Angel

KING

sooraj551729@gmail.com +91-9400635788

1

INDEX

Name : SOORAJ S. Subject :

Std. : Div. : Roll No. :

School / College :

S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		QUANTUM COMPUTATION		

Mathematical postulates of quantum mechanics

There are 5 postulates that fully encapsulate the mathematical modelling of quantum mechanics.

① The state (or wavefunction) of individual quantum systems is described by unit vector living in separate complex Hilbert spaces

A vector space with a well-defined inner product is called a Hilbert space.

② The probability of measuring a system in a given state is given by the modulus squared of the inner product of the o/p state and the current state of the system. This is known as Born's rule. Immediately after the measurement, the wavefunction collapses into that state.

③ Quantum operations are represented by unitary operators on the Hilbert space (a consequence of the Schrodinger eqⁿ).

④ The
is giv
product
Hilbert

⑤ Physic
the e
operat
i.e.,
For
there
acting
When
obtain
operato
Suppose
$c_1|\lambda_1\rangle +$
state,
change
to the

④ The Hilbert space of a composite system is given by the tensor product (Kronecker product) of the separate, individual Hilbert spaces.

⑤ Physical observables are represented by the eigenvalues of a Hermitian operator on the Hilbert space.

i.e., For any physical quantity (i.e., observable) a, there exists a corresp. Hermitian operator 'A' acting on the Hilbert space $\mathcal{H}$.

When we make a measurement of 'a', we obtain one of the eigenvalues λ_j of the operator A.

Suppose the system is in a superposition state $c_1|\lambda_1\rangle + c_2|\lambda_2\rangle$. If we measure 'a' in this state, then the state undergoes an abrupt change to one of the eigenstates corresp. to the observed eigenvalue.

Finite dimension vectors are a great way to represent the state of quantum systems when the physical variables only have discrete possibilities, eg., being here or there, up or down, left or right, etc. This treatment of quantum mechanics is often referred to as "matrix mechanics" and is very well suited to quantum information and quantum cryptography.

But, when we want to describe physical quantities that have continuous values, such as the position of a particle along a line?

In this case, we need a vector space of infinite and continuous dimension.

It's possible to define a Hilbert space on the set of continuous functions, eg., $f(x) = x^3 + 2x + 1$.

This is referred to as "wave mechanics".

☐ Single quantum state & the qubit

Postulate 1: The state of individual quantum systems are described by unit vectors living in separate complex Hilbert spaces.

Quantum state

* The collection of all relevant physical properties of a quantum system (e.g., position, momentum, spin, polarization) is known as the state of the system.

Exclusive states

When modelling the state of a given physical quantity (position, spin, polarization), two states are said to be exclusive if the fact of being in one of the states with certainty implies that there are no chances whatsoever of being in any of the other states.

Ex:- hypothetical situation:

We have a single quantum particle, and 3 quantum boxes. The whole system behaves quantum mechanically. If we know for certain that the particle is in box 1, then it's certainly not in box 2 or 3.

Ex:-

Imagine a particle that can move horizontally, vertically and diagonally. These 3 states are definitely not exclusive since moving diagonally can be thought of as moving horizontally & vertically at the same time.

The concept of orthogonal vectors and projection isn't unlike the concept of exclusivity (ie., a given vector has no component along any of its orthogonal vectors). Therefore, it makes sense to establish a connection b/w exclusive states and orthogonal vectors.

⟹ Given a quantum system with n exclusive states, each state will be represented by a vector from an orthonormal basis of a n-dimensional Hilbert space.

In quantum information, we're using quantum bits or qubits. Like the classical bit, a qubit only has two exclusive states, "quantum-0" and "quantum-1". But unlike the classical bit, the qubit behaves according to the laws of quantum mechanics.

Since a qubit only has 2 different exclusive states, the state/vector representing the quantum-0 and the quantum-1 should be 2-dimensional. The vectors $|0\rangle$ and $|1\rangle$ are conventionally represented by the vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This basis is known as the computational basis.

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

The states $|\pm\rangle$ are linear combinations of the computational basis (i.e., the quantum-0 and quantum-1)

$\longrightarrow$ quantum superposition

<u>Quantum superposition principle</u>

If a quantum system can be in the state $|0\rangle$, and can also be in $|1\rangle$, then quantum mechanics allows the system to be in any arbitrary state

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix}a\\b\end{bmatrix}$$

We say that $|\psi\rangle$ is in a superposition

of $|0\rangle$ a

'a' and

Ex:-

The stat

$|+\rangle$ an

$|0\rangle =$

of $|0\rangle$ and $|1\rangle$ with probability amplitudes 'a' and 'b'.

Ex:-

The state $|0\rangle$ is a superposition of $|+\rangle$ and $|-\rangle$ since

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Quantum measurement

Postulate 2: The probability of measuring a system in a given state is given by the modulus squared of the inner product of the o/p state and the current state of the system (Born's rule). Immediately after the measurement, the wavefunction collapses into that state.

Born's rule

Suppose, we have a quantum state $|\psi\rangle$ and an orthonormal basis $\{|\phi_1\rangle, \ldots, |\phi_n\rangle\}$. Then we can measure $|\psi\rangle$ w.r.t this orthonormal basis, i.e., we ask the quantum system which one of these states it's in.

The probability of measuring the state $|\phi_i\rangle$, $P(\phi_i)$ is:

$$\boxed{P(\phi_i) = |\langle \phi_i | \psi \rangle|^2}$$

Wave collapse

After the measurement is performed the original state collapses in the measured state, i.e., we're left with one of the states $|\phi_1\rangle, \dots, |\phi_n\rangle$.

Quantum measurement

From a mathematical point of view, applying Born's rule and then the wave collapse correspond to a quantum measurement.

Ex: Suppose we have $|\psi\rangle = |+\rangle$ and we measure it in the orthonormal basis $\{|\phi_0\rangle = |0\rangle, |\phi_1\rangle = |1\rangle\}$. Then the state could collapse to:

$$\begin{cases} |0\rangle & \text{with probability } |\langle 0|+\rangle|^2 = \frac{1}{2} \\ |1\rangle & \text{"} \qquad |\langle 1|+\rangle|^2 = \frac{1}{2} \end{cases}$$

This is like flipping a coin.

If we decide to measure in the basis $\{|+\rangle, |-\rangle\}$, then the outcome state will be:

$$\begin{cases} |+\rangle & \text{with probability } |\langle +|+\rangle|^2 = 1 \\ |-\rangle & \text{"} \qquad |\langle -|+\rangle|^2 = 0 \end{cases}$$

Ex:- If we treat the general case of a qubit in an unknown quantum state $|\psi\rangle = a|0\rangle + b|1\rangle$ and we measure in the computational basis, the outcome will be:

$$\begin{cases} |0\rangle & \text{with probability } |\langle 0|\psi\rangle|^2 = |a|^2 \\ |1\rangle & \text{"} \qquad |\langle 1|\psi\rangle|^2 = |b|^2 \end{cases}$$

If we measure the system in the basis $\{|+\rangle, |-\rangle\}$?

$$\langle \pm|\psi\rangle = \frac{1}{\sqrt{2}}(\langle 0| \pm \langle 1|)(a|0\rangle + b|1\rangle)$$
$$= \frac{1}{\sqrt{2}}(a\langle 0|0\rangle + b\langle 1|1\rangle) = \frac{a \pm b}{\sqrt{2}}$$

$$\langle \pm|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \pm 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{a \pm b}{\sqrt{2}}$$

$$\Longrightarrow |\pm\rangle \text{ with probability } \frac{|a \pm b|^2}{2}$$

Quantum Operations

A quantum operation transforms a quantum state to another quantum state, therefore, we must have it so that the norm of the vector is preserved.

* Quantum operations are represented by unitary operators on the Hilbert space

Popular quantum operations:

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Longleftrightarrow$ Pauli matrices

Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

<u>Ex:</u> What'd be the final state if we perform

1. $\mathbb{1}$ on the state $|0\rangle$?
2. X on " $|1\rangle$?
3. Z on " $|+\rangle$?
4. H on " $|1\rangle$?

Ans:

1. $\mathbb{1}|0\rangle = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = |0\rangle$

2. $X|1\rangle = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = |0\rangle$

3. $Z|+\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = |-\rangle$

4. $H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = |-\rangle$

<u>Ex:-</u> Hadam

1. $H|0\rangle = \frac{1}{\sqrt{\ }}$
2. $H|1\rangle =$
3. $H|+\rangle =$
4. $H|-\rangle =$

- The Had
may be

$H = \frac{1}{\sqrt{2}}$

The Had

$H^{\otimes n} = \frac{\ }{\sqrt{\ }}$

* $|\psi\rangle^{\otimes k} : |$

Ex:- $|$

Ex:- <u>Hadamard matrix</u>

1. $H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = |+\rangle$

2. $H|1\rangle = |-\rangle$

3. $H|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$

4. $H|-\rangle = |1\rangle$

- The Hadamard operator on one qubit may be written as:

$$H = \frac{1}{\sqrt{2}}\left\{(|0\rangle+|1\rangle)\langle 0| + (|0\rangle-|1\rangle)\langle 1|\right\}$$

The Hadamard transform on n-qubits,

$$\boxed{H^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y|}$$

* $|\psi\rangle^{\otimes k}$: $|\psi\rangle$ tensored with itself k times

Ex:- $\boxed{|\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle}$

$$H|+\rangle = \frac{1}{\sqrt{2}} H\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(H|0\rangle + H|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle + |1\rangle + |0\rangle - |1\rangle\right)$$

$$= |0\rangle$$

$\longrightarrow$ quantum interference

$|0\rangle$ undergoes constructive interference and $|1\rangle$ undergoes destructive interference.

Interference is a property of waves and it is an example of wave-particle duality inherent to quantum mechanics.

In the la
in variou
system yo
If our
of a ph
wave pl
excited
a laser

In the lab, quantum operations are performed in various ways, depending on the quantum system you're trying to manipulate.

If our qubit is represented by the polarization of a photon, we use quarter and half wave plates. If we use the ground and excited states of an atom, we can use a laser pulse.

State Space

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.

Quantum mechanics does not tell us, for a given physical system, what the state space of that system is, nor does it tell us what the state vector of the system is. Figuring that out for a specific system is a difficult problem for which physicists have developed many intricate and beautiful rules.

QED $\longrightarrow$ tells us what state spaces to use to give quantum descriptions of atoms and light.

The simplest quantum mechanical system is the qubit. A qubit has a 2D state space. Suppose $|0\rangle$ and $|1\rangle$ form an orthonormal basis (computational basis) for that state space. Then an arbitrary state vector in the state space can be written

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where, a, b are complex #.

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

Unlike a classical bit,

we cannot examine a qubit to determine its quantum state, ie, the values of α and β.

Instead, QM tells us that we can only acquire much more restricted information about the quantum state. When we measure a qubit we get either the result 0, with probability $|\alpha|^2$, or the result 1, with probability $|\beta|^2$. Naturally, $|\alpha|^2 + |\beta|^2 = 1$ since the probabilities must sum to 1.

QM 23

QC ② Bloch Sphere

→ a qubit's state is a unit vector in a 2D complex vector space.

- This dichotomy b/w the unobservable state of a qubit and the observations we can make lies at the heart of quantum computation and quantum information.

A qubit can exist in a continuum of states b/w $|0\rangle$ and $|1\rangle$ – until it is observed.

Geometric representation - qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \text{where } \alpha, \beta \in \mathbb{C}$$

with $|\alpha|^2 + |\beta|^2 = 1$

⇒ we may rewrite this as

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$$

OM 23

where θ, ϕ, γ are real numbers.

QC ② → Bloch Sphere

We can ignore the factor of $e^{i\gamma}$ out the front, because in the case of qubits, we know that a quantum state does not change if we multiply with any # of unit norm.

ie., $|\psi\rangle = e^{i\gamma}|\psi\rangle$

$\therefore$

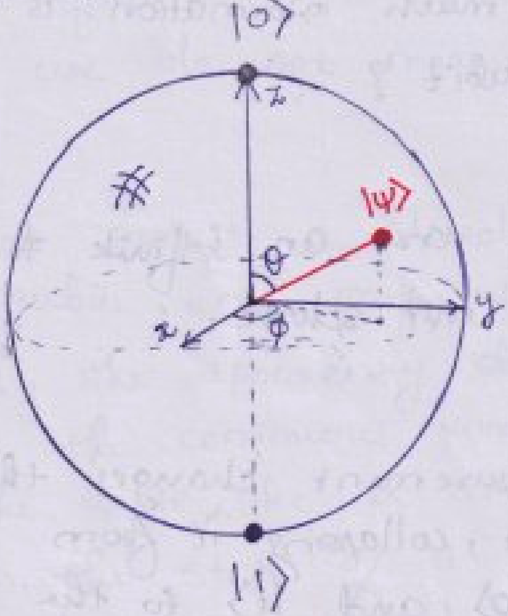
$$\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle}$$

The numbers θ and ϕ define a point on the unit 3D sphere. This sphere is often called the Bloch sphere; it provides a useful means of visualizing the state of a single qubit.

* There is no simple generalization of the Bloch sphere known for multiple qubits.

* Bloch sphere of a qubit

* Bloch sphere representation of a qubit



How much information is represented by a qubit ?

There are an infinite # of points on the unit sphere.

Measurement changes the state of a qubit, collapsing it from its superposition of $|0\rangle$ and $|1\rangle$ to the specific state consistent with the measurement result.

Why does this type of collapse occur ? Nobody knows.

$\Rightarrow$ one obtains only a single bit of information about the state of the qubit.

Only if infinitely many identically prepared qubits were measured would one be able to determine α and β for a qubit

How much
a qubit if

When Natu
system of
'measurement
track of al
describing t
a sense, i
Nature co
information'.

How much information is represented by a qubit if we do not measure it ?

When Nature evolves a closed quantum system of qubits, not performing any 'measurements', she apparently does keep track of all the continuous variables describing the state, like α and β. In a sense, in the state of a qubit, Nature conceals a great deal of 'hidden information'.

Evolution

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation. i.e., the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$|\psi'\rangle = U|\psi\rangle$$

Just as QM does not tell us the state space or quantum state of a particular quantum system, it does not tell us which unitary operators U describe real world quantum dynamics. Q.M merely assures us that the evolution of any closed quantum system may be described in such a way.

Ex:-

Quantum NOT gate : $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(Bit flip matrix)

$$X \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Z gate : $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
(phase flip matrix)

- which leaves $|0\rangle$ unchanged and flips the sign of $|1\rangle$ to give $-|1\rangle$.

$$Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

Hadamard gate : $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

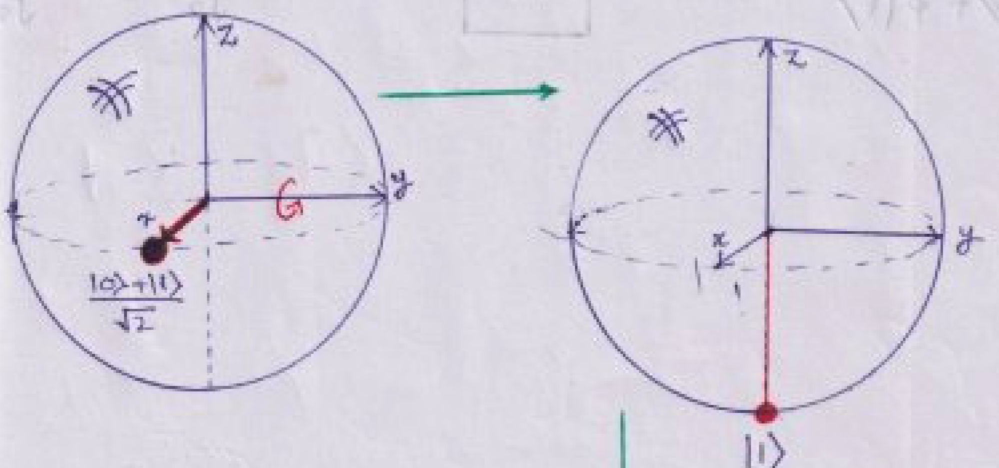
In the Q
gates corres
the sphere

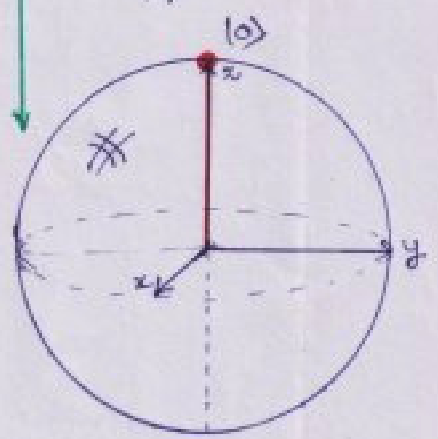
- Hadamard
the sphere
by a rota

* Visualization of
Hadamard gate
Bloch sphere,
the i/p state

In the Bloch sphere picture, single qubit gates corresp. to rotations and reflections of the sphere.

