 + R^2 + QR + RQ = (Q + R)^2
\end{aligned}$$

$$\sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)} = |\rho - \sigma| = Q + R$$

$\therefore$

The trace distance becomes,

$$T(\rho, \sigma) = \frac{1}{2} \operatorname{tr} |\rho - \sigma| = \frac{1}{2} \operatorname{tr} (Q + R) = \frac{1}{2} \operatorname{tr} Q + \frac{1}{2} \operatorname{tr} R$$

$$\implies 2T(\rho,\sigma) = \mathrm{tr}(Q) + \mathrm{tr}(R)$$

Defining $V \equiv R+\rho = Q+\sigma$

where Q, R are +ve semidefinite.

$$2T(\rho,\sigma) = \mathrm{tr}(V-\sigma) + \mathrm{tr}(V-\rho)$$

$$= \mathrm{tr}(2V) - \mathrm{tr}(\rho) - \mathrm{tr}(\sigma)$$

Min-Max theorem

The Rayleigh quotient for any vector $|x\rangle$ is defined to be the ratio $r(x) = \frac{\langle x|V|x\rangle}{\langle x|x\rangle}$,

where $r(x)$ is scaling invariant,

ie, $r(tx) = \frac{\langle tx|V|tx\rangle}{\langle tx|tx\rangle} = \frac{t^2\langle x|V|x\rangle}{t^2\langle x|x\rangle} = \frac{\langle x|V|x\rangle}{\langle x|x\rangle} = r(x)$

ILA (9)

$\therefore$ It is sufficient to study the special case $\langle x|x\rangle = 1$, so that the critical points of the function $\frac{\langle x|V|x\rangle}{\langle x|x\rangle}$ is the same as that of $\langle x|V|x\rangle$ subjected to the constraint $\langle x|x\rangle = 1$.

The critical points of $r(x)$ are the eigenvectors $|u_i\rangle$ of the operator V.

$\therefore \quad t_{min} = \min_{u\neq 0} r(u) \quad$ and $\quad t_{max} = \max_{u\neq 0} r(u)$

If $t_1 \geq t_2 \geq \cdots \geq t_k \geq \cdots \geq t_d$ be the eigenvalues of V in the descending order then this implies:

$$t_d = \min_{u\neq 0} r(u) = \min_{x\neq 0}\{r(x) : |x\rangle \in U \ \& \ \dim(U) = d\}$$

$$t_{d-1} = \min_{u \neq 0 \in u_d^{\perp}} r(u)$$

$$= \max_{U} \left\{ \min_{x \neq 0} \left\{ r(x) : |x\rangle \in U \ \& \ \dim(U) = d-1 \right\} \right\}$$

$\vdots$

$$t_k = \min_{u \neq 0 \in \{u_d, d_{d-1}, \ldots, u_{k+1}\}^{\perp}} r(u)$$

$$= \max_{U} \left\{ \min_{x \neq 0} \left\{ r(x) : |x\rangle \in U \ \& \ \dim(U) = k \right\} \right\}$$

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Let $t_1 \geq t_2 \geq \cdots \geq t_k \geq \cdots \geq t_d$ be the eigenvalues of V in the descending order.

By the min-max theorem,

$$t_k = \min_{u \neq 0 \in \{u_d, u_{d-1}, \ldots, u_{k+1}\}^{\perp}} r(u)$$

$$= \max_{U} \left\{ \min_{x \neq 0} \left\{ r(x) : |x\rangle \in U \ \& \ \dim(U) = k \right\} \right\}$$

$$= \max_{U} \left\{ \min_{x \neq 0} \left\{ \langle x|V|x\rangle : |x\rangle \in U \ \& \ \dim(U) = k \right\} \right\}$$

$$= \max_{U} \left\{ \min_{x \neq 0} \left\{ \langle x|R+P|x\rangle : |x\rangle \in U \ \& \ \dim(U) = k \right\} \right\}$$

$$= \max_{U} \left\{ \min_{x \neq 0} \left\{ \langle x|R|x\rangle + \langle x|P|x\rangle : |x\rangle \in U \ \& \ \dim(U) = k \right\} \right\}$$

$$\geq \max_{U} \left\{ \min_{x \neq 0} \left\{ \langle x|P|x\rangle : |x\rangle \in U \ \& \ \dim(U) = k \right\} \right\}$$

$$= r_k$$

Q. Stack 7/1/2023

Similarly, since $V = Q + \sigma$ we can prove that $t_k \geq s_k$.

$$\left.\begin{aligned} t_k &\geq r_k \\ t_k &\geq s_k \end{aligned}\right\} \quad t_k \geq \max(r_k, s_k)$$

$$2t_k \geq 2\max(r_k, s_k) = r_k + s_k + |r_k - s_k|$$

$\therefore$

$$\begin{aligned} 2T(\rho,\sigma) &= tr(2V) - tr(\rho) - tr(\sigma) \\ &= \left(\sum_k 2t_k\right) - tr(\rho) - tr(\sigma) \\ &\geq \sum_k \left(r_k + s_k + |r_k - s_k|\right) - tr(\rho) - tr(\sigma) \\ &= tr(\rho) + tr(\sigma) + \sum_k |r_k - s_k| - tr(\rho) - tr(\sigma) \\ &= \sum_k |r_k - s_k| \end{aligned}$$

$$\therefore \quad 2T(\rho,\sigma) \geq \sum_i |r_i - s_i| \qquad \text{— (11.46)}$$

$$|\eta(y)-\eta(x)| \le \eta(|y-x|) \quad \text{if } |y-x| \le 1/2 \quad \text{given}$$
$$\eta(x) = -x\log x$$

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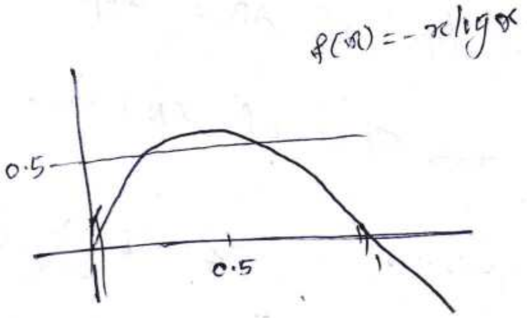
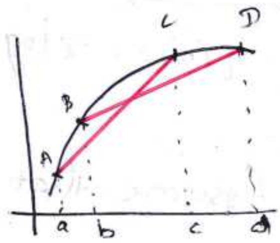
Proof



Method 1

If f is concave and $\underline{\underline{a \le b ; c \le d}}$, where $a \le c$ and $b \le d$, then the slope of the chord joining $(a, f(a)), (c, f(c))$ exceeds that of the chord joining $(b, f(b)), (d, f(d))$



Chordal Slope lemma

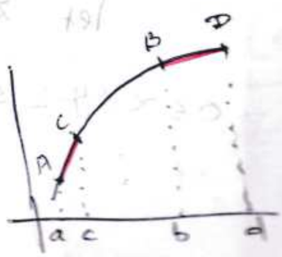
QC

Let $a \le b \le c \le d$,

case 1.

slope of $AB \ge$ slope of $AC \ge$ slope of BC

slope of $BC \ge$ slope of $BD \ge$ slope of CD



$\Rightarrow$ slope of $AC \ge$ slope of $BC \ge$ slope of BD

$\Rightarrow$ slope $\underline{\underline{\text{of } AC \ge \text{slope of } BD}}$

Case 2. $a \le b$ & $c \le d$ and $c \le b$ & $a \le d$

$a \le c \le b$ & $c \le b \le d$

Slope of $AC \ge$ Slope of CB

Slope of $CB \ge$ Slope of BD

$\Rightarrow$ Slope of $AC \ge$ Slope of $CB \ge$ slope of BD

$\Rightarrow$ Slope of $AC \ge$ Slope of BD

$\therefore$

$$\frac{f(c)-f(a)}{c-a} \ge \frac{f(d)-f(b)}{d-b} \quad \text{if } a \le b;\ c \le d \text{ and } a \le c;\ b \le d.$$

$\eta(x) = -x \log x$ is concave.

Assume that $y > x$, where $x, y \in [0,1]$

let $z = y - x$ so that

$0 \le x = y - z \le 1 - z \le 1$ and $z = y - x \le 1/2$

$x \leq 1-z$; $x+z=y \leq 1$

Concavity of $\eta(x) = -x\log x \Longrightarrow$

Slop of chord joining $(x, \eta(x))$ & $(x+z, \eta(x+z))$ $\geq$ Slope of chord joining $(1-z, \eta(1-z))$ & $(1, \eta(1))$

$$\frac{\eta(y)-\eta(x)}{y-x} \geq \frac{\eta(1)-\eta(1-z)}{z}$$

$$\frac{\eta(y)-\eta(x)}{z} \geq \frac{-\eta(1-z)}{z}$$

$$\eta(y)-\eta(x) \geq -\eta(1-z) \qquad \text{——— (1)}$$

Similarly,

$0 \leq x$; $z \leq x+z = y$

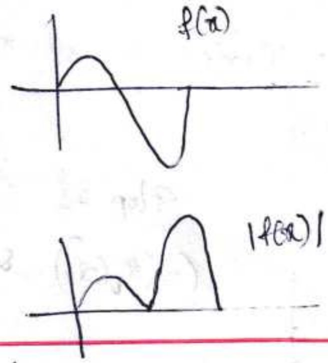
Slope of chord joining $(0, \eta(0))$ & $(z, \eta(z))$ $\geq$ Slope of chord joining $(x, \eta(x))$ & $(x+z, \eta(x+z))$