- Hadamard operation is just a rotation of the sphere about the $\hat{y}$-axis by 90°, followed by a rotation about the $\hat{x}$-axis by 180°.





* Visualization of the Hadamard gate on the Bloch sphere, acting on the i/p state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{X}—\; \beta|0\rangle + \alpha|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{Z}—\; \alpha|0\rangle - \beta|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{H}—\; \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

* Postulate 2 describes how the quantum states of a closed quantum system at 2 different times are related. A more refined version of this postulate can be given which describes the evolution of a quantum system in continuous time.

Postulate 2': The time evolution of the state of a closed quantum system is described by the Schrödinger equation,

$$\boxed{i\hbar \frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle}$$

where,
H is a fixed Hermitian operator known as the Hamiltonian of the closed system.

* If we know the Hamiltonian, then (together with a knowledge of $\hbar$) we understand its dynamics completely, at least in principle.

In general,

figuring out the Hamiltonian needed to describe a particular physical system is a very difficult problem - much of the 20th century physics has been concerned with this problem. - which requires substantial i/p from experiment in order to be answered.

Since the Hamiltonian is a Hermitian operator, it has a spectral decomposition:

$$H = \sum_{E} E|E\rangle\langle E|$$

with eigenvalues E and corresp. normalized eigenvectors $|E\rangle$.

The states
energy eige
States, anc
because th
acquire a

$|E\rangle$ -

Ex:-

Suppose. c

H

ω is a
to be expe

The ener
are the s
namely
corresp.

The grou
and the

The states $|E\rangle$ are referred to as energy eigenstates, or sometimes as stationary states, and E is the energy of the state $|E\rangle$, because their only change in time is to acquire an overall numerical factor,

$$|E\rangle \longrightarrow \exp\left(-iEt/\hbar\right)|E\rangle$$

Ex:-

Suppose a single qubit has Hamiltonian

$$H = \hbar\omega X$$

ω is a parameter that, in practice, needs to be experimentally determined.

The energy eigenstates of this Hamiltonian are the same as the eigenstates of X, namely $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ with corresp. energies $\hbar\omega$ and $-\hbar\omega$.

The ground state is therefore $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ and the ground state energy $-\hbar\omega$.

What's the connection b/w the Hamiltonian picture of dynamics & the unitary operator picture ?

The solution to the Schrodinger's eqⁿ.

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle \quad \text{is :}$$

$$|\psi(t_2)\rangle = \exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]|\psi(t_1)\rangle$$

$$= U(t_1, t_2)|\psi(t_1)\rangle$$

where we define :

$$U(t_1, t_2) = \exp\left[\frac{-iH(t_2-t_1)}{\hbar}\right]$$

which is unitary.

Any uni
in the f
Hermitian

There i
b/w the
dynamics
the contin
Hamiltonian

Any unitary operator U can be realized in the form $U = \exp(iK)$ for some Hermitian operator K.

$\therefore$ There is a one-to-one correspondence b/w the discrete-time description of dynamics using unitary operators, and the continuous time description using Hamiltonians.

In quantum computation & quantum information, we often speak of appying a unitary operator to a particular quantum system. Ex:- we may speak of applying the unitary gate X to a single qubit. Doesn't this contradict the postulate about unitary operators describing the evolution of a closed quantum system?

If we are 'applying' a unitary operator, then that implies that there is an external 'we' who is interacting with the quantum system, and the system is not closed ?

More gene

for

be possible

Hamiltonian

the Hamilt

constant,

parameters

control, &

the course

is not, t

according

a time-va

good appro

- Measuremen

most cases

- Energy en

More generally,

for many systems it turns out to be possible to write down a time-varying Hamiltonian for a quantum system, in which the Hamiltonian for the system is not a constant, but varies according to some parameters which are under an experimentalist's control, & which may be changed during the course of an experiment. The system is not, therefore, closed, but it does evolve according to Schrödinger's equation with a time-varying Hamiltonian, to some good approximation.

- Measurements destroy the quantum state in most cases
- Energy enters & leaves the system

Quantum Measurement

Closed quantum systems evolve according to unitary evolution.

When we observe the system to find out what is going on inside the system, an interaction which makes the system no longer closed, and thus not necessarily subject to unitary evolution.

Postulate 3: Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that the result m occurs is given by

$$p(m) = \| M_m|\psi\rangle \|^2 = \langle\psi| M_m^\dagger M_m |\psi\rangle$$

The state of the system after the measurement is:

$$|\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

The measurement operators satisfy the completeness equation,

$$\sum_m M_m^\dagger M_m = I$$

The completeness eq^n. expresses the fact that probabilities sum to one:

$$1 = \sum_m P(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle$$

* Measureme

of measure

not necessa

$\sum_m M_m^\dagger M =$

upon meas

after applic

* Measurements are described by a collection of measurement operators $\{M_m\}$ that are not necessarily projections but fulfill $\sum_m M_m^\dagger M = I$. The post measurement state upon measurement of m is the state after application of M_m.

Ex:- Measurement of a qubit in the computational basis.

This is a measurement on a single qubit with 2 outcomes defined by the two measurement operators: $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$.

Each measurement operator is Hermitian & $M_0^2 = M_0$, $M_1^2 = M_1$.

$$M_0^\dagger M_0 + M_1^\dagger M = M_0 + M_1 = I$$

$\rightarrow$ completeness relation is obeyed.

Suppose, the state being measured is $|\psi\rangle = a|0\rangle + b|1\rangle$. Then the probability of obtaining measurement outcome 0 is :

$$P(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = |a|^2$$

Similarly,

$$P(1) = |b|^2$$

The state is :

$$\frac{M_0|\psi\rangle}{|a|} =$$

$$\frac{M_1|\psi\rangle}{|b|}$$

* $\frac{a}{|a|}, \frac{b}{|b|}$ effectively states are

The state after measurement in the two cases is :

$$\frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle$$

$$\frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle$$

* $\frac{a}{|a|}$, $\frac{b}{|b|}$, which have modulus 1, can effectively be ignored, so the post-measurement states are effectively $|0\rangle$ and $|1\rangle$.

* The status of Postulate 3 as a fundamental postulate intrigues many people.

Paradox

Measuring devices are quantum mechanical systems, so the quantum system being measured and the measuring device together are part of a larger, isolated, quantum mechanical system. (It may be necessary to include quantum systems other than the system being measured and the measuring device to obtain a completely isolated system, but the point is that this can be done).

According to postulate 2, the evolution of this larger isolated system can be described by a unitary evolution.

Might it be possible to derive postulate 3 as a consequence of this picture?

Distinguishing Quantum States

Alice chooses a state $|\psi_i\rangle$ $(1 \le i \le n)$ from some fixed set of states known to both parties. She gives the state $|\psi_i\rangle$ to Bob, whose task it is to identify the index i of the state Alice has given him.

Suppose,

the states $|\psi_i\rangle$ are orthonormal;

then Bob can do a quantum measurement to distinguish these states, using the following procedure –

Define measurement operators $M_i = |\psi_i\rangle\langle\psi_i|$, one for each possible index i

If the state $|\psi_i\rangle$ is prepared then $P(i) = \langle\psi_i|M_i|\psi_i\rangle = 1$, so the result occurs with certainty.

$\therefore$ It is possible to reliably distinguish the orthonormal states $|\psi_i\rangle$.

If the states $|\psi_i\rangle$ are not orthonormal,

The idea is that, Bob will do a measurement described by measurement operators M_j, with outcome j. Depending on the outcome of the measurement Bob tries to guess what the index i was using some rule, $i = f(j)$, where $f(\cdot)$ represents the rule he uses to make the guess.

The key to why Bob can't distinguish non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ is the observation that $|\psi_2\rangle$ can be decomposed into a (non-zero) component ||el to $|\psi_1\rangle$ & a component orthogonal to $|\psi_1\rangle$.

Suppose, j is a measurement outcome such that $f(j) = 1$, that is, Bob guesses that the state was $|\psi_1\rangle$ when he observes j. But because of the component of $|\psi_2\rangle$ parallel to $|\psi_1\rangle$, there is a non-zero probability of getting outcome j when $|\psi_2\rangle$ is prepared, so sometimes Bob will make an error identifying which

state was

$\Rightarrow$ Non-orthog
distinguished

Proof

Suppose there
distinguishing
$|\psi_1\rangle$ and $|\psi$

If the state
the probability
$f(j) = 1$ $(f(j)$

Defining $\langle\phi|E$

these observat

$\langle\psi_1|E_1|\psi_1\rangle =$

state was prepared.

⇒ Non-orthogonal states can't be reliably distinguished.

Proof

Suppose there exists a measurement distinguishing the non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$. ~~is~~

If the state $|\psi_1\rangle$ $(|\psi_2\rangle)$ is prepared then the probability of measuring j such that $f(j)=1$ $(f(j)=2)$ must be 1.

Defining $E_i = \sum_{j \cdot f(j)=i} M_j^{\dagger} M_j$,

these observations may be written as:

$\langle\psi_1|E_1|\psi_1\rangle = 1$ & $\langle\psi_2|E_2|\psi_2\rangle = 1$

$$\sum_i E_i = I \implies \sum_i \langle \psi_1 | E_i | \psi_1 \rangle = 1$$

$$\langle \psi_1 | E_1 | \psi_1 \rangle = 1 \implies \langle \psi_1 | E_2 | \psi_1 \rangle = 0$$

$$\implies \sqrt{E_2}\, |\psi_1\rangle = 0$$

Suppose, we decompose $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle$

where $|\phi\rangle$ is orthonormal to $|\psi_1\rangle$.

$|\alpha|^2 + |\beta|^2 = 1$. and $|\beta| < 1$., since $|\psi_1\rangle$ and $|\psi_2\rangle$ are not orthogonal.

Then,

$$\sqrt{E_2}\, |\psi_2\rangle = \beta \sqrt{E_2}\, |\phi\rangle$$

$$\Rightarrow \langle \psi_2 | E_2 | \psi_2 \rangle = |\beta|^2 \langle \phi | E_2 | \phi \rangle \leq |\beta|^2 < 1$$

where we have used,

Since,

$$\langle \phi | E_2 | \phi \rangle \leq \sum_i \langle \phi | E_i | \phi \rangle = \langle \phi | \sum_i E_i | \phi \rangle$$

$$= \langle \phi | \phi \rangle = 1$$

This is a

This is a contradiction to $\langle \psi_2 | E_2 | \psi_2 \rangle = 1$

(A) Distinguishing Orthogonal States

- Consider 2 parties, Alice and Bob. Alice chooses a state $|\psi_i\rangle$ $(1 \leq i \leq n)$ from a fixed set of states known to both parties

- Alice gives Bob the state $|\psi_i\rangle$ and he must identify the index i of the state Alice has given him.

- Bob uses the following set of measurement operators $\{M_i\}_{i=1}^{n}$ where $M_i = |\psi_i\rangle\langle\psi_i|$ and $M_0 = I - \sum_{i\neq 0} |\psi_i\rangle\langle\psi_i|$, so that $\sum_m M_m^\dagger M_m = I$.

- If the state $|\psi_i\rangle$ was prepared by Alice, then the probability that measurement outcome i occurs with probability $P_i = \langle\psi_i|M_i|\psi_i\rangle = 1$, so that outcome i occurs with certainty and one can reliably distinguish orthonormal states.

(B) Distinguishing non-orthogonal states

Two different, non-orthogonal states are given, $|\psi_1\rangle$ and $|\psi_2\rangle$.

Suppose a distinguishing measurement is possible. If the possible measurement outcomes are denoted by m,
i.e., we can define measurement operators $\{M_1, M_2\}$ such that

- If state is prepared in $|\psi_1\rangle$, the probability of measuring $m=1$ is:

  $p(m=1) = \langle\psi_1|M_1^\dagger M_1|\psi_1\rangle = 1$

- If state is prepared in $|\psi_2\rangle$, the probability of measuring $m=2$ is:

Strictly speaking, there could be more than one measurement output that could characterize each states,

i.e., there could be more M_i's in the measurement operator set.

But, we'll take a simpler proof.

Completeness

$\Longrightarrow \langle \psi_1|$

$\langle \psi_1 | M_1^\dagger M_1 | \psi_1 \rangle$

Decompose

$|\phi\rangle$ is or

and $|\beta|$

$M_2|\psi_2\rangle =$

$=$

$\langle \phi | M_2^\dagger M_2 |$

Completeness equations, $M_1^\dagger M_1 + M_2^\dagger M_2 = \sum_{i=1}^{2} M_i^\dagger M_i = 1$

$$\Rightarrow \langle\psi_1|M_1^\dagger M_1|\psi_1\rangle + \langle\psi_1|M_2^\dagger M_2|\psi_1\rangle = 1$$

$$\langle\psi_1|M_1^\dagger M_1|\psi_1\rangle = 1 \quad \Rightarrow \langle\psi_1|M_2^\dagger M_2|\psi_1\rangle = 0$$

$$\Rightarrow M_2|\psi_1\rangle = 0$$

Decompose $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\phi\rangle$ where $|\phi\rangle$ is orthogonal to $|\psi_1\rangle$ and $|\alpha|^2+|\beta|^2 = 1$. and $|\beta| < 1$. since $\langle\psi_1|\psi_2\rangle < 1$.

$$\left.\begin{aligned} M_2|\psi_2\rangle &= \alpha M_2|\psi_1\rangle + \beta M_2|\phi\rangle \\ &= \beta M_2|\phi\rangle \end{aligned}\right\} \underline{\underline{M_2|\psi_2\rangle = \beta M_2|\phi\rangle}}$$

$$\begin{aligned} \langle\phi|M_2^\dagger M_2|\phi\rangle &\leq \langle\phi|M_1^\dagger M_1|\phi\rangle + \langle\phi|M_2^\dagger M_2|\phi\rangle \\ &= \langle\phi|M_1^\dagger M_1 + M_2^\dagger M_2|\phi\rangle \\ &= \langle\phi|\phi\rangle = 1 \end{aligned}$$

$M_2|\psi_2\rangle = \beta M_2|\phi\rangle$

$\therefore$

$$\langle\psi_2|M_2^\dagger M_2|\psi_2\rangle = |\beta|^2 \langle\phi|M_2^\dagger M_2|\phi\rangle$$

$$\leq |\beta|^2 < 1$$

since $\langle\phi|M_2^\dagger M_2|\phi\rangle \leq 1$.

Projective Measurements

A projective measurement is described by an observable, M, a Hermitian operator on the state space of the system being observed.

The observable has a spectral decomposition,

$$M = \sum_m m P_m = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

where P_m : projector onto the eigenspace of M with eigenvalue m.

The possible outcomes of the measurement correspond to the eigenvalues, m, of the observable.

upon measuring the state $|\psi\rangle$, the probability of getting result m is given by: $\boxed{p(m) = \langle\psi|P_m|\psi\rangle}$

Proof $\rightarrow$ $|\psi\rangle = \sum_i a_i |\lambda_i\rangle$ & $M = \sum \lambda_i |\lambda_i\rangle\langle\lambda_i| = \sum \lambda_i \Pi_i$

$a_i = \Pi_i |\psi\rangle = |\lambda_i\rangle\langle\lambda_i|\psi\rangle$

$Pr(\text{outcome } i) = |a_i|^2 = \| \Pi_i|\psi\rangle \|^2 = \langle\psi|\Pi_i|\psi\rangle$

Given that outcome m occurred, the state of the quantum system immediately after the measurement is:

$$|\psi'\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}}$$

$\frac{1 \le n(i)}{9}$ check last

Projective measurements can be understood as a special case of Postulate 3. Suppose, the measurement operators in Postulate 3, in addition to satisfying the completeness relation $\sum_m M_m^\dagger M_m = I$, also satisfy the conditions that M_m are orthogonal projectors, that is, M_m are Hermitian, and $M_m M_{m'} = \delta_{mm'} M_m$.

Two widely used nomenclatures for measurements are:

- Rather than giving an observable to describe a projective measurement, often people simply list a complete set of orthogonal projectors P_m satisfying the relations $\sum_m P_m = I$ and $P_m P_{m'} = \delta_{mm'} P_m$

The corresponding observable implicit in this usage is, $M = \sum_m m P_m$

- Measure in a basis $|m\rangle$

where $|m\rangle$ form an orthonormal basis

— simply means to perform the projective measurements with projectors $P_m = |m\rangle\langle m|$.

Expectation Value

The avg. value of the measurement is,

$$E(M) = \sum_m m\, p(m)$$
$$= \sum_m m \langle \psi | P_m | \psi \rangle$$
$$= \langle \psi | \sum_m m P_m | \psi \rangle$$

$$m|m\rangle = M|m\rangle$$
$$\sum_m m P_m = \sum_m m|m\rangle\langle m| = \sum_m M|m\rangle\langle m|$$
$$= M \sum_m |m\rangle\langle m| = M$$

$$= \langle \psi | M | \psi \rangle$$

$$\boxed{\therefore \langle M \rangle = \langle \psi | M | \psi \rangle = tr\left(M|\psi\rangle\langle\psi|\right) = tr(M\rho)}$$

Uncertainty

The standard deviation is a measure of the typical spread of the observed values upon measurement of M.

ie.,
The uncertainty ΔM of the observable M is a measure of the spread of results around the mean $\langle M \rangle$.

If we perform a large # of experiments in which the state $|\psi\rangle$ is prepared and the observable M is measured, then the standard deviation ΔM of the observed values is determined by the formula:

$$\boxed{\begin{aligned}\Delta M &= \sqrt{\langle (M - \langle M \rangle)^2 \rangle} \\ &= \sqrt{\langle M^2 \rangle - \langle M \rangle^2}\end{aligned}}$$

where, $\Delta M = \sqrt{(M - \langle M \rangle I)^2}$

Proof

$$(\Delta M)^2 = \sum_m \left(m - \langle M \rangle\right)^2 p(m)$$

$$= \sum_m \left(m - \langle M \rangle\right)^2 \langle \psi | P_m | \psi \rangle$$

$$= \sum_m \left(m^2 - 2m\langle M \rangle + \langle M \rangle^2\right) \langle \psi | P_m | \psi \rangle$$

$$= \sum_m m^2 \langle \psi | P_m | \psi \rangle - 2\langle M \rangle \sum_m m \langle \psi | P_m | \psi \rangle + \langle M \rangle^2 \sum_m \langle \psi | P_m | \psi \rangle$$

$$= \langle \psi | \sum_m m^2 P_m | \psi \rangle - 2\langle M \rangle \langle \psi | \sum_m m P_m | \psi \rangle + \langle M \rangle^2 \langle \psi | \sum_m P_m | \psi \rangle$$

$$A|m\rangle = m|m\rangle \implies A^2|m\rangle = m^2|m\rangle$$

$$\sum_m m^2 P_m = \sum_m m^2 |m\rangle\langle m| = \sum_m M^2 |m\rangle\langle m| = M^2 \sum_m |m\rangle\langle m| = M^2$$

$$= \langle \psi | M^2 | \psi \rangle - 2\langle M \rangle \langle \psi | M | \psi \rangle + \langle M \rangle^2 \langle \psi | \psi \rangle$$

=

$(\Delta M)^2 =$

$(\Delta M)^2 =$

=

=

$(M - \langle M \rangle$

$\Rightarrow (M$

$\sum_m (m -$

=

$(\Delta M)^2 =$

$$= \langle M^2\rangle - 2\langle M\rangle^2 + \langle M\rangle^2$$

$$(\Delta M)^2 = \langle M^2\rangle - \langle M\rangle^2$$

$$(\Delta M)^2 = \sum_m (m - \langle M\rangle)^2 p(m)$$

$$= \sum_m (m - \langle M\rangle)^2 \langle\psi|P_m|\psi\rangle$$

$$= \langle\psi|\sum_m (m - \langle M\rangle)^2 P_m|\psi\rangle$$

$$(M - \langle M\rangle I)|m\rangle = M|m\rangle - \langle M\rangle I|m\rangle$$

$$= m|m\rangle - \langle M\rangle|m\rangle = (m - \langle M\rangle)|m\rangle$$

$$\Rightarrow (M - \langle M\rangle I)^2|m\rangle = (m - \langle M\rangle)^2|m\rangle$$

$$\sum_m (m - \langle M\rangle^2)P_m = \sum_m (m - \langle M\rangle)^2 |m\rangle\langle m|$$

$$= \sum_m (M - \langle M\rangle I)^2 |m\rangle\langle m|$$

$$= (M - \langle M\rangle I)^2 \sum_m |m\rangle\langle m| = (M - \langle M\rangle I)^2$$

$$= \langle\psi|(M - \langle M\rangle I)^2|\psi\rangle$$

$$(\Delta M)^2 = \langle (M - \langle M\rangle I)^2\rangle = \langle (M - \langle M\rangle)\rangle$$

Ex:- of projective measurements on single qubits.

1, Measurement of the observable $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

has eigenvalues +1 and −1 with corresp. eigenvectors $|0\rangle$ and $|1\rangle$.

Measurement of Z on the state $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ gives the result +1 with probability $\langle\psi|0\rangle\langle 0|\psi\rangle = \frac{1}{2}$, and similarly the result −1 with probability ½.

2, We have a qubit in the state $|0\rangle$, & we measure the observable X. What's the avg. value of X ? std of X ?

Ans: $\langle X\rangle = \langle 0|X|0\rangle = \langle 0|\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle = 0$

$\Delta(X) = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{\langle 0|X^2|0\rangle} = \sqrt{\langle 0|0\rangle} = 1$

Suppose,

$\vec{v}$ is any 3D unit vector.

Then we can define an observable

$$\vec{v}\cdot\vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$$

Measurement of this observable is sometimes referred to as a 'measurement of spin along the $\vec{v}$-axis'.

3, Show that $\vec{v}\cdot\vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_{\pm} = (I \pm \vec{v}\cdot\vec{\sigma})/2$.

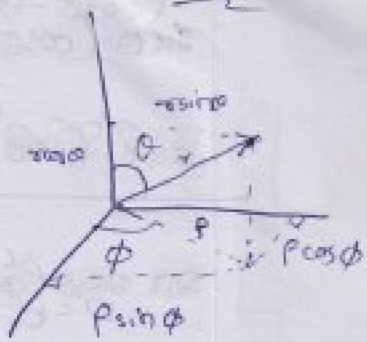
Ans: $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$r = 1$ here

$V_x = \sin\theta \cos\phi$

$V_y = \sin\theta \sin\phi$

$V_z = \cos\theta$



$$\vec{V}\cdot\vec{\sigma} = V_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + V_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + V_z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} V_z & V_x - iV_y \\ V_x + iV_y & -V_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}$$

$$\det(\vec{V}\cdot\vec{\sigma} - \lambda I) = 0 = \begin{vmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \lambda \end{vmatrix}$$

$$\cos^2\theta - \lambda^2 + \sin^2\theta = 1 - \lambda^2 = 0$$

$$\Longrightarrow \lambda = \pm 1$$

Solving $(\vec{V}\cdot\vec{\sigma} - \lambda I)\vec{x} = 0$

For $\lambda = 1$,

$$\begin{bmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$(\cos\theta - 1)a + \sin\theta\, e^{-i\phi}\, b = 0$

$\sin\theta\, e^{i\phi} a - (\cos\theta + 1)\, b = 0$

$$b = e^{i\phi}\frac{1-\cos\theta}{\sin\theta}\, a = e^{i\phi}\frac{\sin\theta/2}{\cos\theta/2}\, a$$

$$|a|^2 + |b^2| = 1 = |a|^2\left[1 + \frac{\sin^2\theta/2}{\cos^2\theta/2}\right]$$

$$|a|^2\left[\frac{1}{\cos^2\theta/2}\right] = 1$$

$$\longrightarrow \quad |a| = |\cos\theta/2|$$

The overall phase of the eigenstate is not observable, so we take the simplest option for a :

$$a = \cos\theta/2 \;;\; b = e^{i\phi}\sin\theta/2$$

$$\therefore \quad \boxed{|\vec{v};+\rangle = \cos\theta/2\,|+\rangle + \sin\theta/2\, e^{i\phi}\,|-\rangle}$$

Similarly,

$$\boxed{|\vec{v};-\rangle = \sin\theta/2\,|+\rangle - \cos\theta/2\, e^{i\phi}\,|-\rangle}$$

Projector

$|\vec{v};+\rangle\langle\vec{v};+$

$=$

$=$

$= \frac{1}{2}$

$= \frac{1}{2}$

$= \frac{1}{2}$

Projector onto $|\vec{v};+\rangle$ is:

$$|\vec{v};+\rangle\langle\vec{v};+| = \begin{bmatrix}\cos\theta/2 \\ \sin\theta/2\, e^{i\phi}\end{bmatrix}\begin{bmatrix}\cos\theta/2 & \sin\theta/2\, e^{-i\phi}\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta/2 & \sin\theta/2\cos\theta/2\, e^{-i\phi} \\ \sin\theta/2\cos\theta/2\, e^{i\phi} & \sin^2\theta/2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1+\cos\theta}{2} & \frac{\sin\theta\, e^{-i\phi}}{2} \\ \frac{\sin\theta\, e^{i\phi}}{2} & \frac{1-\cos\theta}{2}\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1+\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & 1-\cos\theta\end{bmatrix}$$

$$= \frac{1}{2}\left(I + \begin{bmatrix}\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(I + \vec{v}\cdot\vec{\sigma}\right)$$

The projectors onto the eigenvectors are:

$$\boxed{\begin{aligned} P_+ &= |\vec{v};+\rangle\langle\vec{v};+| = \frac{I+\vec{v}\cdot\vec{\sigma}}{2} \\ P_- &= |\vec{v};-\rangle\langle\vec{v};-| = \frac{I-\vec{v}\cdot\vec{\sigma}}{2} \end{aligned}}$$

4. Calculate the probability of obtaining the result +1 for a measurement of $\vec{v}\cdot\vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What's the state of the system after the measurement of +1 is obtained?

$$P_r(m=+1) = \langle +|\left(|\vec{v};+\rangle\langle\vec{v};+|\right)|+\rangle$$

$$= \langle +|\vec{v};+\rangle\langle\vec{v};+|+\rangle$$

$|+\rangle = \begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$

$$= \langle +|\frac{I+\vec{v}\cdot\vec{\sigma}}{2}|+\rangle$$

$$= \frac{1}{2}\langle +|+\rangle + \frac{1}{2}\langle +|\vec{v}\cdot\vec{\sigma}|+\rangle$$

$$= \frac{1}{2} + \frac{1}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \frac{1}{2} + \frac{1}{2}\cos\theta = \underline{\cos^2\theta/2}$$

State of

if +1

$|\psi'\rangle$

=

=

=

=

State of the system after the measurement if +1 is obtained:

$$|\psi'\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}} = \frac{|\vec{v};+\rangle\langle\vec{v};+|+\rangle}{\sqrt{\cos^2\theta/2}}$$

$$= \frac{\frac{1}{2}(I + \vec{v}\cdot\vec{\sigma})|+\rangle}{|\cos\theta/2|}$$

$$= \frac{1}{2}\,\frac{|+\rangle + (\cos\theta, \sin\theta\, e^{i\phi})}{|\cos\theta/2|}$$

$$= \frac{1}{2|\cos\theta/2|}\begin{bmatrix}\cos\theta+1\\ \sin\theta\, e^{i\phi}\end{bmatrix} = \frac{1}{2|\cos\theta/2|}\begin{bmatrix}2\cos^2\theta/2\\ 2\sin\theta/2\cos\theta/2\; e^{i\phi}\end{bmatrix}$$

$$= \pm 1\begin{bmatrix}\cos\theta/2\\ \sin\theta/2\, e^{i\phi}\end{bmatrix}$$

The Heisenberg uncertainty principle

Suppose A & B are 2 Hermitian operators. and $|\psi\rangle$ is a quantum state.

Suppose,

$$\langle\psi|AB|\psi\rangle = x+iy \text{ , where } x,y \in \mathbb{R}$$

Taking the conjugate transpose

$$\Rightarrow \langle\psi|BA|\psi\rangle = x-iy$$

$$\Rightarrow \langle\psi|[A,B]|\psi\rangle = 2iy \quad \& \quad \langle\psi|\{A,B\}|\psi\rangle = 2x$$

$$|\langle\psi|[A,B]|\psi\rangle|^2 + |\langle\psi|\{A,B\}|\psi\rangle|^2 = 4|\langle\psi|AB|\psi\rangle|^2$$

Cauchy-Schwarz inequality, $|u\cdot v|^2 \le |u|^2|v|^2$, $u = A|\psi\rangle$, $v = B|\psi\rangle$

$$|\langle\psi|AB|\psi\rangle|^2 \le \langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle$$

$$|2iy|^2 + |2x|^2 = 4(x^2+y^2) = 4|\langle\psi|AB|\psi\rangle|^2$$

$$\Rightarrow |\langle\psi|[A,B]|\psi\rangle|^2 = |2iy| = 4y^2 \le 4(x^2+y^2) = 4|\langle\psi|AB|\psi\rangle|^2$$

$$|\langle\psi|[A,B]|\psi\rangle|^2 \le 4\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle$$

Suppose C & D are 2 observables.

Let,

$$A = C - \langle C \rangle \quad \text{and} \quad B = D - \langle D \rangle$$
$$= C - \langle C \rangle I \qquad\qquad = D - \langle D \rangle I$$

$$\Rightarrow [A,B] = [C - \langle C \rangle, D - \langle D \rangle] = [C,D]$$

$$\Rightarrow \left| \langle \psi | [C,D] | \psi \rangle \right|^2 \leq 4 [\Delta C]^2 [\Delta D]^2$$

$$\boxed{\Delta(C)\Delta(D) \geq \frac{\left| \langle \psi | [C,D] | \psi \rangle \right|}{2}}$$

$\Rightarrow$ Heisenberg's uncertainty principle

- If we prepare a large # of quantum systems in identical states, $|\psi\rangle$, and then perform measurements of C on some of these systems, and of D in others, then the standard deviation $\Delta(C)$ of the C results times the standard deviation $\Delta(D)$ of the results for D will satisfy the inequality above.