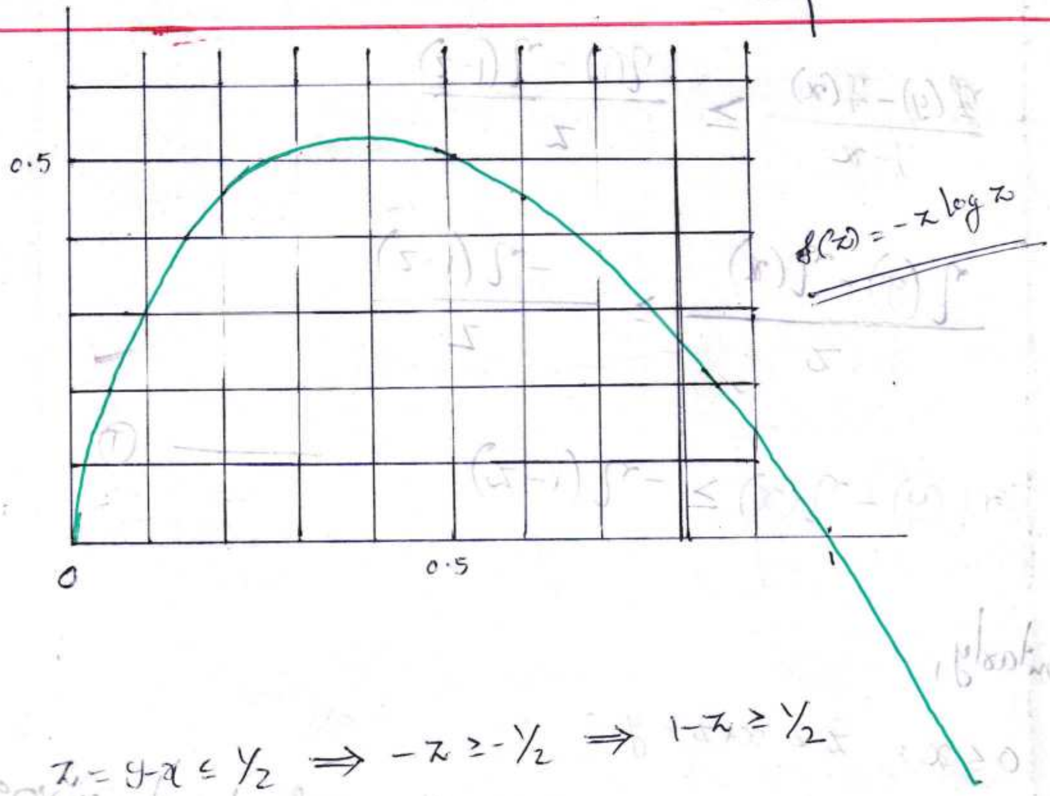
$$\frac{\eta(z)-\eta(0)}{z-0} \geq \frac{\eta(y)-\eta(x)}{y-x} \Rightarrow \frac{\eta(z)}{z} \geq \frac{\eta(y)-\eta(x)}{z}$$

$$\eta(y)-\eta(x) \leq \eta(z) \qquad \text{——— (2)}$$

① & ② $\Longrightarrow -\eta(1-z) \le \eta(y)-\eta(x) \le \eta(z)$

$$|\eta(y)-\eta(x)| \le \max\{\eta(z),\eta(1-z)\}$$





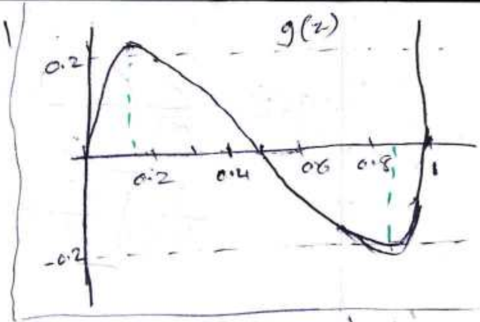
① $z = y-x \le 1/2 \Rightarrow -z \ge -1/2 \Rightarrow 1-z \ge 1/2$

$\therefore 0 \le z \le 1-z \le 1$

Let $g(z) = \eta(z) - \eta(1-z)$

$= -z\log z + (1-z)\log(1-z)$

$g(z) = 0$ when $z = 0, \frac{1}{2}, 1$



$$\Rightarrow g(0) = g(\tfrac{1}{2}) = g(1) = 0$$

$$\frac{d}{dz}\log_2 z = \frac{d}{dz}\left(\frac{\ln z}{\ln 2}\right) = \frac{1}{\ln(2)}\frac{d}{dz}(\ln z) = \frac{1}{\ln 2} * \frac{1}{z}$$

$$g'(z) = -\log z - \cancel{z} * \frac{1}{\ln 2 * \cancel{z}} - \log(1-z) - \cancel{(1-z)} * \frac{1}{\ln 2 \cancel{(1-z)}}$$

$$= \frac{-2}{\ln 2} - \log\big(z(1-z)\big)$$

$$g'(z) = 0 \Rightarrow \ln\big(z(1-z)\big) = -2 = \ln(1/e^2)$$

$$\Rightarrow z(1-z) = z - z^2 = 1/e^2$$

$$\Rightarrow z^2 - z + 1/e^2 = 0$$

$$\Delta = 1 - \frac{4}{e^2} = \frac{e^2-4}{e^2}$$

$$z = \frac{1 \pm \frac{\sqrt{e^2-4}}{e}}{2} = \frac{1 \pm \sqrt{1 - 4/e^2}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{e^2}}$$

$$4 > e > 2 \Rightarrow \tfrac{1}{2} > \tfrac{1}{e} > \tfrac{1}{4} \Rightarrow \tfrac{1}{16} < \tfrac{1}{e^2} < \tfrac{1}{4} \Rightarrow -\tfrac{1}{4} < -\tfrac{1}{e^2} < -\tfrac{1}{16}$$

$$0 < \frac{1}{4} - \frac{1}{e^2} < \frac{3}{16} < \frac{4}{16} = \frac{1}{4} \Rightarrow \underline{\underline{0 < \sqrt{\tfrac{1}{4} - \tfrac{1}{e^2}} < \tfrac{1}{2}}}$$

$\therefore$ $g'(z) = 0$ at $z_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{e^2}}$ with $0 < z_1 < \frac{1}{2}$

and $z_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{e^2}}$ with $\frac{1}{2} < z_2 < 1$

$$g''(z) = \frac{d}{dz}\left(-2 - \log z - \log(1-z)\right)$$

$$= \frac{1}{\ln 2}\left[\frac{-1}{z} + \frac{1}{1-z}\right] = \frac{1}{\ln 2}\left[\frac{2z-1}{z(1-z)}\right]$$

$z \leq 1/2 \Rightarrow 2z \leq 1 \Rightarrow \underline{\underline{2z - 1 \leq 0}}$

$\therefore\ g''(z) \leq 0 \quad$ for $\quad z \leq 1/2$

Therefore,

$g(z) = \eta(z) - \eta(1-z) = -z\log z + (1-z)\log(1-z)$ is differentiable & hence continuous on $z \in [0, 1/2]$

$g(z)$ crosses the z-axis at $z=0$ and $z = 1/2$

$g'(z)$ changes sign exactly once in $[0, 1/2]$ at $z_1 = \frac{1}{2} - \sqrt{1/4 - 1/e^2}$.

$g''(z) \leq 0$ for $z \leq 1/2$ $\Rightarrow$ $z_1 = \frac{1}{2} - \sqrt{1/4 - 1/e^2} \in [0, 1/2]$ is a maxima of $g(z)$.

$$g(x) = \int_{x}^{1-x} \log t \, dt + \frac{(1-x)-x}{\ln 2}$$

$$= \frac{1}{\ln 2} \int_{x}^{1-x} (\ln t + 1) \, dt$$

$\ln(t)+1=0$ at $t = 1/e < 1/2$

$\therefore$ For $1/e < t < 1/2$ we have $\ln(t)+1>0$

$\therefore$ For x near the right end of $[0, 1/2]$ we have $g(x)>0$.

$\Rightarrow$ $g(x)$ increases after crossing the x-axis at $x=0$ until maximum at $x_1 = \frac{1}{2} - \sqrt{1/4 - 1/e^2}$ and then decreases until it hits the x-axis again at $x = 1/2$.

$\therefore$ $g(x) = \eta(x) - \eta(x-1) \geq 0$ for $x \in [0, 1/2]$

$\therefore$ $\eta(x) \geq \eta(x-1)$ for $x \in [0, 1/2]$

(2) $\log(z)$ is concave on $(0,\infty)$.

$$0 \le z \le 1/2 \implies 0 < z \le 1-z \le 1$$

$$z \le 1-z \le 1$$

$\therefore$ Slope of the chord from $(z, \log(z))$ to $(1-z, \log(1-z))$ $\ge$ Slope of the chord from $(1-z, \log(1-z))$ to $(1, \log(1))$.

$$\frac{\log(1-z) - \log(z)}{1-2z} \ge \frac{0 - \log(1-z)}{1-(1-z)}$$

$$\frac{\log(1-z) - \log(z)}{1-2z} \ge \frac{-\log(1-z)}{z}$$

$$0 \le z \le 1/2 \Rightarrow -1 \le -2z \le 0 \implies \underline{0 \le 1-2z \le 1}.$$

$$z\log(1-z) - z\log(z) \ge -(1-2z)\log(1-z)$$

$$-z\log(z) \ge (-1+2z-z)\log(1-z)$$

$$-z\log(z) \ge -(1-z)\log(1-z)$$

$$\underline{\eta(z) \ge \eta(1-z)}$$

$\boxed{\text{Method 2}}$

Assume that $y \geq x$,

we need to prove that

$$\left| -y \log y + x \log x \right| \leq -(y-x) \log (y-x)$$

where $x, y \in [0,1]$ with $y - x \leq 1/2$.