Ex:- Consid

cohn me

$[x, y] = 2i$

Uncertainty P

$\Delta(X) \cdot \Delta$

$\Rightarrow$ $\Delta(X)$ an

greater th

Ex:- Consider the observables X and Y when measured for the quantum state $|0\rangle$.

$$[X, Y] = 2iZ$$

Uncertainty principle $\longrightarrow$

$$\Delta(X)\cdot\Delta(Y) \geq |\langle 0|Z|0\rangle| = |\langle 0|0\rangle| = 1$$

$$\boxed{\Delta(X)\cdot\Delta(Y) \geq 1}$$

$\Rightarrow$ $\Delta(X)$ and $\Delta(Y)$ must both be strictly greater than 0.

POVM measurements

(Positive Operator-valued Measure)

The quantum measurement postulate, Postulate 3, involves 2 elements. First, it gives a rule describing the measurement statistics, i.e., the respective probabilities of the different possible measurement outcomes. Second, it gives a rule describing the post-measurement state of the system.

However, for some applications the post-measurement state of the system is of little interest, with the main item of interest being the probabilities of the respective measurement outcomes.

Ex:-

in an experiment where the system is measured only once, upon conclusion of the experiment.

In such instances there is a mathematical tool known as the POVM formalism which is especially well adapted to the analysis of the measurements.

Suppose a measurement described by measurement operators M_m is performed upon a quantum system in the state $|\psi\rangle$. Then the probability of outcome m is given by $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$.

Suppose we define,

$$E_m = M_m^\dagger M_m$$

which is a positive operator, such that $\sum_m E_m = I$ and $p(m) = \langle\psi|E_m|\psi\rangle$. 1LQ(8)

The set of operators E_m are sufficient to determine the probabilities of the different measurement outcomes. The operators E_m are known as the POVM elements associated with the measurement. The complete set $\{E_m\}$ is known as a POVM.

Ex:-

Consider

measure

P_m are

and $\sum_m$

In this

elements

measurem

$E_m = P_m^\dagger$

* Any m

operators

is a proj

Ex:-

Consider a projective measurement described by measurement operators P_m, where the P_m are projectors such that $P_m P_{m'} = \delta_{mm'} P_m$ and $\sum_m P_m = I$.

In this instance (only this) all the POVM elements are the same as the measurement operators themselves, since

$E_m = P_m^+ P_m = P_m$.

* Any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Suppose,

$\{E_m\}$ is some arbitrary set of positive operators such that $\sum_m E_m = I$.

Defining $M_m \equiv \sqrt{E_m}$,

$$\sum_m M_m^{\dagger} M_m = \sum_m E_m = I$$

$\therefore$ The set $\{M_m\}$ describes a measurement with POVM $\{E_m\}$.

For this reason it is convenient to define a POVM to be any set of operators $\{E_m\}$ such that:

(a) Each operator E_m is positive

(b) the completeness relation $\sum_m E_m = I$ is obeyed, expressing the fact that probabilities sum to one.

Given a POVM $\{E_m\}$, the probability of outcome m is given by

$$p(m) = \langle \psi | E_m | \psi \rangle$$

Ex:-

Suppose Alice gives Bob a qubit prepared in one of the 2 states,

$$|\psi_1\rangle = |0\rangle \text{ (or) } |\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

It is impossible for Bob to determine whether he has been given $|\psi_1\rangle$ or $|\psi_2\rangle$ with perfect reliability. However,

it is possible for him to perform a measurement which distinguishes the states some of the time, but never makes an error of mis-identification.

Consider a POVM containing 3 elements,

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1| = \frac{1}{1+|\langle\psi_1|\psi_2\rangle|} |\psi_1^{\perp}\rangle\langle\psi_1^{\perp}|$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}$$

$$= \frac{1}{1+|\langle\psi_1|\psi_2\rangle|} |\psi_2^{\perp}\rangle\langle\psi_2^{\perp}|$$

$$E_3 = I - E_1 - E_2$$

These are positive operators which satisfy the completeness relation $\sum_m E_m = I$, and therefore form a legitimate POVM.

Suppose,

Bob is given the state $|\psi_1\rangle = |0\rangle$. He performs the measurement described by the POVM $\{E_1, E_2, E_3\}$. There is zero probability that he will observe the result E_1, since E_1 has been cleverly chosen to ensure that $\langle\psi_1|E_1|\psi_1\rangle = 0$

$\therefore$ If the result of his measurement is E_1 then Bob can safely conclude that the state he received must have been $|\psi_2\rangle$.

Similarly,
if th
then it
that Bob

Some of
the meas
can infer
the state

$\Longrightarrow$
identi
given

This infall
sometimes
about the

• This de
POVM
way of
measureme
only the

Similarly,

if the measurement outcome E_2 occurs then it must have been the state $|\psi_1\rangle$ that Bob received.

Some of the time, however, Bob will obtain the measurement outcome E_3, and he can infer nothing about the identity of the state he was given.

$\Longrightarrow$ Bob never makes a mistake identifying the state he has been given.

This infallibility comes at the price that sometimes Bob obtains no information about the identity of the state.

- This demonstrates the utility of the POVM formalism as a simple & intuitive way of gaining insight into quantum measurements in ~~...~~ instances where only the measurement statistics matter.

Ex(2.63). Suppose a measurement is described by measurement operators M_m. Show that there exist unitary operators U_m such that $M_m = U_m\sqrt{E_m}$, where E_m is the POVM associated to the measurement.

Ans: M_m is Hermitian $\Rightarrow M_m = V_m D_m V_m^\dagger$

Polar decomposition: $A = Q\Sigma V^\dagger = UV^\dagger (V\Sigma V^\dagger)$
$= QS$

ILA (19)

$Q = UV^\dagger$: unitary

$S = V\Sigma V^\dagger$: positive semi definite.

If A is invertible, S is +ve definite.

$$M_m = V_m D_m V_m^\dagger = \underbrace{V_m V_m^\dagger}\ \underbrace{V_m D_m V_m^\dagger}$$

$$= U_m J_m = U_m\sqrt{M_m^\dagger M_m} = U_m\sqrt{E_m}$$

Ex(2.64). Sup
state chos
of linearly
POVM $\{E$
E_i occurs
certainty
$|\psi_i\rangle$. (Th
$\langle\psi_i|E_i|\psi_i\rangle$ =

Ans: Constru
to all s

$|\psi_i'\rangle =$

Then.

$E_m =$

where A

$E_{m+1} =$

is posi

Ex(2.64). Suppose Bob is given a quantum state chosen from a set $|\psi_1\rangle, \ldots, |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, \ldots, E_{m+1}\}$ such that if outcome E_i occurs, $1 \le i \le m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$. (The POVM must be such that $\langle\psi_i|E_i|\psi_i\rangle = 0$ for each i).

Ans: Construct $|\psi_i'\rangle$ which is orthogonal to all states except $|\psi_i\rangle$:

$$|\psi_i'\rangle = |\psi_i\rangle - \sum_{\substack{k=1 \\ k\neq i}}^{m} \frac{\langle\psi_i|\psi_k\rangle|\psi_k\rangle}{\||\psi_k\rangle\|^2}$$

Then.

$$E_m = A|\psi_m'\rangle\langle\psi_m'|$$

where A is chosen such that

$$E_{m+1} = I - \sum_{j=1}^{m} E_j$$

is positive.

(OR) To distinguish the states $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_m\rangle\}$ we require

$$p(j|i) = \langle \psi_i | E_j | \psi_i \rangle = \mathrm{tr}\left(E_j |\psi_i\rangle\langle\psi_i|\right) = p_i \delta_{ij}$$

where, $p_i > 0$ and $1 \le i, j \le m$.

For each state $|\psi_i\rangle$, we can find a state $|\psi_i'\rangle$ that is orthogonal to the space spanned by $\{|\psi_j\rangle\}_{j\neq i}$, but not orthogonal to $|\psi_i\rangle$.

This is possible since all $|\psi_i\rangle$ are linearly independent, and we can construct an orthonormal basis using the Gram-Schmidt process.

$$|\psi_i'\rangle = |\psi_i\rangle - \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{|\psi_j\rangle\langle\psi_j|}{\| |\psi_j\rangle \|^2} |\psi_i\rangle$$

Construct

$E_i = A|\psi_i'\rangle$

where

$E_{m+1} = I -$

- Creating ... in the ... P_{m+1}.

Construct the operators,

$$E_i = A|\psi_i'\rangle\langle\psi_i'|, \quad 1 \le i \le m$$

where A is chosen such that,

$$E_{m+1} = I - \sum_{j=1}^{m} E_j$$

- Creating an optimal POVM is much trickier, in the sense of minimizing the probability P_{m+1}.

□ General measurements, Projective Measurements,
(Projection valued measure or PVM)

Positive Operator Valued Measure (POVM)

- When the other axioms of quantum mechanics are taken into account, projective measurements augmented by unitary operations turn out to be completely equivalent to general measurements.

Check composite systems

So a physicist trained in the use of projective measurements might ask to what end we start with the general formalism, Postulate 3?

Reasons.

1) mathematically general measurements are in some sense simpler than projective measurements, since they involve fewer restrictions on the measurement operators.

Ex:-

there is no requirement for ~~negative~~ general measurements analogous to the condition $P_i P_j = \delta_{ij} P_i$ for projective measurements.

2) There are important problems in ~~QM~~ QC & QI - such as the optimal way to distinguish a set of quantum states - the ans. to which involves a general measurement, rather than a projective measurement.

3) Projectiv
the sens
projective
outcome
gives th
change
Ex-
Suppo
After the
$|\psi\rangle$

Applying
it, so c

This re
tips us
measureme
measure

3) Projective measurements are repeatable in the sense that if we perform a projective measurement once, & obtain the outcome m, repeating the measurement gives the outcome m again & does not change the state.

Ex:-

Suppose $|\psi\rangle$: initial state

After the 1st measurement the state is,

$$|\psi_m\rangle = \frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}$$

Applying P_m to $|\psi_m\rangle$ does not change it, so we have $\langle\psi_m|P_m|\psi_m\rangle = 1$.

This repeatability of projective measurements tips us off to the fact that many important measurements in Q.M are not projective measurements.

For instance,

If we use a silvered screen to measure the position of a photon, we destroy the photon in the process. This makes it impossible to repeat the measurement of the photon's position!

For such QM measurements that are not repeatable, the general measurement postulate, Postulate 3, must be employed.

Where do POVMs fit in this picture?

POVMs are best viewed as a special case of the general measurement formalism, providing the simplest means by which one can study general measurement statistics, without the necessity for knowing the post-measurement state.

Phase

The state $e^{i\theta}|\psi\rangle$ is equal to $|\psi\rangle$, up to the global phase factor $e^{i\theta}$.

The statistics of measurement predicted for these 2 states are the same.

Suppose,

M_m is a measurement operator associated to some quantum measurement, the respective probabilities for outcome m occuring are $\langle\psi|M_m^\dagger M_m|\psi\rangle$ and $\langle\psi|e^{-i\theta}M_m^\dagger M_m e^{i\theta}|\psi\rangle = \langle\psi|M_m^\dagger M_m|\psi\rangle$.

$\therefore$

From an observational point of view these 2 states are identical.

$\Longrightarrow$ We may ignore global phase factors as being irrelevant to the observed properties of the physical system.

Composite Systems

Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 thro' n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

Ex. 2.66 Show that the avg. value of the observable $X_1 Z_2$ for a two qubit system measured in the state $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ is zero.

$$X\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\beta\\ \alpha\end{bmatrix}$$

$$Z = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\alpha\\ -\beta\end{bmatrix}$$

Ans: $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$E(X_1 Z_2) = \langle\psi|X_1 Z_2|\psi\rangle$$

$$= \frac{1}{2}\langle 00|X_1 Z_2|00\rangle + \frac{1}{2}\langle 00|X_1 Z_2|11\rangle$$

$$+ \frac{1}{2}\langle 11|X_1 Z_2|00\rangle + \frac{1}{2}\langle 11|X_1 Z_2|11\rangle$$

$$= \frac{1}{2}\langle 00|10\rangle - \frac{1}{2}\langle 00|01\rangle + \frac{1}{2}\langle 11|10\rangle$$

$$- \frac{1}{2}\langle 11|01\rangle$$

$$= \frac{1}{2}\, 0 - 0 + \frac{1}{2}\langle 1|1\rangle \otimes \langle 1|0\rangle - \frac{1}{2}\langle 1|0\rangle \otimes \langle 1|1\rangle$$

$$= 0 - 0 + 0 - 0 = 0$$

* Projective measurements together with unitary dynamics are sufficient to implement a general measurement.

<u>Proof</u>

Suppose, we have a quantum system with Hilbert Space, $\mathcal{H}_A$

We want to perform a general measurement described by measurement operators $\{M_m\}$ with $m = 1, 2, \cdots, n$ on the system $\mathcal{H}_A$

i.e.,
$\{M_m\}$ are a collection of measurement operators that are not projections but fulfill $\sum_m M_m^\dagger M_m = I$.

[illegible] to [illegible] measurements

BCO:H1

To do this,

we introduce an ancilla system, with state space. $M = \mathbb{C}^n$, having an orthonormal Basis $|m\rangle$, in one-to-one correspondence with the possible outcomes of the measurement we wish to implement.

Define an operator U

on products $|\psi\rangle|0\rangle = |\psi\rangle \otimes |0\rangle$ of states $|\psi\rangle$ from $\mathcal{H}_A$ with the state $|0\rangle \in \mathbb{C}^n = \mathcal{H}_B$ as :

$$U\left(|\psi\rangle \otimes |0\rangle\right) = \sum_{m=1}^{n} \left(M_m|\psi\rangle\right) \otimes |m\rangle$$

where $|m\rangle$ is the standard orthonormal basis of $\mathbb{C}^n$. & M_m satisfy the completeness relation $\sum_m M_m^\dagger M_m = I$.

$$\big(\langle\phi|\langle 0|\big)\big(|\psi\rangle|0\rangle\big) = \big(\langle\phi|\otimes\langle 0|\big)\big(|\psi\rangle\otimes|0\rangle\big)$$

$$= \langle\phi|\psi\rangle\otimes\langle 0|0\rangle \qquad \big[(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$$

$$= \langle\phi|\psi\rangle$$

$$\langle\phi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \Big(\sum_{m'}\langle\phi|M_{m'}^\dagger\,\langle m'|\Big)\Big(\sum_m M_m|\psi\rangle|m\rangle\Big)$$

$$= \sum_{m',m}\Big(\langle\phi|M_{m'}^\dagger\otimes\langle m'|\Big)\Big(M_m|\psi\rangle\otimes|m\rangle\Big)$$

$$= \sum_{m',m}\langle\phi|M_{m'}^\dagger M_m|\psi\rangle\otimes\langle m'|m\rangle$$

$$\langle m'|m\rangle = \delta_{m'm}$$

$$= \sum_m\langle\phi|M_m^\dagger M_m|\psi\rangle$$

$$= \langle\phi|\sum_m M_m^\dagger M_m|\psi\rangle$$

$$= \langle\phi|\psi\rangle = \big(\langle\phi|\langle 0|\big)\big(|\psi\rangle|0\rangle\big)$$

$\Rightarrow$ U

the fo

Look middle

Stuck 2

can we recover the bases of 2 infinitely dimensional vector spaces into tensor product?

If $\mathcal{H}_A$ & $\mathcal{H}$

and $\{|m\rangle\}$

$\langle\psi$

$\Rightarrow\big(\langle\psi_i|\otimes\langle m|$

stuck 3/9/21

If $\{|\psi_i\rangle\}$

for $\mathcal{H}_A$ and

orthonormal

$\rightarrow \{|\psi_i\rangle\otimes|0$

thus form

$\big(\langle\psi_i|\otimes\langle 0|U^\dagger\big)\big(U|\psi$

for all $|\psi_i\rangle$

$\Rightarrow$ U is a

spanned

$\rightarrow$ U preserves inner products of the form $|\psi\rangle|0\rangle$.