With $u \geq 0$,

$$\int_0^1 \frac{u(u-1)}{1+(u-1)t}\,dt = \frac{1}{\cancel{u-1}} \int_0^1 \frac{u(\cancel{u-1})}{1+(u-1)t}\,dt = u\Big[\log\big(1+(u-1)t\big)\Big]_0^1$$

$$= u \log u.$$

$$\boxed{u \log u = \int_0^1 \frac{u(u-1)}{1+(u-1)t}\,dt}$$

$$\left| \int_0^1 \left(\frac{y(1-y)}{1+(y-1)t} - \frac{x(1-x)}{1+(x-1)t} \right) dt \right| \leq \int_0^1 \frac{(y-x)(1-(y-x))}{1+(y-x-1)t}\,dt$$

$$\int_0^1 \left| \frac{y(1-y)}{1+(y-1)t} - \frac{x(1-x)}{1+(x-1)t} \right| dt \leq \int_0^1 \frac{(y-x)(1-(y-x))}{1+(y-x-1)t}\,dt$$

It suffices to prove that, for all $t \in [0,1]$ and $0 \le x \le y \le 1$ with $y - x \le 1/2$,

$$\left| \frac{y(1-y)}{1+(y-1)t} - \frac{x(1-x)}{1+(x-1)t} \right| \le \frac{(y-x)(1-(y-x))}{1+(y-x-1)t}$$

$$\left(\frac{y(1-y)}{1+(y-1)t} - \frac{x(1-x)}{1+(x-1)t} \right)^2 \le \left(\frac{(y-x)(1-(y-x))}{1+(y-x-1)t} \right)^2$$

$$S(\rho) = -\operatorname{tr}(\rho \log \rho) = -\sum_i r_i \log r_i$$

$$S(\sigma) = -\sum_i s_i \log s_i$$

$$|S(\rho) - S(\sigma)| = \left| \sum_i \left(-r_i \log r_i + s_i \log s_i \right) \right|$$

$$= \left| \sum_i \left(\eta(r_i) - \eta(s_i) \right) \right|$$

* Whenever $|r-s| \le 1/2$, we have

$$|\eta(r) - \eta(s)| \le \eta(|r-s|)$$

When $2T(\rho,\sigma) \le 1/e$,

$$\left. \begin{aligned} 2T(\rho,\sigma) &\le 1/e < 1/2 \\ 2T(\rho,\sigma) &\ge \sum_i |r_i - s_i| \end{aligned} \right\} \sum_i |r_i - s_i| < 1/2$$

$$\therefore |r_i - s_i| \le 1/2 \quad \forall\, i$$

$$|S(\rho)-S(\sigma)| = \left|\sum_i \left(\eta(r_i)-\eta(s_i)\right)\right| \leq \sum_i \eta\left(|r_i-s_i|\right) \qquad \text{—— (11.47)}$$

Setting $\Delta \equiv \sum_i |r_i - s_i|$,

$$\Delta\eta\left(|r_i-s_i|/\Delta\right) - |r_i-s_i|\log\Delta =$$

$$= \cancel{\Delta}\left(-|r_i-s_i|/\cancel{\Delta}\right)\log\left(|r_i-s_i|/\Delta\right) - |r_i-s_i|\log\Delta$$

$$= -|r_i-s_i|\left(\log\left(\frac{|r_i-s_i|}{\Delta}\times\Delta\right)\right)$$

$$= -|r_i-s_i|\log\left(|r_i-s_i|\right) = \eta\left(|r_i-s_i|\right)$$

$\therefore$

$$|S(\rho)-S(\sigma)| \leq \sum_i \left\{\Delta\eta\left(|r_i-s_i|/\Delta\right) - |r_i-s_i|\log\Delta\right\}$$

$$= \Delta\sum_i \eta\left(|r_i-s_i|/\Delta\right) - \log\Delta\sum_i |r_i-s_i|$$

$$= \Delta\sum_i \eta\left(|r_i-s_i|/\Delta\right) - \Delta\log\Delta$$

$$\left|S(\rho)-S(\sigma)\right| \le \Delta \sum_i \eta\left(|r_i-s_i|/\Delta\right)+\eta(\Delta)$$

where $\Delta = \sum_i |r_i - s_i|$

<u>Theorem 11.2:</u> If X is a random variable with d outcomes. Then $H(X) = -p(x)\sum_x \log p(x) \le \log d$.

$$\sum_i \left(\frac{|r_i-s_i|}{\Delta}\right) = \frac{\sum_i |r_i-s_i|}{\Delta} = 1$$

$$\frac{|r_i-s_i|}{\Delta} = \frac{|r_i-s_i|}{\sum_i |r_i-s_i|} \le 1$$

$\therefore \frac{|r_i-s_i|}{\Delta}$ form a probability distribution.

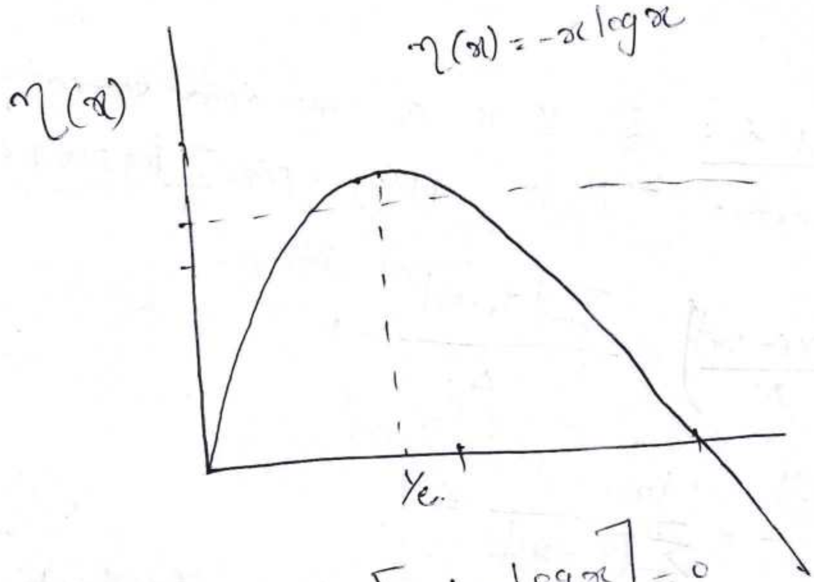
$$\Rightarrow -\sum_i \frac{|r_i-s_i|}{\Delta}\log\left(|r_i-s_i|/\Delta\right) = \sum_i \eta\left(|r_i-s_i|\right) \le \log(d)$$

$\therefore$

$$\left|S(\rho)-S(\sigma)\right| \le \Delta \log(d)+\eta(\Delta) \quad \text{—— (11.48)}$$

Eqⁿ (11.46) $\Rightarrow \quad 2T(\rho,\sigma) \geq \sum_i |r_i - s_i| = \Delta$

$\Rightarrow \quad \Delta \leq 2T(\rho,\sigma)$



$\eta'(x) = 0 \Rightarrow \frac{1}{\ln(2)}\left[-1 - \log x\right] = 0$

$\ln(x) = -1 = \ln(1/e) \Rightarrow \underline{x = 1/e}$

$\eta(x)$ is increasing (strictly) on the interval $[0, 1/e]$.

$|S(\rho) - S(\sigma)| \leq \Delta\log(d) + \eta(\Delta)$

$\leq 2T(\rho,\sigma)\log(d) + \eta(2T(\rho,\sigma))$

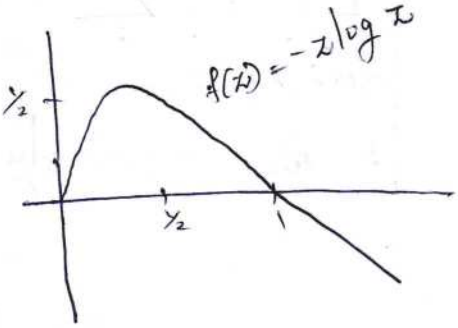
Basic properties of von Neumann entropy

① The entropy is non-negative. The entropy is zero iff the state is pure.

$$\boxed{\begin{array}{l} S(\rho) \geq 0 \\ S(\rho) = 0 \text{ iff } \rho \text{ is pure} \end{array}}$$

Proof

$$S(\rho) = -\operatorname{tr}(\rho \operatorname{tr} \rho) = -\sum_x \lambda_x \log \lambda_x$$



$$S(\rho) = -\sum_x \lambda_x \log \lambda_x = 0$$

$$\Longrightarrow \lambda_x = 0 \text{ (or) } 1$$

② In a d-dimensional Hilbert space the entropy is at most $\log d$. The entropy is equal to $\log d$ iff the system is in the completely mixed state I/d.

$$\boxed{\begin{array}{l} S(\rho) \leq \log d \\ S(\rho) = \log d \quad \text{iff} \quad \rho = I/d \end{array}}$$

Proof

Let $\rho = \sum_i p_i |i\rangle\langle i|$ is the orthonormal decomposition of ρ and $\sigma = I/d = \frac{1}{d}\sum_i |i\rangle\langle i|$ is the completely mixed state.

$$\begin{aligned} S(\rho\|\sigma) &= tr(\rho \log \rho) - tr(\rho \log \sigma) \\ &= -S(\rho) - tr(\rho \log \sigma) \\ &= -S(\rho) - \sum_i \langle i|\rho \log \sigma|i\rangle \\ &= -S(\rho) - \sum_i p_i \langle i|\log \sigma|i\rangle \end{aligned}$$

$$= -S(\rho) - \sum_i p_i \log \tfrac{1}{d} \langle i | i \rangle$$

$$= -S(\rho) + \log(d) \sum_i p_i$$

$$S(\rho \| \mathbb{I}/d) = -S(\rho) + \log d \geq 0$$

$$\Rightarrow \underline{\underline{S(\rho) \leq \log(d)}}$$

③ Suppose a composite system AB is in a pure state. Then $S(A) = S(B)$

$$\boxed{AB \text{ is a pure state} \implies S(A) = S(B)}$$

<u>Proof</u> An arbitrary bipartite pure state is given as,

$$|\psi\rangle = \sum_i \sum_j c_{ij} |e_{1,i}\rangle \otimes |e_{2,j}\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$$

The coefficients c_{ij} form a matrix $C_{d_1 \times d_2}$.

The SVD of C is, $C = U \Sigma V^{\dagger}$, where U, V are unitary operators.

QC② QC①

$$|\psi\rangle = \sum_{i,j} \sum_{kl} U_{ik} \Sigma_{kl} V^*_{jl} |e_{1,i}\rangle \otimes |e_{2,j}\rangle$$

$$= \sum_{i,j,k,l} \Sigma_{kl} \cdot U_{ik} |e_{1,i}\rangle \otimes V^*_{jl} |e_{2,j}\rangle$$

$$= \sum_{k,l} \Sigma_{kl} \left(\sum_i U_{ik} |e_{1,i}\rangle \otimes \sum_j V^*_{jl} |e_{2,j}\rangle \right)$$

$$= \sum_{k,l} \Sigma_{kl} |f_{1,k}\rangle \otimes |f_{2,l}\rangle$$

where, $|f_{1,k}\rangle = \sum_i U_{ik} |e_{1,i}\rangle$ and $|f_{2,k}\rangle = \sum_j V^*_{jl} |e_{2,j}\rangle$

such that $\| |f_{1,k}\rangle \|^2 = \| |f_{2,l}\rangle \|^2 = 1$, $\langle f_{1,k} | f_{1,m} \rangle = \delta_{km}$,

$\langle f_{2,k} | f_{2,m} \rangle = \delta_{km}$

$\therefore$ $|f_{1,k}\rangle$, $|f_{2,l}\rangle$ form an orthonormal basis for $\mathbb{C}^{d_1}$, $\mathbb{C}^{d_2}$, respectively.

$$\Sigma_{kl} = d_k \delta_{kl}$$

$$|\psi\rangle = \sum_{k,l} d_k \delta_{kl} |f_{1,k}\rangle \otimes |f_{2,l}\rangle$$

$$= \sum_{i=1}^{r} d_i |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

where, r : # of non-vanishing diagonal elements in Σ.