Stuck middle

Stuck 2
Can we recover the bases of 2 infinitely dimensional vector spaces into inner product?

If $\mathcal{H}_A$ & $\mathcal{H}_B$ are vector spaces with basis $\{|\psi_i\rangle\}$ and $\{|m\rangle\}$ then $\mathcal{H}_A \otimes \mathcal{H}_B$ has a basis $\{|\psi_i\rangle \otimes |m\rangle\}$

$$\langle \psi_i | \psi_j \rangle = \delta_{ij} \quad \& \quad \langle m | m' \rangle = \delta_{mm'}$$

$$\Rightarrow \Big(\langle \psi_i | \otimes \langle m |\Big) \cdot \Big(|\psi_j\rangle \otimes |m'\rangle\Big) = \langle \psi_i | \psi_j \rangle \otimes \langle m | m' \rangle$$

Stuck
3/9/21

If $\{|\psi_i\rangle\}$ and $\{|m\rangle\}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, then $\{|\psi_i\rangle \otimes |m\rangle\}$ is an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$.

$\rightarrow$ $\{|\psi_i\rangle \otimes |0\rangle\}$ is a set of orthonormal vectors, thus form a subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\Big(\langle \psi_i | \otimes \langle 0 | U^\dagger\Big)\Big(U |\psi_j\rangle \otimes |0\rangle\Big) = \Big(\langle \psi_i | \otimes \langle 0 |\Big)\Big(|\psi_j\rangle \otimes |0\rangle\Big)$$

for all $|\psi_i\rangle, |\psi_j\rangle$

$\Longrightarrow$ U is a unitary operator in the subspace spanned by $\{|\psi_i\rangle \otimes |0\rangle\}$.

$\boxed{\text{Ex:- 2.67}}$

Suppose V is a Hilbert space with a subspace W. Suppose $U: W \to V$ is a linear operator which preserves inner products, ie., for any $|w_1\rangle$ and $|w_2\rangle$ in W,

$$\langle w_1 | U^{\dagger} U | w_2 \rangle = \langle w_1 | w_2 \rangle .$$

There exists a unitary operator $U': V \to V$ which extends U. i.e.,

$U' |w\rangle = U |w\rangle \quad \forall \; |w\rangle \in W$, but U' is defined on the entire space V.

Proof

~~U preserves inner products $\Longrightarrow U: W \to V$ sends an orthonormal basis of W to an orthonormal basis of V.~~

Let,

W is an

the n-dim

$\{|w_1\rangle, \ldots, |w_m\rangle\}$

of W, a

orthonormal

vectors $\{|w$

If $U|w_i\rangle =$

$\langle v_i | v_j \rangle =$

$\rightarrow U: W \to V$ ma

W to an

which is

$U = \sum_{i=}^{m}$

Let

W is an m-dimensional subspace of the n-dimensional space V.

$\{|W_1\rangle, \ldots, |W_m\rangle\}$ be the orthonormal basis of W, and it can be extended to an orthonormal basis of V by adding some vectors $\{|W_{m+1}\rangle, \ldots, |W_n\rangle\}$.

If $U|W_i\rangle = |V_i\rangle$, then we have

$$\langle V_i|V_j\rangle = \langle W_i|U^{\dagger}U|W_j\rangle = \langle W_i|W_j\rangle = \delta_{ij}$$

$\Rightarrow$ $U: W \to V$ maps the orthonormal basis of W to an orthonormal set in V, which is $\{|V_1\rangle, \ldots, |V_m\rangle\}$.

$$U = \sum_{i=1}^{m} |V_i\rangle\langle W_i|$$

$\{|v_1\rangle, \cdots |v_m\rangle\}$ are an orthonormal set hence it can be extended to an orthonormal basis of V by adding some orthonormal vectors $\{|v_{m+1}\rangle, \ldots, |v_n\rangle\}$.

We can write down the unitary extension U' by setting

$$U'|w_j\rangle = |v_j\rangle \quad \text{for } j = m+1, \ldots, n$$

$$U' = \sum_{j=1}^{n} |w_j\rangle\langle v_j|$$

$\Longrightarrow U$ can be extended to a unitary operator on the space $\mathcal{H}_A \otimes \mathcal{H}_B$. $(\mathcal{H}_A \otimes \mathbb{C}^n)$

i.e., $U \in \mathcal{L}(\mathcal{H}_A \otimes \mathbb{C}^n)$

Now,

suppose we perform a projective measurement on the two systems described by projectors

$P_m = I_A \otimes |m\rangle\langle m|$.

Outcome m occurs with probability,

$$p(m) = \langle\psi|\langle 0| U^\dagger P_m U |\psi\rangle|0\rangle$$

$$= \sum_{m', m''} \langle\psi| M_{m'}^\dagger \otimes \langle m'| \left(I_A \otimes |m\rangle\langle m|\right) M_{m''} |\psi\rangle \otimes |m''\rangle$$

$$= \langle\psi| M_m^\dagger M_m |\psi\rangle$$

just as given in Postulate 3.

The joint state of the system $\mathcal{H}_A \otimes \mathbb{C}^n$ after measurement, conditional on result m occuring, is given by:

$$\frac{P_m U|\psi\rangle \otimes |0\rangle}{\sqrt{\langle\psi|\otimes\langle 0|U^{\dagger} P U|\psi\rangle \otimes |0\rangle}} = \frac{(I_A \otimes |m\rangle\langle m|)\sum_{m=1}^{n} (M_m|\psi\rangle)\otimes|m\rangle}{\sqrt{\langle\psi|M_m^{\dagger} M_m|\psi\rangle}}$$

$$= \frac{(M_m|\psi\rangle)\otimes|m\rangle}{\sqrt{\langle\psi|M_m^{\dagger} M_m|\psi\rangle}}$$

It follows that,

the state of the system $\mathcal{H}_B$ after the measurement is $|m\rangle$.
and the state of the system $\mathcal{H}_A$ is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^{\dagger}M_m|\psi\rangle}}$$

⇒ First performing U and then measuring the projective measurement P and tracing out the system $\mathbb{C}^n = \mathcal{H}_B$ (forgetting about the system) is equivalent to performing the general measurement M_m.

Naimark theorem: is proved.

An indirect measurement may be seen as the physical implementation of a POVM and any POVM may be realized by an indirect measurement.

In measuring a quantity of interest on a physical system one generally deals with a larger system that involves additional degrees of freedom, besides those of the system itself. These additional physical entities are globally referred to as the apparatus or the ancilla.

As a matter of fact, the measured quantity may be always described by a standard Observable, however, on a larger Hilbert space describing both the systems & the apparatus. When we trace out the degrees of freedom of the apparatus we are generally left with a POVM rather than a PVM.

Conversely, any generalized ... implement... in a large...

Conversely,

any conceivable POVM describe a generalized measurement, may be always implemented as a standard measurement in a larger Hilbert space.

Superdense coding

- example of the information processing tasks that can be accomplished using QM.

Two parties 'Alice' and 'Bob', who are a long way away from one another. Their goal is to transmit some classical information ~~which~~ from Alice to Bob. Suppose, Alice is in possession of 2 classical bits of information which she wishes to send Bob, but is only allowed to send a single qubit to Bob. Can she achieve her goal?

Suppose, Alice and Bob initially share a pair of qubits in the entangled state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice is initially in possession of the 1st qubit, while Bob has possession of the 2nd qubit, as in Fig.

Note that $|\psi\rangle$ is fixed state; there is no need for Alice to have sent Bob any qubits in order to prepare this state. Instead some 3rd party may prepare the entangled state ahead of time, sending one of the qubits to Alice and the other to Bob.

By sending the single qubit in her possession to Bob, Alice can communicate 2 bits of classical information to Bob.

If she wishes to send '00' to Bob then she does nothing at all to her qubit; If she wishes to send '01' then she applies the phase flip Z to her qubit. If she wishes to send '10' then she applies the quantum NOT gate, X, to her qubit. If she wishes to send

'11' then she applies the iY gate to her qubit.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ; \quad Z\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \quad X\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \Longrightarrow iY = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$iY\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}$$

The 4 resulting states are seen to be:

$$00: |\psi\rangle \longrightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$01: |\psi\rangle \xrightarrow{Z} \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$10: |\psi\rangle \xrightarrow{X} \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$11: |\psi\rangle \xrightarrow{iY} \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

→ Bell basis / Bell states / EPR pairs

Note: The Bell states form an orthonormal basis, and can therefore be distinguished by an appropriate quantum measurement.

If Alice
giving Bo
then by
Bell basis
the 4 pos

→

Alice, int
is able to
to Bob.
in the p
to interact

If Alice sends her qubit to Bob, giving Bob possession of both qubits, then by doing a measurement in the Bell basis Bob can determine which of the 4 possible bit strings Alice sent.

→

Alice, interacting with only a single bit, is able to transmit 2 bits of information to Bob. Of course, 2 qubits are involved in the protocol, but Alice never need to interact with the 2^{nd} qubit.

Ex: 2.70

Suppose, E is any positive operator acting on Alice's qubit. Show that $\langle\psi|E\otimes I|\psi\rangle$ takes the same value when $|\psi\rangle$ is any of the 4 Bell states.

Suppose some malevolent 3rd party ('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the 4 possible bit strings 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?

Ans: $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}, X=\begin{bmatrix}0&1\\1&0\end{bmatrix}, Y=\begin{bmatrix}0&-i\\i&0\end{bmatrix}, Z=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Each Pauli matrix is Hermitian, and together with the identity matrix, the Pauli matrices form a basis for the real vectorspace of 2×2 Hermitian matrices

B C (1) Bloch sphere
ILA (5) 82-82
QM (2) spinors

Any single qubit operation E (positive operator) can be expressed as a linear combination of the Pauli matrices & the identity matrix,

$$E = c_I I + c_x X + c_y Y + c_z Z$$

The Bell basis are:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$(Z\otimes I)|\Phi^+\rangle$

$(X\otimes I)|\Phi^+\rangle$

$(iY\otimes I)|\Phi^+\rangle$

$\langle\Phi^+|E\otimes I|\Phi$

$= c_I$

$= c$

$\langle\Phi^-|E\otimes I|$

σ_j

$$(Z\otimes I)|\phi^+\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\phi^-\rangle$$

$$(X\otimes I)|\phi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} = |\psi^+\rangle$$

$$(iY\otimes I)|\phi^+\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\psi^-\rangle$$

$$\langle\phi^+|E\otimes I|\phi^+\rangle = \langle\phi^+|c_I I\otimes I|\phi^+\rangle + \langle\phi^+|c_x X\otimes I|\phi^+\rangle$$
$$+ \langle\phi^+|c_y Y\otimes I|\phi^+\rangle + \langle\phi^+|c_z Z\otimes I|\phi^+\rangle$$
$$= c_I\langle\phi^+|\phi^+\rangle + 0+0+0$$

[Bell basis are orthogonal to each other.]

$$= c_I$$

$$\langle\phi^-|E\otimes I|\phi^-\rangle = \langle\phi^+|(Z\otimes I)^\dagger(E\otimes I)(Z\otimes I)|\phi^+\rangle$$
$$= \langle\phi^+|(Z\otimes I)((c_I I + c_x X + c_y Y + c_z Z)\otimes I)(Z\otimes I)|\phi^+\rangle$$

$$\sigma_j\sigma_k = \delta_{jk} I + i\epsilon_{jkl}\sigma_l$$

$$= \langle\phi^+|(Z\otimes I)(c_I Z - ic_x Y + ic_y X + c_z I)\otimes I|\phi^+\rangle$$
$$= \langle\phi^+|(c_I I - c_x X - c_y Y + c_z Z)\otimes I|\phi^+\rangle$$
$$= c_I\langle\phi^+|\phi^+\rangle + 0 + 0 + 0 = c_I.$$

Similarly:

Similarly,

$\langle \psi^+ | E \otimes I | \psi^+ \rangle = C_I$

$\langle \psi^- | E \otimes I | \psi^- \rangle = C_I$

$\Rightarrow \langle \psi | E \otimes I | \psi \rangle$ takes the same value when $|\psi\rangle$ is any of the Bell states.

If Eve gets the qubit that Alice sent to Bob and performs the measurement M_m, the result Eve gets with probability,

$P(m) = \langle \psi | M_m^\dagger M_m \otimes I | \psi \rangle$

where $M_m^\dagger M_m$ is a positive operator.

The measurement result doesn't depend on the state Alice prepared.

$\therefore$ Eve cannot infer anything about the information Alice sent.

□ Multiple quantum states

* The Hilbert space of a composite system is given by the tensor product (Kronecker product) of the separate individual Hilbert spaces.

Joint quantum systems must follow the following properties:

1. Dimensions

We should be able to see a composite system made of 2 qubits (2D each) as a single quantum system with 4 dimensions.

$\therefore$

Since each qubit has 2 exclusive states each ($|0\rangle_2$ and $|1\rangle_2$), then the full system will have 4 distinct states namely:

$|00\rangle_4 = |0\rangle_2 \otimes |0\rangle_2 \longleftrightarrow$ qubit 1 in $|0\rangle_2$ and qubit 2 in $|0\rangle_2$

$|01\rangle_4 = |0\rangle_2 \otimes |1\rangle_2 \longleftrightarrow$ qubit 1 in $|0\rangle_2$ and qubit 2 in $|1\rangle_2$

$|10\rangle_4 = |1\rangle_2 \otimes |0\rangle_2 \longleftrightarrow$ qubit 1 in $|1\rangle_2$ and qubit 2 in $|0\rangle_2$

$|11\rangle_4 = |1\rangle_2 \otimes |1\rangle_2 \longleftrightarrow$ qubit 1 in $|1\rangle_2$ and qubit 2 in $|1\rangle_2$

$$|00\rangle_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad |01\rangle_4 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad |10\rangle_4 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad |11\rangle_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

2. Measurement Probabilities

Let's assume that,

qubit 1 is in $|\psi_1\rangle_2$ and qubit 2 is in $|\psi_2\rangle_2$ such that the joint 4-dimensional state is abstractly given by

$$|\psi\rangle_4 = |\psi_1 \psi_2\rangle_4 = |\psi_1\rangle_2 \otimes |\psi_2\rangle_2$$

Since, the joint system can be seen as a single, larger system of higher dimension Born's rule still applies.

$\therefore$ the probability of measuring qubit 1 in $|\phi_1\rangle$ and qubit 2 in $|\phi_2\rangle$

ie., measuring the joint system in $|\phi\rangle_4 = |\phi_1 \phi_2\rangle_4 = |$

will be given by :

$$P(\phi) = \left|\langle\phi|_4 |\psi\rangle_4\right|^2$$

$$= \left|\langle\phi_1\phi_2|_4 \, |\psi_1\psi_2\rangle_4\right|^2$$

$$= \left|\left[\langle\phi_1|_2 \otimes \langle\phi_2|_2\right]\left[|\psi_1\rangle_2 \otimes |\psi_2\rangle_2\right]\right|^2$$

If we think of each qubit as their own separate system, then the probability of measuring qubit 1 in $|\phi_1\rangle_2$ and the prob. of measuring qubit 2 in $|\phi_2\rangle_2$ is given by $P(\phi_1) = |\langle\phi_1|\psi_1\rangle|^2$ and $P(\phi_2) = |\langle\phi_2|\psi_2\rangle|^2$, respectively.