Normalization condition $\Rightarrow \langle\psi|\psi\rangle = \sum_i d_i^2 = 1$

$\therefore$ Give a pure state $|\psi\rangle$ of a composite system AB, there exists orthonormal states $|f_{1,i}\rangle$ for system A, and orthonormal states $|f_{2,i}\rangle$ of system B such that,

$$|\psi\rangle = \sum_i d_i |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

where d_i are non-negative real numbers satisfying $\sum_i d_i^2 = 1$ known as Schmidt coefficients.

$$\rho_A = tr_B(\rho) = tr_B\left(|\psi\rangle\langle\psi|\right)$$

$$= tr_B\left[\left(\sum_{i=1}^{r} d_i |f_{1,i}\rangle \otimes |f_{2,i}\rangle\right)\left(\sum_{j=1}^{r} d_j \langle f_{1,j}| \otimes \langle f_{2,j}|\right)\right]$$

$$= tr_B\left[\sum_{i,j} d_i d_j \left(|f_{1,i}\rangle \otimes |f_{2,i}\rangle\right)\left(\langle f_{1,j}| \otimes \langle f_{2,j}|\right)\right]$$

$$= tr_B\left[\sum_{i,j=1}^{r} d_i d_j \, |f_{1,i}\rangle\langle f_{1,j}| \otimes |f_{2,i}\rangle\langle f_{2,j}|\right]$$

$$tr(A+B) = tr(A) + tr(B)$$

$$= \sum_{i,j=1}^{r} tr_B\left[d_i d_j \, |f_{1,i}\rangle\langle f_{1,j}| \otimes |f_{2,i}\rangle\langle f_{2,j}|\right]$$

Partial trace defined on the Schmidt basis,

$$tr_B(\rho) = \sum_i \left(I_A \otimes \langle f_{2,i}|\right) \rho \left(I_A \otimes |f_{2,i}\rangle\right)$$

$$= \sum_{i,j=1}^{r}\left[\sum_{k=1}^{r}\left(I_A \otimes \langle f_{2,k}|\right)\left(d_i d_j \, |f_{1,i}\rangle\langle f_{1,j}| \otimes |f_{2,i}\rangle\langle f_{2,j}|\right)\left(I_A \otimes |f_{2,k}\rangle\right)\right]$$

$$= \sum_{i,j=1}^{r}\left[\sum_{k=1}^{r} d_i d_j \, |f_{1,i}\rangle\langle f_{1,j}| \otimes \langle f_{2,k}|f_{2,i}\rangle\langle f_{2,j}|f_{2,k}\rangle\right]$$

$$= \sum_{i,j=1}^{r}\left[\sum_{k=1}^{r} d_i d_j \, |f_{1,i}\rangle\langle f_{1,j}| \otimes \delta_{ki}\, \delta_{jk}\right]$$

$$= \sum_{i=1}^{r} d_i^2 \, |f_{1,i}\rangle\langle f_{1,i}|$$

$$\therefore \quad \rho_A = \sum_{i=1}^{r} d_i^2 \, |f_{1,i}\rangle\langle f_{1,i}|$$

$$\rho_B = \sum_{i=1}^{r} d_i^2 \, |f_{2,i}\rangle\langle f_{2,i}|$$

$\Rightarrow$ The eigenvalues of ρ_A and ρ_B are identical and is the square of the Schmidt coefficients, namely d_i^2 for both density matrices.

and the entropy is determined completely by the eigenvalues, $S(\rho) = -\mathrm{tr}(\rho \log \rho) = -\sum_x \lambda_x \log \lambda_x$.

$$\therefore \quad \underline{\underline{S(A) = S(B)}}$$

(4) Suppose p_i are probabilities, and the states ρ_i have support on orthogonal subspaces. Then,

$$\boxed{S\left(\sum_i p_i \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i)}$$

- The support of a matrix M is the orthogonal complement of its kernel, i.e., $\text{supp}(M) = \ker(M)^{\perp}$.

<u>Proof</u> $S(\rho) = -\text{tr}(\rho \log \rho) = -\sum_x \lambda_x \log \lambda_x$

Let λ_i^j and $|\lambda_i^j\rangle$ be the eigenvalues and corresp. eigenvectors of ρ_i

$$\left(\sum_i p_i \rho_i\right)|\lambda_i^j\rangle = p_i \lambda_i^j |\lambda_i^j\rangle$$ Since the states ρ_i have support on orthogonal subspaces

$\Rightarrow p_i \lambda_i^j$ and $|\lambda_i^j\rangle$ are the eigenvalues and eigenvectors of $\sum_i p_i \rho_i$.

$$\therefore\; S\left(\sum_i p_i \rho_i\right) = -\sum_{i,j} p_i \lambda_i^j \log(p_i \lambda_i^j)$$

$$= -\sum_{i,j} p_i \lambda_i^j \left[\log(p_i) + \log(\lambda_i^j)\right]$$

$$= -\left(\sum_i p_i \log p_i\right) \sum_j \lambda_i^j - \sum_i p_i \sum_j \lambda_i^j \log \lambda_i^j$$

$$= -\sum_i p_i \log p_i - \sum_i p_i \sum_j \lambda_i^j \log \lambda_i^j$$

$$= H(p_i) + \sum_i p_i S(\rho_i)$$

- ρ_i have support on orthogonal subspaces.

$\Longrightarrow$ If a given ρ_i has an eigenvector $|\lambda_i^j\rangle$ with a non-zero eigenvalue λ_i^j, then $|\lambda_i^j\rangle$ is also an eigenvector of all other ρ_k for $k \neq i$, but $\lambda_k^j = 0$.

⑤ Joint entropy theorem

Suppose p_i are probabilities, $|i\rangle$ are orthogonal states for a system A, and ρ_i is any set of density operators for another system B. Then,

$$\boxed{S\left(\sum_i p_i |i\rangle\langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i)}$$

Proof

$$\rho_{AB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i \quad \text{and} \quad \rho_i |\lambda_i^j\rangle = \lambda_i^j |\lambda_i^j\rangle$$

$$\left(|i\rangle\langle i| \otimes \rho_i\right)\left(|i\rangle \otimes |\lambda_i^j\rangle\right) = |i\rangle \otimes \rho_i |\lambda_i^j\rangle = \lambda_i^j \left(|i\rangle \otimes |\lambda_i^j\rangle\right)$$

$$\rho_{AB}\left(|i\rangle \otimes |\lambda_i^j\rangle\right) = \left(\sum_i p_i |i\rangle\langle i| \otimes \rho_i\right)\left(|i\rangle \otimes |\lambda_i^j\rangle\right)$$

$$= p_i \lambda_i^j \left(|i\rangle \otimes |\lambda_i^j\rangle\right)$$

$\Rightarrow$ The eigenvectors of $\rho_{AB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i$ are $|i\rangle|\lambda_i^j\rangle$ with eigenvalue $p_i \lambda_i^j$.

$\therefore$

$$S\left(\sum_i p_i |i\rangle\langle i| \otimes \rho_i\right) = -\sum_{i,j} p_i \lambda_i^j \log p_i \lambda_i^j$$

$$= -\sum_{i,j} p_i \lambda_i^j \left(\log p_i + \log \lambda_i^j\right)$$

$$= -\sum_i p_i \log p_i \sum_j \lambda_i^j - \sum_i p_i \sum_j \lambda_i^j \log \lambda_i^j$$

$$= -\sum_i p_i \log p_i - \sum_i p_i \sum_j \lambda_i^j \log \lambda_i^j$$

$$= H(p_i) + \sum_i p_i \, S(\rho_i)$$

Ex: 11.13 Entropy of a tensor product

Use the joint entropy theorem to show that $\boxed{S(\rho\otimes\sigma) = S(\rho) + S(\sigma)}$. Prove this result directly from the definition of the entropy.

Ans:

Joint entropy theorem $\Rightarrow$

Method :)

$$S\left(\sum_i p_i |i\rangle\langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i)$$

Let $\rho = \sum_i \lambda_i |i\rangle\langle i|$ be the spectral decomposition

Set $p_i = \lambda_i$, $\rho_i = \sigma$ for all i,

$$S(\rho\otimes\sigma) = S\left(\sum_i \lambda_i |i\rangle\langle i| \otimes \sigma\right) = H(\lambda_i) + \sum_i \lambda_i S(\sigma)$$

$$= -\sum_i \lambda_i \log \lambda_i + S(\sigma) \sum_i \lambda_i$$

$$= S(\rho) + S(\sigma)$$

Method: 2 Let $\rho = \sum_i \lambda_i |i\rangle\langle i|$ and $\sigma = \sum_j \mu_j |j\rangle\langle j|$

$$\rho \otimes \sigma = \left(\sum_i \lambda_i |i\rangle\langle i|\right) \otimes \left(\sum_j \mu_j |j\rangle\langle j|\right)$$

$$= \sum_{i,j} \lambda_i |i\rangle\langle i| \otimes \mu_j |j\rangle\langle j|$$

$$= \sum_{i,j} \lambda_i \mu_j |i\rangle\langle i| \otimes |j\rangle\langle j|$$

$$S(\rho \otimes \sigma) = -\sum_{i,j} \lambda_i \mu_j \log(\lambda_i \mu_j)$$

$$= -\sum_{i,j} \lambda_i \mu_j \left[\log(\lambda_i) + \log(\mu_j)\right]$$

$$= -\sum_i \lambda_i \log \lambda_i \sum_j \mu_j - \sum_j \mu_j \log \mu_j \sum_i \lambda_i$$

$$= -\sum_i \lambda_i \log \lambda_i - \sum_i \mu_j \log \mu_j$$

$$= \underline{\underline{S(\rho) + S(\sigma)}}$$

Method: 2 Let $\rho = \sum_i \lambda_i |i\rangle\langle i|$ and $\sigma = \sum_j \mu_j |j\rangle\langle j|$

$$\rho \otimes \sigma = \left(\sum_i \lambda_i |i\rangle\langle i|\right) \otimes \left(\sum_j \mu_j |j\rangle\langle j|\right)$$

$$= \sum_{i,j} \lambda_i |i\rangle\langle i| \otimes \mu_j |j\rangle\langle j|$$

$$= \sum_{i,j} \lambda_i \mu_j \, |i\rangle\langle i| \otimes |j\rangle\langle j|$$

$$S(\rho \otimes \sigma) = -\sum_{i,j} \lambda_i \mu_j \log(\lambda_i \mu_j)$$

$$= -\sum_{i,j} \lambda_i \mu_j \left[\log(\lambda_i) + \log(\mu_j)\right]$$

$$= -\sum_i \lambda_i \log \lambda_i \sum_j \mu_j - \sum_j \mu_j \log \mu_j \sum_i \lambda_i$$

$$= -\sum_i \lambda_i \log \lambda_i - \sum_i \mu_j \log \mu_j$$

$$= \underline{\underline{S(\rho) + S(\sigma)}}$$

The joint entropy $S(A,B)$ for a composite system with 2 components A and B is defined as,

$$S(A,B) \equiv -\mathrm{tr}\left(\rho^{AB} \log \rho^{AB}\right)$$

ρ_{AB} : density matrix of the system AB

We define the conditional entropy and mutual information by:

$$S(A|B) \equiv S(A,B) - S(B)$$

$$S(A:B) \equiv S(A) + S(B) - S(A,B)$$

$$= S(A) - S(A|B) = S(B) - S(B|A)$$

Some properties of the Shannon entropy fail to hold for the von Neumann entropy, and this has many interesting consequences for quantum information theory.

Ex:-

For random variables X and Y, the inequality $H(X) \leq H(X,Y)$ holds.

This makes intuitive sense: surely we can't be more uncertain about the state of X than we are about the joint state of X and Y.
This intuition fails for quantum states!

Consider a system AB of 2 qubits in the entangled state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, which is a pure state.

$$\Longrightarrow S(A,B) = 0$$

The system A has density operator $I/2$, and thus has entropy equal to 1.

$$S(A) = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = -\log \tfrac{1}{2} = \log 2 = 1$$

$\Longrightarrow S(B|A) = S(A,B) - S(A)$ is $-$ve.

Ex: 11-

Ans:

Ex: 11.14 Entanglement and negative conditional entropy

Suppose $|AB\rangle$ is a pure state of a composite system belonging to Alice and Bob. Show that $|AB\rangle$ is entangled iff $S(B|A) < 0$.

$|AB\rangle$ is entangled iff $S(B|A) < 0$ where $|AB\rangle$ is a pure state of a composite system.

ρ is pure $\Rightarrow S(\rho) = 0$

Ans: $|AB\rangle$ is a pure state $\Longrightarrow S(A,B) = 0$

$S(B|A) = S(A,B) - S(A) = -S(A) < 0$

- If the initial multipartite system was entangled, the reduced system is left in a mixed state.

A is pure state $\Longleftrightarrow S(A) = 0$

AB is entangled $\Longleftrightarrow S(A) > 0$

(A is mixed) $\Longleftrightarrow S(B|A) < 0$

□ Measurements & Entropy

Suppose,

a projective measurement described by projectors P_i is performed on a quantum system, but we never learn the result of the measurement. If the state of the system before the measurement was ρ then the state after is given by,

$$\rho' = \sum_i P(i) \rho_i'$$

$$= \sum_i tr\left(P_i^\dagger P_i \rho\right) \frac{P_i \rho P_i^\dagger}{tr\left(P_i^\dagger P_i \rho\right)}$$

$$= \sum_i P_i \rho P_i^\dagger$$

$$= \sum_i P_i \rho P_i$$

The following result shows that the entropy is never decreased by this procedure, and remains constant only if the state is not changed by the measurement.

Theorem 11.9 : $\boxed{\text{Projective measurements increase entropy}}$

Suppose P_i is a complete set of orthogonal projectors and ρ is a density operator. Then the entropy of the state $\rho' \equiv \sum_i P_i \rho P_i$ of the system after the measurement is atleast as great as the original entropy,

$$\boxed{S(\rho') \geq S(\rho)}$$

with equality if and only if $\rho = \rho'$.

Proof

Applying Klein's inequality to ρ and ρ',

$$S(\rho \| \rho') = -S(\rho) - tr(\rho \log \rho') \geq 0$$

The result will follow if we can prove that $-tr(\rho \log \rho') = S(\rho')$.

Applying the completeness relation $\sum_i P_i = I$, the relation $P_i^2 = P_i$, and the cyclic property of the trace:

$$-tr(\rho \log \rho') = -tr(I \rho \log \rho')$$
$$= -tr\left(\sum_i P_i \rho \log \rho'\right)$$
$$= -tr\left(\sum_i P_i^2 \rho \log \rho'\right)$$
$$= -tr\left(\sum_i P_i \rho \log \rho' P_i\right)$$

$$\rho' P_i = \left(\sum_j P_j \rho P_j\right) P_i = P_i \rho P_i^2 = P_i \rho P_i = P_i \rho'$$

$$\log(B) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(B-I)^k}{k}$$
$$= (B-I) - \frac{(B-I)^2}{2} + \frac{(B-I)^3}{3} - \frac{(B-I)^4}{4} + \cdots$$

if $\|B-I\| < 1$ then the series convergest and $e^{\log(B)} = B$.

For ρ' , $0 \leq \lambda_i \leq 1$

$$\Rightarrow -1 \leq \lambda_i - 1 \leq 0$$

$$\Rightarrow \|\rho' - I\| \leq 1$$

where, $(\rho')^\dagger = \rho' \Rightarrow (\rho' - I)^\dagger = \rho' - I$

$\therefore$ $\rho' - I$ is hermitian

$\therefore$ $\sigma^{max}_{(\rho' - I)} = |\lambda_i|_{max}$

• P_i commutes with $\rho' \Rightarrow P_i$ commutes with $\log \rho'$

$$\therefore -tr(\rho \log \rho') = -tr\left(\sum_i P_i \rho \log \rho' P_i\right)$$

$$= -tr\left(\sum_i P_i \rho P_i \log \rho'\right)$$

$$= -tr\left(\rho' \log \rho'\right) = S(\rho')$$

$$S(\rho\|\rho') = -S(\rho) - tr(\rho \log \rho')$$

$$= -S(\rho) + S(\rho') \geq 0$$

$$\Longrightarrow S(\rho') \geq S(\rho)$$

Note:

Example

Start with the state $|+\rangle$ and measure it in the Z basis, so you get answers $|0\rangle$ and $|1\rangle$ with ½ probability each since

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

If we knew which result we have got, then you have a pure state and 0 entropy. But, if we don't know the result, our best description of the state is $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ which has 1 bit of entropy.

Ex: 11.15 Generalized measurements can decrease entropy.

Suppose a qubit in the state ρ is measured using the measurement operators $M_1 = |0\rangle\langle 0|$ and $M_2 = |0\rangle\langle 1|$. If the result of the measurement is unknown to us then the state of the system afterwards is, $\rho' = M_1 \rho M_1^\dagger + M_2 \rho M_2^\dagger$. Show that this procedure can decrease the entropy of the qubit.

Ans:

$$\rho' = M_1 \rho M_1^\dagger + M_2 \rho M_2^\dagger$$

$$= |0\rangle\underbrace{\langle 0|\rho|0\rangle}\langle 0| + |0\rangle\underbrace{\langle 1|\rho|1\rangle}\langle 0|$$

$$= \rho_{00} |0\rangle\langle 0| + \rho_{11} |0\rangle\langle 0|$$

$$= \begin{bmatrix} \rho_{00} + \rho_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho_{00} + \rho_{11} = tr(\rho) = 1$

$$S(\rho') = -tr(\rho' \log \rho')$$

$$= -\sum_x \lambda_x \log \lambda_x = 0$$

Subadditivity.

Suppose distinct quantum systems A and B have a joint state ρ^{AB}. Then the joint entropy of the 2 systems satisfies the inequalities:

(i) Subadditivity inequality for von Neumann entropy

$$\boxed{\begin{array}{l} S(A,B) \leq S(A) + S(B) \\ \text{with equality iff } \rho^{AB} = \rho^{A} \otimes \rho^{B} \\ \text{ie., systems A and B are uncorrelated} \end{array}}$$

Proof

Klein's inequality

$$\Rightarrow S(\rho \| \sigma) = -S(\rho) - tr(\rho \log \sigma) \geq 0$$

$$S(\rho) \leq -tr(\rho \log \sigma)$$

Setting $\rho = \rho^{AB}$ and $\sigma = \rho^{A} \otimes \rho^{B}$,

$$\boxed{\log\left(\rho^A \otimes \rho^B\right) = \log\left(\rho^A\right) \otimes I_B + I_A \otimes \log\left(\rho^B\right)}$$

Proof

Let $\rho^A = \sum_i \lambda_i^A |i\rangle\langle i|$ and $\rho^B = \sum_j \lambda_j^B |j\rangle\langle j|$

$$\rho^A \otimes \rho^B = \left(\sum_i \lambda_i^A |i\rangle\langle i|\right) \otimes \left(\sum_j \lambda_j^B |j\rangle\langle j|\right)$$

$$= \sum_{i,j} \lambda_i^A |i\rangle\langle i| \otimes \lambda_j^B |j\rangle\langle j|$$

$$= \sum_{i,j} \lambda_i^A \lambda_j^B |i\rangle\langle i| \otimes |j\rangle\langle j|$$

$$= \sum_{i,j} \lambda_i^A \lambda_j^B \left(|i\rangle \otimes |j\rangle\right)\left(\langle i| \otimes \langle j|\right)$$

which is the spectral decomposition of $\rho^A \otimes \rho^B$.

$$\log\left(\rho^A \otimes \rho^B\right) = \sum_{i,j} \log\left(\lambda_i^A \lambda_j^B\right) |i\rangle\langle i| \otimes |j\rangle\langle j|$$

$$= \sum_{i,j} \left(\log(\lambda_i^A) + \log(\lambda_j^B)\right) |i\rangle\langle i| \otimes |j\rangle\langle j|$$

$$= \sum_{i,j} \left[\log(\lambda_i^A) |i\rangle\langle i| \otimes |j\rangle\langle j|\right] + \left[|i\rangle\langle i| \otimes \log(\lambda_j^B) |j\rangle\langle j|\right]$$

$$= \sum_i \log(\lambda_i^A) |i\rangle\langle i| \otimes \sum_j |j\rangle\langle j|$$

$$+ \sum_i |i\rangle\langle i| \otimes \sum_j \log(\lambda_j^B) |j\rangle\langle j|$$

$$= \log(\rho^A) \otimes I_B + I_A \otimes \log(\rho^B)$$

Q: Stack
ANS: 3/3/2023

$$\boxed{tr\left(\rho^{AB}(\sigma^A \otimes I)\right) = tr\left(\rho^A \sigma^A\right)}$$

QC ②

Proof

Let's write $\rho^{AB} = \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'|$

and $\sigma^A = \sum_{a,a'} \sigma^A_{a,a'} |a\rangle\langle a'|$ where $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthonormal basis.