Probability theory : two independent things happening is given by the product of the individual probabilities.

$$P(\phi) = P(\phi_1)\, P(\phi_2)$$

$$\boxed{\left|\left[\langle\phi_1|_2 \otimes \langle\phi_2|_2\right]\left[|\psi_1\rangle_2 \otimes |\psi_2\rangle_2\right]\right|^2 = |\langle\phi_1|\psi_1\rangle|^2 \, |\langle\phi_2|\psi_2\rangle|^2}$$

3 Joint quantum operations

If U_1 is a unitary operator on qubit 1 and U_2 is a unitary operator on qubit 2, then the joint operation $U_1 \otimes U_2$ must have the property that:

$$\boxed{\begin{aligned}\left[U_1 \otimes U_2\right]|\psi_1\psi_2\rangle_4 &= \left[U_1 \otimes U_2\right]\left[|\psi_1\rangle_2 \otimes |\psi_2\rangle_2\right] \\ &= \left[U_1|\psi_1\rangle_2\right] \otimes \left[U_2|\psi_2\rangle_2\right]\end{aligned}}$$

→ the resulting joint state after applying the joint quantum operation $[U_1 \otimes U_2]|\psi_1\psi_2\rangle_4$ must be equal to the joint state of the individual state after the individual operations, $U_1|\psi_1\rangle_2$ and $U_2|\psi_2\rangle_2$ respectively.

Kronecker product / Tensor product

We'd like to have an explicit representation of the operator $\otimes$, so that we can have an explicit representation of states like $|\psi,\psi\rangle$ and operators $U_1 \otimes U_2$. This is where the Kronecker, or tensor, product comes in.

* The Kronecker product (tensor product) is a special way to join vectors together to make bigger vectors.

Say we have $|v\rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $|w\rangle = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

The Kronecker product is defined as

$$|v\rangle \otimes |w\rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ v_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{bmatrix}$$

We can generalize the Kronecker product to matrices :

$$M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \quad \& \quad N = \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix}$$

$$M \otimes N = \begin{bmatrix} m_1 \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} & m_2 \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} \\ m_3 \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} & m_4 \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 n_1 & m_1 n_2 & m_2 n_1 & m_2 n_2 \\ m_1 n_3 & m_1 n_4 & m_2 n_3 & m_2 n_4 \\ m_3 n_1 & m_3 n_2 & m_4 n_1 & m_4 n_2 \\ m_3 n_3 & m_3 n_4 & m_4 n_3 & m_4 n_4 \end{bmatrix}$$

Let 'A' be an $m \times n$ matrix and let B be a $p \times q$ matrix. Then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots\cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots\cdots & a_{2n}B \\ \vdots & & & \\ a_{m1}B & a_{m2}B & \cdots\cdots & a_{mn}B \end{bmatrix}$$

is an $(mp) \times (nq)$ matrix called the tensor product (Kronecker product) of A and B.

*

1. $(A \otimes B)(C$

2. $(A_1 \otimes B_1)(A_2 \otimes$

3. $(\alpha A) \otimes B =$

4. $A \otimes (B \otimes C)$

5. $A \otimes O = O \otimes$

6. $I_m \otimes I_n = I$

7. $A \otimes B \neq B$

*

1. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

2. $(A_1 \otimes B_1)(A_2 \otimes B_2)(A_3 \otimes B_3) = (A_1 A_2 A_3) \otimes (B_1 B_2 B_3)$

3. $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$

4. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

5. $A \otimes O = O \otimes A = O$ $\left[A \otimes B = O \text{ iff either } A = 0 \text{ or } B = 0 \right]$

6. $I_m \otimes I_n = I_{mn}$

7. $A \otimes B \neq B \otimes A$

Proof

4. $$(A\otimes B)\otimes C = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \otimes C = \begin{bmatrix} (a_{11}B)\otimes C & \cdots & (a_{1n}B)\otimes C \\ \vdots & \ddots & \vdots \\ (a_{m1}B)\otimes C & \cdots & (a_{mn}B)\otimes C \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(B\otimes C) & \cdots & a_{1n}(B\otimes C) \\ \vdots & & \vdots \\ a_{m1}(B\otimes C) & \cdots & a_{mn}(B\otimes C) \end{bmatrix} = A\otimes(B\otimes C)$$

1. $A = [a_{ik}]$ and $C = [c_{kj}]$

$(A\otimes B) = [a_{ik}B]$ & $(C\otimes D) = [c_{kj}D]$

$$(ab)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

$$\big((A\otimes B)(C\otimes D)\big)_{ij} = \sum_{k=1}^{n} (A\otimes B)_{ik}(C\otimes D)_{kj}$$

$$= \sum_{k=1}^{n} (a_{ik}B)(c_{kj}D) = \sum_{k=1}^{n} (a_{ik}c_{kj})(BD)$$

$$= \left[\sum_{k=1}^{n} a_{ik}c_{kj}\right] BD = (AC)_{ij}\, BD$$

$$= (AC\otimes BD)_{ij}$$

$$\boxed{\begin{gathered}(A_1\otimes A_2\otimes\cdots\otimes A_k)(B_1\otimes B_2\otimes\cdots\otimes B_k) = A_1B_1\otimes A_2B_2\otimes\cdots\otimes A_kB_k \\ (A_1\otimes B_1)(A_2\otimes B_2)\cdots(A_k\otimes B_k) = (A_1A_2\cdots A_k)\otimes(B_1B_2\cdots B_k)\end{gathered}}$$

*

1. $A\otimes(B+C$

$(A+B)\otimes C$

2. $(A\otimes B)^{\dagger} =$

3. $(A\otimes B)^{-1} =$

4. $(A\otimes B)^{*} =$

* 1. $A \otimes (B+C) = A \otimes B + A \otimes C$

$(A+B) \otimes C = A \otimes C + B \otimes C$

2. $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$

3. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

If $A \in M_m$ & $B = M_n$ are non-singular, then $A \otimes B$ is also non-singular with $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

4. $(A \otimes B)^{*} = A^{*} \otimes B^{*}$ & $(A \otimes B)^{T} = A^{T} \otimes B^{T}$

Proof

4.

$$(A\otimes B)^T = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B\end{bmatrix}^T = \begin{bmatrix} a_{11}B^T & \cdots & a_{m1}B^T \\ \vdots & \ddots & \vdots \\ a_{1n}B^T & \cdots & a_{mn}B^T \end{bmatrix} = A^T\otimes B^T$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T = \begin{bmatrix} \left[\dfrac{A}{C}\right]^T \\ \left[\dfrac{B}{D}\right]^T \end{bmatrix} = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$$

3. $(A\otimes B)(A^{-1}\otimes B^{-1}) = (AA^{-1})\otimes(BB^{-1}) = I\otimes I = I$

$(A^{-1}\otimes B^{-1})(A\otimes B) = (A^{-1}A)\otimes(B^{-1}B) = I\otimes I = I$

$\rightarrow$ $A^{-1}\otimes B^{-1}$ is the unique inverse of $A\otimes B$ under conventional matrix multiplication.

$\therefore$ $A\otimes B$ is non-singular

* Let A ...
matrix

$tr(A\otimes B$ ...

$det(A\otimes B$ ...

Proof

Let $A\in M$ ...

$Ax = \lambda x$

$By = \mu y$

Proof. $(A\otimes B)($ ...

$trace(A\otimes B) =$ ...

$det(A\otimes B) =$ ...

* Let A be an $m \times m$ matrix and B be a $n \times n$ matrix.

$$\boxed{\begin{aligned} tr(A \otimes B) &= tr(A) \cdot tr(B) = tr(B \otimes A) \\ \det(A \otimes B) &= (\det A)^n (\det B)^m = \det(B \otimes A) \end{aligned}}$$

Proof

Let $A \in M_n$ and $B \in M_m$,

$$\boxed{\begin{matrix} Ax = \lambda x \\ By = \mu y \end{matrix} \Longrightarrow (A \otimes B)(x \otimes y) = \lambda\mu(x \otimes y)}$$

Proof: $(A \otimes B)(x \otimes y) = (Ax) \otimes (By) = (\lambda x) \otimes (\mu y)$

$$= \lambda\mu(x \otimes y)$$

$$\text{trace}(A \otimes B) = \sum_k \gamma_k = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j$$

$$= \sum_{i=1}^{m} \lambda_i \times \sum_{j=1}^{n} \mu_j = \text{trace}(A) \cdot \text{trace}(B)$$

$$\det(A \otimes B) = \prod_{k=1}^{mn} \gamma_k = \prod_{i=1}^{m} \prod_{j=1}^{n} (\lambda_i \mu_j)$$

$$= \left(\prod_{i=1}^{m} \lambda_i^n\right)\left(\prod_{j=1}^{n} \mu_j^m\right) = [\det(A)]^n [\det(B)]^m$$

* If $A \in M_n$ is similar to $B \in M_n$ via $S \in M_n$ and $C \in M_m$ similar to $E \in M_m$ via $T \in M_m$, then $A \otimes C$ is similar to $B \otimes C$ via $S \otimes T$

$$\left.\begin{array}{l} A = S^{-1}BS \\ \& \\ C = T^{-1}ET \end{array}\right\} A \otimes C = (S \otimes T)^{-1}(B \otimes E)(S \otimes T)$$

Proof

$A = S^{-1}BS$ and $C = T^{-1}ET$

$$(S \otimes T)^{-1}(B \otimes E)(S \otimes T) = (S^{-1} \otimes T^{-1})(BS \otimes ET)$$

$$= (S^{-1}BS) \otimes (T^{-1}ET) = A \otimes C$$

* Let $U \in M_n$ and $V \in M_m$ be unitary matrices. Then $U \otimes V$ is a unitary matrix

$$\left.\begin{array}{l} U^{\dagger} = U^{-1} \\ V^{\dagger} = V^{-1} \\ (U, V \text{ are unitary}) \end{array}\right\} (U \otimes V)^{\dagger} = (U \otimes V)^{-1}$$

Proof

$U^{\dagger} = U^{-1}$

$(U \otimes V)^{\dagger} =$

* Let $A \in$ triangular

Proof.

$$U^{\dagger} = U^{-1} \quad \& \quad V^{\dagger} = V^{-1}$$

$$(U \otimes V)^{\dagger} = U^{\dagger} \otimes V^{\dagger} = U^{-1} \otimes V^{-1} = (U \otimes V)^{-1}$$

* Let $A \in M_n$ and $B \in M_m$ be upper triangular. Then, $A \otimes B$ is upper triangular.

* Let $A \in M_n$ and $B \in M_m$ be positive (semi) definite Hermitian matrices.

Then,

$A \otimes B$ is also positive (semi) definite Hermitian.

<u>Proof</u>.

A & B are Hermitian matrices.

$A^\dagger = A$ & $B^\dagger = B$, then

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

$\Longrightarrow$ $A \otimes B$ is Hermitian

$$\left.\begin{aligned} Ax &= \lambda x \\ By &= \mu y \end{aligned}\right\} \quad (A \otimes B)(x \otimes y) = \lambda \mu (x \otimes y)$$

Since A & B are positive (semi) definite - the eigenvalues of both A and B are all positive (non-negative)

$\lambda_i, \mu_j \geq 0$

The product of any eigenvalue from A with any eigenvalue from B will also be positive (non-negative).

$\lambda_i \mu_j \geq 0$

* Let $A \in M_{mn}$ and $B \in M_{pq}$ have singular value decompositions $A = U_A \Sigma_A V_A^{\dagger}$ and $B = U_B \Sigma_B V_B^{\dagger}$. If rank$(A) = r_1$ and rank$(B) = r_2$ then the nonzero singular values of $A \otimes B$ are the $r_1 r_2$ positive numbers $\{\sigma_i(A)\sigma_j(B) : i = 1, 2, \dots, r_1 \ \& \ j = 1, 2, \dots, r_2\}$, where $\sigma_i(A)$ is a non-zero singular value of A and $\sigma_j(B)$ is a non-zero singular value of B.

$$\left.\begin{aligned} A &= U_A \Sigma_A V_A^{\dagger} \\ B &= U_B \Sigma_B V_B^{\dagger} \end{aligned}\right\} \quad A \otimes B = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)^{\dagger}$$

with, $\sigma(A \otimes B) = \sigma_i(A)\,\sigma_j(B)$

Proof

. A

$U_A \in M_m$, V_A

unitary.

$A \otimes B = (U_A \Sigma$

$= (U_A \otimes U$

$= (U_A \otimes$

$U_A \otimes U_B \in M_m$

$(A \otimes B)^{\dagger}(A$

The singular val

the eigenvalues

$= \sqrt{\text{pairwise}}$
of $A^{\dagger}A$

Proof

. $A = U_A \Sigma_A V_A^\dagger$ and $B = U_B \Sigma_B V_B^\dagger$

$U_A \in M_m$, $V_A \in M_n$, $U_B \in M_p$, $V_B \in M_q$ are unitary.

$$A \otimes B = (U_A \Sigma_A V_A^\dagger) \otimes (U_B \Sigma_B V_B^\dagger)$$

$$= (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A^\dagger \otimes V_B^\dagger)$$

$$= (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)^\dagger$$

$U_A \otimes U_B \in M_{mp}$, $V_A \otimes V_B \in M_{nq}$: unitary

---

$$(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B) = A^\dagger A \otimes B^\dagger B$$

The singular values of $A \otimes B$ are the sq. roots of the eigenvalues of $(A \otimes B)^\dagger (A \otimes B) = A^\dagger A \otimes B^\dagger B$.

$= \sqrt{\text{pairwise products of the eigenvalues of } A^\dagger A \text{ and } B^\dagger B}$.

* Let $A, B \in M_{m,n}$. Then,

$A \otimes B = B \otimes A$ if and only if either $A = cB$ or $B = cA$ for some $c \in \mathbb{F}$.

Proof

Assume that, $A \otimes B = B \otimes A$ with $A, B \in M_{m,n}$

$\Longrightarrow a_{ij} B = b_{ij} A$

If $a_{ij} = 0$ for all $i(j) = 1, 2, \dots, m(n)$, then $A = 0$ and $A = cB$ for $c = 0$.

$0 \otimes B = B \otimes 0 = 0$

For each non-zero a_{ij}, $\Longleftarrow$

$B = \frac{b_{ij}}{a_{ij}} A$ (or) $B = cA$ where $c = \frac{b_{ij}}{a_{ij}}$

* Let $A \in M_n$ and $B \in M_m$, then $A \otimes B = I_{mn}$ if and only if $A = \alpha I_n$ and $B = \frac{1}{\alpha} I_m$.

Proof

Assume that $A = \alpha I_n$ and $B = \frac{1}{\alpha} I_m$,

$$A \otimes B = (\alpha I_n) \otimes \left(\frac{1}{\alpha} I_m\right) = \left(\alpha \frac{1}{\alpha}\right)(I_n \otimes I_m) = I_{mn}$$

Assume that $A \otimes B = I_{mn}$,

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} = \begin{bmatrix} I & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I \end{bmatrix}$$

$\Longrightarrow$ $a_{ii} B = I_m$ for all $i = 1, 2, \dots, n$

$a_{ij} B = 0$ for all $i \neq j$

$\Longrightarrow$ $a_{ij} = 0$ for $i \neq j$ & $B = \frac{1}{a_{ii}} I$

If $\alpha =$

$A =$

If $\alpha = a_{ii}$,

$$A = \alpha I_n \quad \& \quad B = \frac{1}{\alpha} I_m$$

Ex:- The Kronecker product of the following 2 vectors.

$$|v\rangle = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad |w\rangle = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$ is given by :

Ans- $$|v\rangle \otimes |w\rangle = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ -1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -3 \\ -4 \\ 6 \\ 8 \end{bmatrix}$$

$$\boxed{\dim\left(|v\rangle \otimes |w\rangle\right) = \dim\left(|v\rangle\right) \dim\left(|w\rangle\right)}$$

$$= 3 \times 2 = 6$$

Ex:-

$$X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Applying t

$|0+\rangle =$

$X \otimes Z|0$

Applying this unitary operation on the state:

$$|0+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{yields:}$$

$$X \otimes Z|0+\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = |1-\rangle$$

Multiple qubits

If we have 2 qubits with individual states $|\psi\rangle$ and $|\phi\rangle$, their joint quantum state $|\Psi\rangle$ is given by:

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$$

where $\otimes$ represents the Kronecker product.

The state of 2 qubits is given by a 4-D vector. In general, any 4D unit vector can represent the state of 2 qubits.

If we have n qubits with individual states $|\psi_1\rangle, \ldots, |\psi_n\rangle$ the joint state will be given by $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$. The final vector will have a dimension of 2^n.

Any 2^n-dimensional unit vector can be seen as as a n-qubit state.

Ex:-

$$|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Ex:- If ⟨Alice

Bob has

combined

$|\psi\rangle \otimes |\phi\rangle$

Ex:- Suppose A

the state |

is in $|\chi\rangle$ =

given by:

$|\psi\rangle \otimes |\phi\rangle \otimes$

=

$= \frac{1}{2}$

Ex:- If Alice has the quantum state $|\psi\rangle = |0\rangle$ and Bob has the state $|\phi\rangle = |+\rangle$ then their combined state is given by:

$$|\psi\rangle \otimes |\phi\rangle = |0\rangle \otimes |+\rangle$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Ex:- Suppose Alice is in the state $|\psi\rangle = |+\rangle$, Bob in the state $|\phi\rangle = |+\rangle$ and their new pal, Charlie is in $|\chi\rangle = |-\rangle$ then their combined state is given by:

$$|\psi\rangle \otimes |\phi\rangle \otimes |\chi\rangle = |+\rangle \otimes |+\rangle \otimes |-\rangle$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Ex:- If $|\psi\rangle = |+1\rangle$ and $|\phi\rangle = |1-\rangle$, calculate the outcome probability of measuring the state $|\psi\rangle$ in the state $|\phi\rangle$.

$$|\psi\rangle = |+1\rangle = |+\rangle \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

$$|\phi\rangle = |1-\rangle = |1\rangle \otimes |-\rangle$$

$$= \begin{bmatrix}0\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

Ans:

Born's rule

$$P(\phi) = |\langle\phi|\psi\rangle|^2 = \left|\frac{1}{2}\begin{bmatrix}0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right|^2$$

$$= \left|\frac{-1}{2}\right|^2 = \frac{1}{4}$$

Inner product property of the Kronecker product

$$P(\phi) = \left|\langle +1|1-\rangle\right|^2 = \left|\left[\langle +|\otimes\langle 1|\right]\left[|1\rangle\otimes|-\rangle\right]\right|^2$$

$$= \left|\langle +|1\rangle\right|^2 \cdot \left|\langle 1|-\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right|^2 \cdot \left|\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\right|^2 \cdot \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{4}$$

$(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$

Not every 4-dimensional vector can be written as a Kronecker product of two 2-dimensional vectors. e.g.,

- You can have a 2-qubit state $|\Phi\rangle$ such that:

$$|\Phi\rangle \neq |\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle$$

These types of states are called entangled states.

* If a two-qubit state can be written as:

$$|\Phi\rangle = |\psi\rangle|\phi\rangle = |\psi\rangle \otimes |\phi\rangle$$

for some one-qubit states $|\psi\rangle$ and $|\phi\rangle$ then we say that $|\Phi\rangle$ is a separable state. In any other case, we call them entangled states.

Quantum Entanglement

* If a two-qubit ~~state~~ quantum state $|\Psi\rangle$ cannot be written as $|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle$ for any possible choice of $|\psi\rangle$ and $|\phi\rangle$, then $|\Psi\rangle$ is said to be entangled.

Ex: Are

$|\Psi\rangle = $

states

Ans:

① Assu

$\therefore$ there

$|\psi\rangle = [$

such th

$|\Phi\rangle = \frac{1}{2}$

$\frac{\phi_1}{\phi_2} = 1,\ \frac{\psi}{\psi}$

$\cancel{\phi_1^2 \phi_2^2}$

$\phi_1 = \phi_2 = \psi$

Ex: Are $|\Phi\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$ and $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ separable quantum states ?

Ans:

① Assuming that $|\Phi\rangle$ is separable .

$\therefore$ there must exist 2 vectors :

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad \text{and} \quad |\phi\rangle = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

such that $|\Phi\rangle = |\psi\rangle|\phi\rangle$

$$|\Phi\rangle = \frac{1}{2}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_1 \\ \psi_2\phi_2 \end{bmatrix}$$

$\frac{\phi_1}{\phi_2} = 1$, $\frac{\psi_1}{\psi_2} = -1 \longrightarrow \phi_1 = \phi_2$ & $\psi_1 = -\psi_2$

$\cancel{\phi_1^2\phi_2^2\psi_1^2\psi_2^2 = \phi_1^4\psi_1^4 = \frac{1}{16} \longrightarrow \phi_1\psi_1 = \frac{1}{2}}$

$\phi_1 = \phi_2 = \psi_1 = 1/\sqrt{2}$ & $\psi_2 = -1/\sqrt{2}$

$$|\Phi\rangle = |\psi\rangle|\phi\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = |-+\rangle$$

$\Longrightarrow |\Phi\rangle$ is separable

(2)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}\psi_1\phi_1\\ \psi_1\phi_2\\ \psi_2\phi_1\\ \psi_2\phi_2\end{bmatrix}$$

$\psi_1\phi_2$

$\psi_1 = 0$ or

If $\psi_1 = 0$,

If $\phi_2 = 0$,

$\Longrightarrow |$

$\psi_1 \phi_2 = 0$ & $\psi_2 \phi_1 = 0$

$\psi_1 = 0$ (or) $\phi_2 = 0$ & $\psi_2 = 0$ (or) $\phi_1 = 0$

If $\psi_1 = 0$, then $\psi_1 \phi_1 = 0$, which is a contradiction

If $\phi_2 = 0$, then $\psi_2 \phi_2 = 0$, "

$\Longrightarrow |\Psi\rangle$ is not separable

~~Ex~~ Sam

(OR)

We want to know if 2 states exist,

$|\psi\rangle = \psi_1|0\rangle + \psi_2|1\rangle$ and $|\phi\rangle = \phi_1|0\rangle + \phi_2|1\rangle$

such that $|\psi\rangle \otimes |\phi\rangle = |\Psi\rangle$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$= \left(\psi_1|0\rangle + \psi_2|1\rangle\right) \otimes \left(\phi_1|0\rangle + \phi_2|1\rangle\right)$$

$$= \psi_1\phi_1|00\rangle + \psi_1\phi_2|01\rangle + \psi_2\phi_1|10\rangle + \psi_2\phi_2|11\rangle$$

$$\Rightarrow \psi_1\phi_1 = \frac{1}{\sqrt{2}},\ \psi_1\phi_2 = 0,\ \psi_2\phi_1 = 0,\ \psi_2\phi_2 = \frac{1}{\sqrt{2}}$$

Ex: Given

and $|\Phi$

the prob

in $|\Phi$

in $|\Psi$

Ans: N

(i) $|\Psi\rangle$

Born's

$P(\Phi) = |$

$=$

$=$

Ex: Given the 2-qubit state $|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$ and $|\Phi\rangle = \frac{1}{\sqrt{6}}(|00\rangle + i|01\rangle + 2|10\rangle)$, what is the probability of measuring the system in $|\Phi\rangle$ given that it is originally in $|\Psi\rangle$?

Ans: Not separable.

$$|\Psi\rangle = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \text{ and } |\Phi\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\i\\2\\0\end{bmatrix}$$

Born's rule,

$$P(\Phi) = |\langle\Phi|\Psi\rangle|^2 =$$

$$= \left|\frac{1}{2\sqrt{6}}(1-i-2)\right|^2 = \left|\frac{-1-i}{2\sqrt{6}}\right|^2$$

$$= \frac{1}{12}$$

(2) $\langle \Phi | \Psi \rangle = \frac{1}{2\sqrt{6}} \left[\langle 00| + i\langle 01| + 2\langle 10| \right]$

$\left[|00\rangle + |01\rangle - |10\rangle + |11\rangle \right]$

the inner product is written using an orthonormal basis

$\begin{bmatrix} 1 \\ i \\ 2 \\ 0 \end{bmatrix}$ $\Longrightarrow$ multiply the coef. of the same terms

$\langle \Phi | \Psi \rangle = \frac{1 - i - 2}{2\sqrt{6}} = \frac{-1 - i}{2\sqrt{6}}$

$\Longrightarrow P(\Phi) = \frac{1}{12}$

The following 2-qubit states are known as the Bell's states. They represent an orthonormal, entangled basis for two qubits :

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Quantum operation on multiple qubits

Unitary matrices acting on two qubits

We have two unitary matrices U_1 and U_2.
Then:

$$U = U_1 \otimes U_2$$

is a bigger matrix which satisfies:

$$\boxed{\begin{aligned} U(|\psi\rangle \otimes |\phi\rangle) &= (U_1 \otimes U_2)(|\psi\rangle \otimes |\phi\rangle) \\ &= U_1|\psi\rangle \otimes U_2|\phi\rangle \end{aligned}}$$

Ex:- $U = X \otimes Z$

$$U|0+\rangle = (X \otimes Z)|0+\rangle$$

$$= (X \otimes Z)(|0\rangle \otimes |+\rangle)$$

$$= X|0\rangle \otimes Z|+\rangle$$

$$= |1\rangle \otimes |-\rangle$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

(OR) $X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$|0+\rangle = |0\rangle \otimes |+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(X \otimes Z)|0+\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = |1-\rangle$$

Arbitrary 2-qubit operation

Similar to the state representation of multiple qubits, not every 4x4 unitary matrix can be written as the Kronecker product of two 2x2 matrices.

On the other hand,

any 4x4 unitary matrix can be thought of as a quantum operation of 2 qubits.

If the operation cannot be written as a Kronecker product of 2 matrices, we say that this is an <u>entangling operation</u>.

Ex: Take the 2-qubit quantum operation - represented by

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Consider,

two qubits in the joint state $|\psi_0\rangle = |00\rangle$. First you apply a Hadamard operation on the 1st qubit and then you applied the operation U. What would be the final state?

Ans:

The Hadamard is a one-qubit operation. But we want to apply it to a two-qubit state.

Appying Hadamard to the 1st qubit only is the exact same thing as saying apply Hadamard to the 1st qubit & nothing to the 2nd qubit.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Applying a Hadamard on only the 1st qubit is equivalent to the quantum operation on 2 qubits given by:

$$H \otimes \mathbb{1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$|\psi_1\rangle = (H \otimes \mathbb{1})|00\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|\psi_2\rangle = U\,|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$= |\Phi^+\rangle$$ (Bell's state)

(OR)

$|\psi_1\rangle = (H \otimes \mathbb{1})|00\rangle =$

$= H|0\rangle \otimes \mathbb{1}|0\rangle$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \\ \\ \end{bmatrix}$

(OR) $H|0\rangle =$

$|\psi_1\rangle =$

$= H$

$= |+\rangle$

$= \frac{1}{\sqrt{2}}($

$= \frac{1}{\sqrt{2}}($

(OR) $H|0\rangle = |+\rangle$

$$|\psi_1\rangle = \cancel{H|00\rangle} \; (H \otimes \mathbb{1})|00\rangle$$

$$= H|0\rangle \otimes \mathbb{1}|0\rangle$$

$$= |+\rangle \otimes |0\rangle = |+\rangle|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$\boxed{U}$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$U|00\rangle = |00\rangle$

$U|01\rangle = |01\rangle$

$U|10\rangle = |11\rangle$

$U|11\rangle = |10\rangle$

$$|\psi_2\rangle = U|\psi_1\rangle$$

$$= \frac{1}{\sqrt{2}} U \left(|00\rangle + |10\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \left(U|00\rangle + U|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right)$$

QC – from Linear Algebra to Physical Realizations.

– Mikio Nakahara & Tetsuo Ohmi

2.3 Multipartite System, Tensor Product & Entangled State

Suppose a system is made of 2 components : one lives in a Hilbert space $\mathcal{H}_1$ and the other in another Hilbert space $\mathcal{H}_2$.

A system composed of 2 separate components is called bipartite. Then the system as a whole lives in a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, whose general vector is written as :

$$|\psi\rangle = \sum_{i,j} c_{ij} |e_{1,i}\rangle \otimes |e_{2,j}\rangle \quad \text{— (2.29)}$$

where $\{|e_{a,i}\rangle\}$ $(a=1,2)$ is an orthonormal basis in $\mathcal{H}_a$ and $\sum_{i,j}|C_{ij}|^2=1$

A state $|\psi\rangle \in \mathcal{H}$ written as a tensor product of 2 vectors as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, $(|\psi_a\rangle \in \mathcal{H}_a)$ is called a separable state or tensor product state.

A separable state admits a classical interpretation such as "The 1st system is in the state $|\psi_1\rangle$, while the 2nd system is in $|\psi_2\rangle$".

The set of separable states has dimension $= \dim \mathcal{H}_1 + \dim \mathcal{H}_2$

Ex:- $|\phi\rangle = a|\phi_1\rangle + b|\phi_2\rangle + c|\phi_3\rangle$

$|\gamma\rangle = d|\gamma_1\rangle + e|\gamma_2\rangle + f|\gamma_3\rangle$

$|\phi\rangle \otimes |\gamma\rangle$

$= ($

of

dimension o

The dimen can be coefficients

$|\phi\rangle \otimes |\gamma\rangle = |\phi\rangle|\gamma\rangle = |\phi\gamma\rangle$

$= \left(a|\phi_1\rangle + b|\phi_2\rangle + c|\phi_3\rangle\right) \otimes \left(d|\gamma_1\rangle + e|\gamma_2\rangle + f|\gamma_3\rangle\right)$

# of coefficients = 3+3 = 6

= dimension of separable states

dimension of the total space = 3×3 = 9.

The dimension of the total space $\mathcal{H}$ can be found by counting the # of coefficients in 2.29,

$= \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2$

$\Rightarrow$ dimension of the total space $\mathcal{H}$ is considerably larger than the dimension of the separable states, when $\dim \mathcal{H}_a \; (a = 1, 2)$ are large..

What are the missing states ?

Consider the spin state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle\right)$$

of 2 separated electrons.

Suppose $|\psi\rangle$ may be decomposed as:

$$|\psi\rangle = \left(c_1|\uparrow\rangle + c_2|\uparrow\rangle\right) \otimes \left(d_1|\uparrow\rangle + d_2|\downarrow\rangle\right)$$

$$= c_1 d_1 |\uparrow\rangle \otimes |\uparrow\rangle + c_1 d_2 |\uparrow\rangle \otimes |\downarrow\rangle + c_2 d_1 |\downarrow\rangle \otimes |\uparrow\rangle + d_1 d_2 |\downarrow\rangle \otimes |\downarrow\rangle$$

$$= \frac{1}{\sqrt{2}} |\uparrow\rangle \otimes |\uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow\rangle \otimes |\downarrow\rangle$$

$c_1 d_2 = c_2 d_1 = 0$, $c_1 d_1 = c_2 d_2 = \frac{1}{\sqrt{2}}$

$\Longrightarrow$ This decomposition is not possible.

$\therefore$

The state $|\psi\rangle$ is not separable

Such non-separable states are called entangled in quantum theory.

$$\dim \mathcal{H}_1 \dim \mathcal{H}_2 \gg \dim \mathcal{H}_1 + \dim \mathcal{H}_2$$

$\Longrightarrow$ Most states in a Hilbert space of a bipartite system are entangled, when the constituent Hilbert spaces are higher dimensional.