We can also write $\rho^A = \sum_{a,a'} \rho^A_{a,a'} |a\rangle\langle a'|$.

$\Rightarrow \langle a'|\rho^A|a\rangle = \rho^A_{a',a}$

$$\mathrm{tr}\left(\rho^A \sigma^A\right) = \sum_{a,a'} \rho^A_{a,a'} \,\mathrm{tr}\left(|a\rangle\langle a'|\,\sigma^A\right)$$

$$= \sum_{a,a'} \rho^A_{a,a'} \langle a'|\sigma^A|a\rangle$$

$$= \sum_{a,a'} \rho^A_{a,a'}\, \sigma^A_{a'a}$$

$$\mathrm{tr}\left(\rho^{AB}\left(\sigma^A \otimes I\right)\right) = \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} \,\mathrm{tr}\left(|a\rangle\langle a'|\sigma^A \otimes |b\rangle\langle b'|\right)$$

$$= \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} \,\mathrm{tr}\left(|a\rangle\langle a'|\sigma^A\right) \mathrm{tr}\left(|b\rangle\langle b'|\right)$$

$$= \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} \langle a'|\sigma^A|a\rangle \langle b'|b\rangle$$

$$= \sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} \langle a'|\sigma^A|a\rangle \cdot \delta_{b',b}$$

$$= \sum_{a,a',b} \rho^{AB}_{a,a',b,b}\; \sigma^A_{a',a}$$

$$\rho^A = \mathrm{tr}_B\left(\rho^{AB}\right) = \sum_{b''} \left(I \otimes \langle b''|\right) \rho^{AB} \left(I \otimes |b''\rangle\right)$$

$$= \sum_{b''} \left(I \otimes \langle b''|\right)\left(\sum_{a,a',b,b'} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes |b\rangle\langle b'|\right)\left(I \otimes |b''\rangle\right)$$

$$= \sum_{a,a',b,b',b''} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \otimes \langle b''|b\rangle\langle b'|b''\rangle$$

$$= \sum_{a,a',b,b',b''} \rho^{AB}_{a,a',b,b'} |a\rangle\langle a'| \, \delta_{b'',b} \, \delta_{b',b''}$$

$$= \sum_{a,a',b} \rho^{AB}_{a,a',b,b} |a\rangle\langle a'|$$

$$\rightarrow \quad \rho^{A}_{a,a'} = \langle a|\rho^{A}|a'\rangle = \sum_{b} \rho^{AB}_{a,a',b,b}$$

$\therefore$

$$tr\left(\rho^{AB}(\sigma^{A} \otimes I)\right) = \sum_{a,a',b} \rho^{AB}_{a,a',b,b} \, \sigma^{A}_{a',a}$$

$$= \sum_{a,a'} \left(\sum_{b} \rho^{AB}_{a,a',b,b}\right) \sigma^{A}_{a',a}$$

$$= \sum_{a,a'} \rho^{A}_{a,a'} \, \sigma^{A}_{a',a}$$

$$= tr\left(\rho^{A} \sigma^{A}\right)$$

$$-\mathrm{tr}(\rho \log \sigma) = -\mathrm{tr}\left(\rho^{AB} \log(\rho^A \otimes \rho^B)\right)$$

$$= -\mathrm{tr}\left(\rho^{AB}\left(\log(\rho^A) \otimes I_B + I_A \otimes \log(\rho^B)\right)\right)$$

$$= -\mathrm{tr}\left(\rho^{AB}\left(\log(\rho^A) \otimes I_B\right)\right) - \mathrm{tr}\left(\rho^{AB}\left(I_A \otimes \log(\rho^B)\right)\right)$$

$$= -\mathrm{tr}\left(\rho^A \log(\rho^A)\right) - \mathrm{tr}\left(\rho^B \log(\rho^B)\right)$$

$$= S(\rho^A) + S(\rho^B)$$

$$S(\rho^{AB}) \leq -\mathrm{tr}\left(\rho^{AB} \log \sigma\right) = S(\rho^A) + S(\rho^B)$$

$$\therefore \quad \underline{S(A,B) \leq S(A) + S(B)}$$

The equality conditions $\rho = \sigma$ for Klein's inequality give equality conditions $\rho^{AB} = \rho^A \otimes \rho^B$ for subadditivity.

(ii) Triangle inequality / Araki-Lieb inequality

$$\boxed{S(A,B) \geq |S(A) - S(B)|}$$

- It is the quantum analogue of the inequality $H(X,Y) \geq H(X)$

Proof.

Introduce a system R which purifies systems A and B.

* Applying subadditivity we have,

$$S(R) + S(A) \geq S(A,R)$$

ABR is in a pure state

$\Longrightarrow S(A,R) = S(B)$ and $S(R) = S(A,B)$

$$S(A,B) + S(A) \geq S(B)$$

$$\therefore \quad \underline{S(A,B) \geq S(B) - S(A)}.$$

By symmetry b/w the systems A and B we have, $S(A,B) \geq S(A) - S(B)$

$$\left.\begin{aligned} S(A,B) &\geq S(A) - S(B) \\ S(A,B) &\geq S(B) - S(A) \end{aligned}\right\} \quad \underline{S(A,B) \geq |S(A) - S(B)|}$$

Concavity of the entropy

- The entropy is a concave function of its inputs.

ie.,

Given probabilities p_i - non negative real numbers such that $\sum_i p_i = 1$, and the corresponding density operators ρ_i, the entropy satisfies the inequality:

$$\boxed{S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i)}$$

Intuition: $\sum_i p_i \rho_i$ expresses the state of a quantum system which is in an unknown state ρ_i with probability p_i. Our uncertainty about this mixture of states should be higher than the average uncertainty of the states ρ_i, since the state $\sum_i p_i \rho_i$ expresses ignorance not only due to the states ρ_i, but also a contribution due to our ignorance of the index i.

- Equality holds iff all the states ρ_i for which $p_i > 0$ are identical

ie.,
the entropy is a strictly concave function of its inputs.

Proof Suppose the ρ_i are states of a system A.

Introduce an auxiliary system B whose state space has an orthonormal basis $|i\rangle$ corresponding to the index i on the density operators ρ_i.

Define a joint state of AB by,

$$\rho^{AB} \equiv \sum_i p_i \rho_i \otimes |i\rangle\langle i|$$

$$Tr_B(\rho_1 \otimes \rho_2) = \rho_1 Tr(\rho_2) = \rho_1$$

$$\rho_A = Tr_B\left(\sum_i p_i \rho_i \otimes |i\rangle\langle i|\right)$$

$$= \sum_i p_i Tr_B(\rho_i \otimes |i\rangle\langle i|) = \sum_i p_i \rho_i$$

$$\rho_B = Tr_A\left(\sum_i p_i \rho_i \otimes |i\rangle\langle i|\right) = \sum_i p_i Tr_A(\rho_i \otimes |i\rangle\langle i|)$$

$$= \sum_i p_i |i\rangle\langle i|$$

$$S(A) = S\left(\sum_i p_i \rho_i\right)$$

$$S(B) = S\left(\sum_i p_i |i\rangle\langle i|\right) = -\sum_i p_i \log p_i = H(p_i)$$

$$S(A,B) = S\left(\sum_i p_i \rho_i \otimes |i\rangle\langle i|\right)$$

$$= H(p_i) + \sum_i p_i S(\rho_i)$$

$\left[\begin{array}{l}\text{Joint entropy theorem} \\ \text{(Check 1}^{\text{st}}\text{ half)}\end{array}\right]$

Applying the subadditivity inequality, $S(A,B) \le S(A) + S(B)$ obtains,

$$H(p_i) + \sum_i p_i S(\rho_i) \le S\left(\sum_i p_i \rho_i\right) + H(p_i)$$

$$\sum_i p_i S(\rho_i) \le S\left(\sum_i p_i \rho_i\right)$$

- The intuition behind the introduction of the auxiliary system B
  - We want to find a system of which, part is in the state $\sum_i p_i \rho_i$, where the value of i is not known. System B effectively ~~stores~~ stores the 'true' value of i; if A were truly in state ρ_i, the system B would be in state $|i\rangle\langle i|$, and observing system B in the $|i\rangle$ basis would reveal this fact

Ex : 11.1.8 Prove that equality holds in the concavity inequality iff all the ρ_i's are the same.

Ans: $\rho^{AB} = \sum_i p_i \rho_i \otimes |i\rangle\langle i|$

$\rho^A = \sum_i p_i \rho_i$ and $\rho^B = \sum_i p_i |i\rangle\langle i|$

$S(A,B) = S(A) + S(B)$ iff $\rho^{AB} = \rho^A \otimes \rho^B$

$\therefore$

$$\sum_i p_i \rho_i \otimes |i\rangle\langle i| = \sum_j p_j \rho_j \otimes \sum_k p_k |k\rangle\langle k|$$

Applying $I \otimes \langle m|$ on the left & $I \otimes |m\rangle$ on the right,

$$p_m \rho_m = \sum_j p_j \rho_j * p_m$$

$$P_m = \sum_j p_j P_j \quad \text{for all } m$$

$\therefore$ All P_m's are the same.

The entropy of a mixture of quantum states

For a mixture $\sum_i p_i \rho_i$ of quantum states ρ_i the following inequality holds:

$$\boxed{\sum_i p_i S(\rho_i) \leq S\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i S(\rho_i) + H(p_i)}$$

→ Our uncertainty about the state $\sum_i p_i \rho_i$ is never more than our average uncertainty about the state ρ_i, plus an additional contribution of $H(p_i)$ which represents the maximum possible contribution our uncertainty about the index i contributes to our total uncertainty.

Theorem 11.10: Suppose $\rho = \sum_i p_i \rho_i$, where p_i are some set of probabilities, and the ρ_i are density operators. Then,

$$\boxed{S(\rho) \leq \sum_i p_i S(\rho_i) + H(p_i)}$$

with equality iff the states ρ_i have support on orthogonal subspaces.

Proof.

Case 1 – $\rho_i = |\psi_i\rangle\langle\psi_i|$, pure state.

Suppose the ρ_i are states of a system A, and introduce an auxiliary system B with an orthonormal basis $|i\rangle$ corresponding to the index i on the probabilities p_i.

Define

$$|AB\rangle \equiv \sum_i \sqrt{p_i}\, |\psi_i\rangle |i\rangle$$

$$\rho^A = tr_B(|AB\rangle\langle AB|) = \sum_{i,j} \sqrt{P_i P_j}\, |\psi_i\rangle\langle\psi_j| \cdot tr(|i\rangle\langle j|)$$

$$= \sum_{i,j} \sqrt{P_i P_j}\, |\psi_i\rangle\langle\psi_j| \cdot \langle j|i\rangle = \sum_{i,j} \sqrt{P_i P_j}\, |\psi_i\rangle\langle\psi_j| \cdot \delta_{ij}$$

$$= \sum_i P_i |\psi_i\rangle\langle\psi_i| = \sum_i P_i \rho_i = \rho$$

$|AB\rangle$ is a pure state $\Rightarrow$

$$S(B) = S(A) = S\left(\sum_i P_i |\psi_i\rangle\langle\psi_i|\right) = S\left(\sum_i P_i \rho_i\right) = S(\rho)$$

We also have,

$$(I \otimes P_k)\rho^{AB}(I \otimes P_k) = \sum_{i,j} \sqrt{P_i P_j}\, |\psi_i\rangle\langle\psi_j| \otimes P_k |i\rangle\langle j| P_k$$

$$= P_k |\psi_k\rangle\langle\psi_k| \otimes |k\rangle\langle k|$$

$$\rho^{AB'} = \sum_k P_k \rho_k^{AB'} = \sum_k (I \otimes P_k)\rho^{AB}(I \otimes P_k)$$

$$= \sum_k P_k |\psi_k\rangle\langle\psi_k| \otimes |k\rangle\langle k|$$

$$\rho^{B'} = tr_A\left(\rho^{AB'}\right) = tr_A\left(\sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes |k\rangle\langle k|\right)$$

$$= \sum_k p_k |k\rangle\langle k| \cdot tr\left(|\psi_k\rangle\langle\psi_k|\right)$$

$$= \sum_k p_k |k\rangle\langle k| \cdot \langle\psi_k|\psi_k\rangle$$

$$= \sum_k p_k |k\rangle\langle k|$$

If we perform a projective measurement on the system B in the $|i\rangle$ basis.
After the measurement the state of system B is,

$$\rho^{B'} = \sum_i p_i |i\rangle\langle i|$$

Theorem 11.9 : Projective measurements never decrease entropy.

$\therefore$

$$S(\rho) = S(A) = S(B) \leq S(B') = H(p_i)$$

$S(\rho) \leq H(p_i)$ where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$= \sum_i p_i P_i$$

$S(\rho_i) = 0$ for a pure state $\rho_i = |\psi_i\rangle\langle\psi_i|$

$\therefore$

$S(\rho) \leq H(p_i) + \sum_i p_i S(\rho_i)$ where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

The equality holds if and only if $B = B'$, Theorem 11.9

we have

$$\rho^B = tr_A(|AB\rangle\langle AB|) = tr_A\left(\sum_{i,j} \sqrt{p_i p_j}\, |\psi_i\rangle\langle\psi_j| \otimes |i\rangle\langle j|\right)$$

$$= \sum_{i,j} \sqrt{p_i p_j}\, tr(|\psi_i\rangle\langle\psi_j|)\, |i\rangle\langle j|$$

$$= \sum_{i,j} \sqrt{p_i p_j}\, \langle\psi_j|\psi_i\rangle\, |i\rangle\langle j|$$

$$\rho^{B'} = \sum_i p_i |i\rangle\langle i|$$

$\rho^{B'} = \rho^B$ iff the states $|\psi_i\rangle$ are orthonormal.

Case 2 - ρ_i is a mixed state

Let $\rho_i = \sum_j p_j^i |e_j^i\rangle\langle e_j^i|$ be orthonormal decompositions for the states ρ_i

So, $$\rho = \sum_i p_i \rho_i$$
$$= \sum_i p_i \sum_j p_j^i |e_j^i\rangle\langle e_j^i|$$
$$= \sum_{i,j} p_i p_j^i |e_j^i\rangle\langle e_j^i|$$

Apply the pure state result $S\left(\sum_k p_k |\psi_k\rangle\langle\psi_k|\right) \leq H(p_k)$
and $\sum_j p_j^i = 1$ for each i, and $\sum_{i,j} p_i p_j^i = \sum_i p_i \sum_j p_j^i = \sum_i p_i = 1$

$$S(\rho) = S\left(\sum_{i,j} p_i p_j^i |e_j^i\rangle\langle e_j^i|\right)$$
$$\leq H(p_i p_j^i)$$
$$= -\sum_{i,j} p_i p_j^i \log(p_i p_j^i)$$
$$= -\sum_{i,j} p_i p_j^i \left[\log(p_i) + \log(p_j^i)\right]$$

$$= - \sum_{i,j} p_i p_j^i \log(p_i) - \sum_{i,j} p_i p_j^i \log(p_j^i)$$

$$= - \sum_i p_i \log(p_i) \sum_j p_j^i - \sum_i p_i \sum_j p_j^i \log(p_j^i)$$

$$= - \sum_i p_i \log(p_i) - \sum_i p_i \sum_j p_j^i \log(p_j^i)$$

$$= H(p_i) - \sum_i p_i S(\rho_i)$$

Equality conditions for the pure state case

$S\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) = H(p_i)$ iff $|\psi_i\rangle$ are orthonormal

$\Rightarrow S\left(\sum_{i,j} p_i p_j^i |e_j^i\rangle\langle e_j^i|\right) = H(p_i p_j^i)$ iff $|e_j^i\rangle$ are orthonormal.

$\therefore$ $S(\rho) = \sum_i p_i S(\rho_i) + H(p_i)$ iff the states ρ_i have support on orthogonal subspaces.

Strong subadditivity

The strong subadditivity inequality for von Neumann entropies states that for a trio of quantum systems A, B, C,

$$\boxed{S(A,B,C) + S(B) \leq S(A,B) + S(B,C)}$$

* Suppose $f(A,B)$ is a real-valued function of two matrices A and B. Then f is said to be jointly concave in A and B if for all $0 \le \lambda \le 1$,

$$\boxed{f(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) \ge \lambda f(A_1, B_1) + (1-\lambda) f(A_2, B_2)}$$

* A real valued function $f(x)$ on an interval (a,b) is called concave, provided for each pair of points $x_1, x_2 \in (a,b)$ and each λ with $0 \le \lambda \le 1$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall \lambda \in [0,1]$$

Ex: 11.23 Joint concavity implies concavity in each input

Let $f(A,B)$ be a jointly concave function, then $f(A,B)$ is concave in A, with B held fixed.

Find a function of 2 variables that is concave in each of its i/p, but is not jointly concave.

* For matrices A and B,

$A \leq B$ if $B-A$ is a positive matrix

$A \geq B$ if $B \leq A$.

Ex: A6.1 $\quad$ $\leq$ is preserved under conjugation

(OR)

If $A \leq B$, show that $XAX^\dagger \leq XBX^\dagger$ for all matrices X

$\Rightarrow$

Ans: $A \leq B \implies B - A$ is a positive matrix

$XAX^\dagger \leq XBX^\dagger \implies XBX^\dagger - XAX^\dagger = X(B-A)X^\dagger$ is a positive matrix.

$v^\dagger (B-A) v \geq 0 \quad \forall \ v$

$u^\dagger X(B-A)X^\dagger u = (X^\dagger u)^\dagger (B-A)(X^\dagger u) \geq 0 \quad \forall \ X^\dagger u$

$\therefore \ u^\dagger \left[X(B-A)X^\dagger\right] u \geq 0 \quad \forall \ u$

$\implies X(B-A)X^\dagger$ is a positive matrix

(OR)

$$B - A = M^{\dagger} M$$

$$\Rightarrow X(B-A)X^{\dagger} = X M^{\dagger} M X^{\dagger} = (M X^{\dagger})^{\dagger} (M X^{\dagger})$$

$\therefore$ $X(B-A)X^{\dagger}$ is a positive matrix.

Ex: A 6.2 $\quad A \geq 0$ iff A is a positive operator.

Ans:

Ex: AG3 $\qquad \leq$ is a partial order

$\leq$ is transitive : $A \leq B$ and $B \leq C \Rightarrow A \leq C$

$\leq$ is asymmetric : $A \leq B$ and $B \leq A \Rightarrow A = B$

$\leq$ is reflexive : $A \leq A$

Ans

i) $A \leq B$ and $B \leq C \Rightarrow B - A$ & $C - B$ are positive

$$\Rightarrow v(B-A)v^{\dagger} \geq 0 \quad \forall \; v$$

and $v(C-B)v^{\dagger} \geq 0 \quad \forall \; v$

$$v(C-A)v^{\dagger} \geq 0 \quad \forall \; v$$

$\therefore A \leq C$

ii) $A \leq B$ and $B \leq A \Rightarrow B - A$ & $A - B$ are positive

$$\Rightarrow v(B-A)v^{\dagger} \geq 0 \quad \text{and} \quad v(A-B)v^{\dagger} \geq 0 \quad \forall \; v$$

$$\Rightarrow v(A-B)v^{\dagger} \leq 0 \quad \text{and} \quad v(A-B)v^{\dagger} \geq 0 \quad \forall \; v$$

$$\Rightarrow v(A-B)v^{\dagger} = 0 \quad \forall \; v \Longrightarrow A - B = 0$$

$$\Rightarrow A = B$$

Let A be an arbitrary matrix. We define the norm of A by,

$$\boxed{\|A\| \equiv \max_{\langle u|u\rangle=1} |\langle u|A|u\rangle|}$$

whereas,

$$\sigma_{max} = \max_{|x\rangle \neq 0} \frac{\| A|x\rangle \|}{\| |x\rangle \|} = \max_{\langle x|x\rangle=1} \langle x|A^{\dagger}A|x\rangle$$

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Ex: A 6.4 Suppose A has eigenvalues λ_i. Define λ to be the maximum of the set $|\lambda_i|$.

Then,

① $\|A\| \geq \lambda$

② When A is Hermitian, $\|A\| = \lambda$

③ When

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\|A\| = 3/2 > 1 = \lambda$$

Ans:

① When $|u\rangle$ is an (unit) eigenvector of A, such that $A|u\rangle = \lambda_u |u\rangle$ where $\lambda = |\lambda_u|$ is the largest.

Stuck 2/3/2023. $|\langle u|A|u\rangle| = |\lambda_u \langle u|u\rangle| = \lambda$

$$\therefore \quad \|A\| = \max_{\langle u|u\rangle = 1} |\langle u|A|u\rangle| \geq \lambda$$

2

A is hermitian $\Longrightarrow$ eigenvectors form an orthonormal Basis

$$\{|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle\}$$

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where,

$$A|e_k\rangle = \lambda_k |e_k\rangle \quad \text{and} \quad \langle e_j|e_k\rangle = \delta_{jk}$$

such that $A = \sum_k \lambda_k |e_k\rangle\langle e_k|$

A general unit vector $|u\rangle$ can be written as a linear combination of the eigenvector Basis, $u = \sum_i c_i |e_i\rangle$ such that $\|u\|^2 = \sum_i |c_i|^2 = 1$

We have,

$$\langle u|A|u\rangle = \left(\sum_i c_i^* \langle e_i|\right)\left(\sum_k \lambda_k |e_k\rangle\langle e_k|\right)\left(\sum_j c_j |e_j\rangle\right)$$

$$= \left(\sum_i c_i^* \langle e_i|\right) \sum_{k,j} \lambda_k c_j |e_k\rangle\langle e_k|e_j\rangle$$

$$= \left(\sum_i c_i^* \langle e_i|\right)\left(\sum_k \lambda_k c_k |e_k\rangle\right)$$

$$= \sum_{i,k} c_i^* c_k \lambda_k \langle e_i|e_k\rangle$$

$$= \sum_i |c_i|^2 \lambda_i$$

$\therefore$

$\|A\| = \max_{\langle u|u\rangle=1} |\langle u|A|u\rangle|$ is the maximum value of $\left|\sum_i |c_i|^2 \lambda_i\right|$, subjected to the constraint that $\sum_i |c_i|^2 = 1$.

Numbering the λ_i's and $|e_i\rangle$ so that λ_1 is the eigenvalue with highest magnitude, and the maximum value of $\left|\sum_i |c_i|^2 \lambda_i\right|$ is achieved when $c_1 = 1$ and $c_2 = \dots = c_n = 0$

The value achieved is $|\lambda_1|$.

$$\left(\sum_i |c_i|^2 \lambda_i\right) - \lambda_{min} = \sum_i |c_i|^2 (\lambda_i - \lambda_{min}) \geq 0$$

$$\Rightarrow \lambda_{min} \leq \sum_i |c_i|^2 \lambda_i$$

$$\lambda_{max} - \left(\sum_i |c_i|^2 \lambda_i\right) = \sum_i |c_i|^2 (\lambda_{max} - \lambda_i) \geq 0$$

$$\Rightarrow \sum_i |c_i|^2 \lambda_i \geq \lambda_{max}$$

$$\lambda_{min} \leq \sum_i |c_i|^2 \lambda_i \leq \lambda_{max} \implies$$

$$\left|\sum_i |c_i|^2 \lambda_i\right| \leq \max(|\lambda_{min}|, |\lambda_{max}|) = |\lambda_i|_{max}$$

(3) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

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$$\langle u|A|u\rangle = [a^* \; b^*] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = [a^* \; b^*] \begin{bmatrix} a \\ a+b \end{bmatrix}$$

$$= |a|^2 + ab^* + |b|^2 = 1 + ab^*$$

We have $\langle u|u\rangle = |a|^2 + |b|^2 = 1$

$$\Rightarrow u = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos t \, e^{i\theta_1} \\ \sin t \, e^{i\theta_2} \end{bmatrix}$$

$$\langle u|A|u\rangle = 1 + ab^* = 1 + \sin t \cos t \, e^{i(\theta_1 - \theta_2)}$$

$$= 1 + \frac{1}{2}\sin 2t \cdot e^{i(\theta_1 - \theta_2)}$$

$$= 1 + \frac{1}{2}\sin 2t \cdot e^{i\theta}$$

where $\theta = \theta_1 - \theta_2$

$$f(t,\theta) := |\langle u|A|u\rangle|^2 = \left(1 + \frac{1}{2}\sin 2t \, e^{i\theta}\right)\left(1 + \frac{1}{2}\sin 2t \, e^{-i\theta}\right)$$

$$= 1 + \frac{1}{2}\sin 2t \left[2\cos\theta\right] + \frac{1}{4}\sin^2 2t$$

$$= 1 + \sin 2t \cdot \cos\theta + \frac{1}{4}\sin^2 2t$$

$$\frac{\partial f}{\partial \theta} = -\sin 2t \cdot \sin\theta = 0 \Rightarrow \sin 2t = 0 \text{ (or) } \sin\theta = 0$$

$$\frac{\partial f}{\partial t} = 2\cos 2t \cos\theta + \frac{1}{4} \times 2\sin 2t \times \cos 2t \times 2$$

$$= 2\cos 2t \cos\theta + \sin 2t \cos 2t$$

$$= \cos 2t\left[2\cos\theta + \sin 2t\right] = 0$$

$\sin 2t = 0$: $\cos 2t = \pm 1$

$\cancel{2}\cos\theta = 0 \longrightarrow$

$\sin\theta = 0$: $\cos\theta = \pm 1$

$$\cos 2t\left[\pm 2 + \sin 2t\right] = 0$$

$\cos 2t = 0$ (or) $\sin 2t = \pm 2$. (Not possible)

$\Rightarrow \sin 2t = \pm 1$

$\sin 2t = 0, \cos\theta = 0 \Rightarrow f(t,\theta) = 1$

$\sin 2t = \pm 1, \cos\theta = \pm 1 \Rightarrow$

$$f(t,\theta) = 1 - 1 + \frac{1}{4} \quad \text{(or)} \quad 1 + 1 + \frac{1}{4}$$

$$= 1/4 \quad \text{(or)} \quad 9/4$$

$$\therefore \max_{\langle u|u\rangle = 1} |\langle u|A|u\rangle| = \sqrt{9/4} = \underline{3/2}$$

Ex: AG.5 $\longrightarrow$ AB and BA have the same eigenvalues.

Ans: Let A and B be square matrices of same order.

Case 1: If one of them is invertible, say A is then

$$A^{-1}(AB)A = BA$$

$\Rightarrow$ AB & BA are similar

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$\Rightarrow$ same eigenvalues.

Let $A = S^{-1}BS$

$$P_A(\lambda) = \det(A-\lambda I) = \det\left[S^{-1}BS - \lambda S^{-1}S\right]$$

$$= \det\left[S^{-1}(B-\lambda I)S\right] = \det(S^{-1})\det(B-\lambda I)\det(S)$$

$$= \det(S^{-1})\det(S)\det(B-\lambda I)$$

$$= \det(S^{-1}S)\det(B-\lambda I)$$

$$= \det(B-\lambda I) = P_B(\lambda)$$

Case 2: If A & B are not invertible.

Define 2 matrices C and D of order $n \times n$ as follows,

$$C = \left[\begin{array}{c|c} xI_n & A \\ \hline B & I_n \end{array}\right] \quad \text{and} \quad D = \left[\begin{array}{c|c} I_n & 0 \\ \hline -B & xI_n \end{array}\right]$$

where x is an indeterminate.

$$\det \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \det(A) \cdot \det(S) = \det(A) \cdot \det(D - CA^{-1}B)$$

$$\det \left[\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right] = \det(A) \det(D)$$

$$\det(CD) = \det \left[\begin{array}{c|c} xI_n - AB & xA \\ \hline 0 & xI_n \end{array}\right]$$

$$= \det(xI_n - AB) \cdot \det(xI_n) = x^n \det(xI_n - AB)$$

$$\det(DC) = \det \left[\begin{array}{c|c} xI_n & A \\ \hline 0 & xI_n - BA \end{array}\right] = \det(xI_n) \det(xI_n - BA)$$

$$= x^n \det(xI_n - BA)$$

$$\det(CD) = \det(DC) \Rightarrow \det(xI_n - AB) = \det(xI_n - BA)$$

~~Hence proved~~

$$P_{AB}(x) = P_{BA}(x)$$

Ex: A6.6 Suppose A & B are such that AB is hermitian. Show that $\|AB\| \le \|BA\|$

Ans: $\|A\| = \max_{\langle u|u\rangle=1} |\langle u|A|u\rangle|$

A is hermitian $\Longrightarrow$ $\|A\| = |\lambda|_{max}$

$P_{AB}(\lambda) = P_{BA}(\lambda)$

$\therefore \|AB\| = \|BA\|$

Ex: A6.7 Suppose A is +ve. Show that $\|A\| \leq 1$ if and only if $A \leq I$

Ans: $\|A\| = \max_{\langle u|u\rangle = 1} |\langle u|A|u\rangle| = |\lambda|_{max}$ since A is +ve

$\lambda_i \geq 0$

If $A \leq I$,

case 1

$(I - A) \geq 0 \Rightarrow 1 - \lambda_i \geq 0 \quad \forall i$

$\lambda_i \leq 1 \quad \forall i$

$\therefore \|A\| = |\lambda|_{max} \leq 1$

case 2 If $\|A\| \leq 1$,

$\|A\| = |\lambda|_{max} \leq 1$

A is +ve $\Rightarrow \lambda_i \geq 0$

$\lambda_i \leq \lambda_{max} \leq 1 \Rightarrow \lambda_i \leq 1 \quad \forall i$

$1 - \lambda_i \geq 0 \quad \forall i$

$\therefore I - A \geq 0 \Rightarrow \underline{\underline{A \leq I}}$

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Quality Manager, ITC Ltd.- ESPB,
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