• Note :

The space of separable states is not a vector space, and in particular not a subspace of the total Hilbert space: the sum of 2 separable states is unlikely to be separable. ie.,
& the dimension here means something more general than vector space dimension.

To specify a separable state, we can supply an element of each ~~of~~ of $\mathcal{H}_1$ and $\mathcal{H}_2$, ie., $\dim \mathcal{H}_1 + \dim \mathcal{H}_2$ complex numbers.

* If V and W are vector spaces with basis $\{v_i\}$ and $\{w_j\}$ respectively then $V \otimes W$ has a basis $\{v_i \otimes w_j\}$.

$$\Updownarrow$$

If $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$ then $\{v_i\}$ and $\{w_j\}$ are bases of V and W, respectively.

* If $\mathcal{H}_1$ and $\mathcal{H}_2$ have orthonormal bases $\{\phi_i\}$ and $\{\psi_j\}$ respectively, then $\{\phi_i \otimes \psi_j\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof : (stack) Can we discover the bases of two-infinitely dimensional vector spaces into a tensor product ?

① Assume, $V, W \neq 0$, i.e., $\{\vec{w}_j\} \neq \phi$

W is some fixed vector from the indexed set $\{\vec{w}_j\}$.

Let $g: V \otimes W \to V$ be the linear function defined by $\vec{v}_i \otimes \vec{w}_j \to \vec{v}_j$.

Let $f: V \to V \otimes W$ be the linear function $f(\vec{v}_i) = \vec{v}_i \otimes \vec{w}$.

S.

suppose,

$a_1 \vec{v}_1 + \cdots + a_k \vec{v}_k = 0$ for some

scalars $a_1, \ldots, a_k$.

Then,

$0 = f(a_1 \vec{v}_1$

$= a_1 (\vec{v}$

Since $\{\vec{v}_i \otimes$

$\Longrightarrow a_1 =$

$\therefore v_1, \ldots$

Similarly,

$\{w_j\}$

Then,

$$0 = f(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k)$$

$$= a_1(\vec{v}_1 \otimes \vec{w}) + \cdots\cdots + a_k(\vec{v}_k \otimes w)$$

Since $\{\vec{v}_i \otimes \vec{w}_j\}$ is linearly independent

$$\Longrightarrow a_1 = \cdots\cdots = a_k = 0$$

$\therefore$ $v_1, \cdots, v_k$ are linearly independent.

Similarly,

$\{w_j\}$ are linearly independent.

(2)

Theorem 1

Let V and W are normed spaces such that $v_i \in V$ and $w_i \in W$.

12/9/21

If $\sum_i v_i \otimes w_i = 0$ and $\{v_i\}$ are linearly independent, then $w_i = 0$ for all i.

Proof

Let V, W are normed vector spaces over the field $\mathbb{F}$ such that $v_i \in V$ and $w_i \in W$.

Let V^*, W^* are algebraic dual spaces of the vector spaces V, W respectively., are defined as the set of all linear maps $f : V \to \mathbb{F}$ and $g : W \to \mathbb{F}$.

Let's define a bilinear map $h: V \times W \longrightarrow \mathbb{F}$ such that $h(v,w) = f(v)\,g(w)$ for all $v \in V$ and $w \in W$.

From Theorem 2,

there exists a unique linear map $h_{\otimes}: V \otimes W \to \mathbb{F}$ such that $h_{\otimes}(v \otimes w) = h(v,w)$, for every $v \in V$ and $w \in W$.

i.e.,

$$h_{\otimes}\Big(\sum_{i,j} c_{ij}\,(v_i \otimes w_j)\Big) = \sum_{i,j} c_{ij}\, h(v_i, w_j)$$

Let V and W be vector spaces over the same field $\mathbb{F}$. A function $f: V \to W$ is said to be a linear map if for any 2 vectors $u, v \in V$ and any scalar $c \in \mathbb{F}$, the following 2 conditions are satisfied:

(i) $f(u+v) = f(u) + f(v)$

(ii) $f(cu) = c\,f(u)$

$h_\otimes$ is a linear map from $V \otimes W$ to $\mathbb{F}$.

$$\Rightarrow h_\otimes(0v) = 0 h_\otimes(v) = 0$$

$$0 = \sum_i v_i \otimes w_i \Longrightarrow$$

$$0 = h_\otimes(0) = h_\otimes(0 \otimes 0) = h_\otimes\left(\sum_i v_i \otimes w_i\right)$$

$$= \sum_i h(v_i, w_i) = \sum_i f(v_i)\, g(w_i)$$

$$= \sum_i f\left(v_i\, g(w_i)\right) = f\left(\sum_i v_i\, g(w_i)\right)$$

Since $f\left(\sum_i v_i\, g(w_i)\right) = 0$ for every $f \in V^*$,

say $f(v) = \|v\|$ then $\|v\| = 0 \Rightarrow v = 0$.

We can conclude that $\sum_i v_i\, g(w_i) = 0$

Since v

$\sum_i v_i g(w$

Since g

$\Longrightarrow$

Theorem 2:

For every

bilinear ma

unique line

the property

$h_\otimes(v \otimes$

Since v_i is a linearly independent set

$$\sum_i v_i\, g(w_i) = 0 \Longrightarrow g(w_i) = 0 \text{ for each } i$$

Since $g(w_i) = 0$ for all $g \in W^*$

$$\Longrightarrow w_i = 0 \text{ for all } i$$

Theorem 2:

For every vector space P, and every bilinear map $h: V \times W \to P$, there is a unique linear map $h_\otimes : V \otimes W \to P$ with the property that

$$h_\otimes(v \otimes w) = h(v, w) \text{ for every } v \in V, w \in W.$$

Back
12/9/21

Proof

Uniqueness is clear, since any linear map on $V \otimes W$ is completely determined by what it does to elements of the form $v \otimes w$.

Define the bilinear map $\phi : V \times W \to V \otimes W$ such that

$$\phi\left(\sum_i a_i v_i, \sum_j b_j w_j\right) = \sum_{i,j} a_i b_j (v_i \otimes w_j)$$

This gives a complete definition because every element of V is a unique linear combination of the $\{v_i\}$ and every element of W is a unique linear combination of the $\{w_j\}$.

If $h : V \times W \to P$ is some bilinear map, we define $h_\otimes : V \otimes W \to P$ as:

$$h_\otimes\left(\sum_{i,j} c_{ij}(v_i \otimes w_j)\right) = \sum_{i,j} c_{ij}\, h(v_i, w_j)$$

such that

$\Rightarrow$ If P

$h : V \times W \to P$

is a unique

such that

such that $h = h_{\otimes} \circ \phi$.

$\Longrightarrow$ If P is a vector space and $h : V \times W \to P$ is a bilinear map, then there is a unique linear map $h_{\otimes} : V \otimes W \to P$ such that $h_{\otimes} \circ \phi = h$.

* Let V and W be normed spaces. If $\{v_i\} \in V$ and $\{w_j\} \in W$ are linearly independent in V and W respectively, then $\{v_i \otimes w_j\}$ is linearly independent in the algebraic tensor product $V \otimes W$.

Proof.

Suppose that,

$$0 = \sum_i \sum_j a_{ij} (v_i \otimes w_j)$$

$$= \sum_i \left(v_i \otimes \sum_j a_{ij} w_j \right)$$

Since $\{v_i\}$ is linearly independent in V. from theorem 1,

$$\sum_j a_{ij} w_j = 0 \quad \text{for each } i$$

Since $\{W_j\}$ is linearly independent,

$$\sum_j a_{ij} W_j = 0 \longrightarrow a_{ij} = 0 \text{ for each } j \text{ as well.}$$

$\Longrightarrow V_i \otimes W_j$ is linearly independent in $V \otimes W$.

Schmidt decomposition

Suppose a bipartite state

$$|\psi\rangle = \sum_{i,j} c_{ij} |e_{1,i}\rangle \otimes |e_{2,j}\rangle$$

is given. We are interested in when the state is separable and when entangled.

The expansion coeff. in the above state of the combined system contain $\dim \mathcal{H}_1 \dim \mathcal{H}_2$ terms and is very difficult to manipulate with.

Solution $\Longrightarrow$ Schmidt decomposition of $|\psi\rangle$

* Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be the Hilbert space of a bipartite system. Then a vector $|\psi\rangle \in \mathcal{H}$ admits the Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i}\, |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

$$\text{with } \sum_i s_i = 1$$

where, s_i : Schmidt coefficients

$\{|f_{a,i}\rangle\}$: an orthonormal set of $\mathcal{H}_a$

$r \in \mathbb{N}$: Schmidt number of $|\psi\rangle$

Proof

$$|\psi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} c_{ij}\, |e_{1,i}\rangle \otimes |e_{2,j}\rangle$$

The coeff. c_{ij} form a $\dim \mathcal{H}_1 \times \dim \mathcal{H}_2$ matrix C.

SVD of C : $C = U \Sigma V^{+}$

U, V : unitary matrices.

$$|\psi\rangle = \sum_{i,j,k,l} U_{ik}\, \Sigma_{kl}\, V^{*}_{jl}\, |e_{1,i}\rangle \otimes |e_{2j}\rangle$$

$$= \sum_{k,l} \Sigma_{kl} \left[\sum_{i=1}^{d_1} U_{ik} |e_{1,i}\rangle \otimes \sum_{j=1}^{d_2} V^{*}_{jl} |e_{2,j}\rangle \right]$$

$l = ?$

Define,

$$\sum_{i=1}^{h_1} U_{ik} |e_{1,i}\rangle = |f_{1,k}\rangle \quad \text{and}$$

$$\sum_{j=1}^{h_2} V^*_{jk} |e_{2,j}\rangle = |f_{2,k}\rangle$$

Unitarity of U & V guarantees that they are orthonormal bases of $\mathcal{H}_1$ & $\mathcal{H}_2$.

Check

$$\||f_{1,k}\rangle\|^2 = \langle f_{1,k}|f_{1,k}\rangle = \sum_i \bar{U}_{ik} \langle e_{1,i}| \cdot \sum_j U_{jk} |e_{1,j}\rangle$$

$$= \sum_{i,j} \bar{U}_{ik} \langle e_{1,i}|e_{1,j}\rangle U_{jk}$$

$$= \sum_i \bar{U}_{ik} U_{ik} \langle e_{1,i}|e_{1,i}\rangle \quad \left[\langle e_{1,i}|e_{1,j}\rangle = \delta_{ij}\right]$$

$$= \sum_i |U_{ik}|^2 = 1 \quad \left[U \text{ is unitary}\right]$$

For $k \neq m$,

$$\langle f_{1,k}|f_{1,m}\rangle = \sum_i \bar{U}_{ik} \langle e_{1,i}| \cdot \sum_j U_{jm} |e_{1,j}\rangle$$

$$= \sum_{i,j} \bar{U}_{ik} \langle e_{1,i}|e_{1,j}\rangle U_{jm}$$

$$= \sum_i \bar{U}_{ik} \langle e_{1,i}|e_{1,i}\rangle U_{im} \quad \left[\langle e_{1,i}|e_{1,j}\rangle = \delta_{ij}\right.$$

$$= \sum_i \bar{U}_{ik} U_{im} = 0 \quad \left[\begin{array}{l} k \neq m \text{ \& } \\ U \text{ is unitary} \end{array}\right.$$

$$\Rightarrow \underline{\langle f_{1,k} | f_{1,m} \rangle = \delta_{km}}$$

$$\Sigma_{kl} = d_k \delta_{kl} ,$$

$$|\psi\rangle = \sum_{i=1}^{r} d_i \, |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

r: # of non-vanishing diagonal elements in Σ.

Replacing the +ve # d_i by $d_i = \sqrt{s_i}$

& ~~Normalization condition~~

~~$\langle\psi|\psi\rangle$~~

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i}\, |f_{1,i}\rangle \otimes |f_{2,i}\rangle$$

Normalization condition,

$$\langle\psi|\psi\rangle = \sum_i s_i = 1$$

$\Rightarrow$ A bipartite state is separable iff its Schmidt number r is 1.

ie, # of non-vanishing diagonal elements in Σ is 1.

Check QC(2)

Schmidt rank of $|\psi\rangle = \xi \otimes \xi$
||
rank of $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$

Ex:- Consider the bipartite state

$$|\psi\rangle = \frac{1}{2}\Big(|e_{1,1}\rangle|e_{2,1}\rangle + |e_{1,1}\rangle|e_{2,2}\rangle + i|e_{1,3}\rangle|e_{2,1}\rangle + i|e_{1,3}\rangle|e_{2,2}\rangle\Big)$$

Ans: The coeff. of $|\psi\rangle$ form a matrix

$$C_{n_1 \times n_2} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ i & i \end{bmatrix} = U \Sigma V^{\dagger}$$

where, $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ i & 0 & -i \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

~~If~~ # of non-vanishing diagonal elts = 1

$$|f_{1,1}\rangle = \sum_{i=1}^{3} U_{i1} |e_{1,i}\rangle = \frac{1}{\sqrt{2}}\left(|e_{1,1}\rangle + |e_{1,3}\rangle\right)$$

$$|f_{2,1}\rangle = \sum_{j=1}^{2} V_{j1}^{*} |e_{2,j}\rangle = \frac{1}{\sqrt{2}}\left(|e_{2,1}\rangle + |e_{2,2}\rangle\right)$$

The Schmidt number is 1 and the state is separable.

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i}\, |f_{1,i}\rangle \otimes |f_{2,i}\rangle = 2\,|f_{1,1}\rangle \otimes |f_{2,1}\rangle$$

Generat
components
be obviou
components

$\mathcal{H} = \mathcal{H}_1$

$\mathcal{H}_a$:

* Classificati
multipartite
and an
decompositio
$N \geq 3$.

Generalization to a system with more components, i.e., a multipartite system, should be obvious. A system composed of N components has a Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$$

$\mathcal{H}_a$: Hilbert space to which the a^{th} comp. belongs.

* Classification of entanglement in a multipartite system is far from obvious, and an analogue of the Schmidt decomposition is not known to date for $N \geq 3$.

Quantum measurement – revisited

How does one mathematically model the act of measuring or observing a quantum system ?

The very act of looking at or observing a quantum system irreversibly alters the state of the system. To model this phenomenon, we shall use the notion of a projective or von Neumann measurement.

Linear maps

Given a vector $|\psi\rangle \in \mathbb{C}^d$, we are interested in how $|\psi\rangle$ can be "mapped" to other vectors. The maps we consider are linear, which means that for map $\Phi : \mathbb{C}^d \to \mathbb{C}^d$ and arbitrary $\sum_i \alpha_i |\psi_i\rangle \in \mathbb{C}^d$,

$$\Phi\left(\sum_i \alpha_i |\psi_i\rangle\right) = \sum_i \alpha_i \Phi(|\psi_i\rangle)$$

The set of linear maps from vector space X to Y is denoted $\mathcal{L}(X, Y)$. i.e., $\mathcal{L}(X) = \mathcal{L}(X, X)$

Hermitian, positive semi-definite & orthogonal projection operators

1. Hermitian operators :

An operator $M \in \mathcal{L}(\mathbb{C}^d)$ is Hermitian iff $M = M^\dagger$.

All of its eigenvalues are real.

$\lambda \in \mathbb{R}$

$\longrightarrow$ Hermitian operators can be thought of as a higher dimensional generalization of the real #.

Ex:- Pauli X, Y and Z gates.
– unitary & Hermitian.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hadamard gate – unitary & Hermitian

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

2. Positive semi-definite operators:

If a Hermitian operator has only non-negative eigenvalues, then it is called positive-semi definite.

$\lambda \in \mathbb{R}_+ + \{0\}$.

$\Longrightarrow$ Positive semi-definite matrices generalize the non-negative real numbers.

3. Orthogonal projection operators:

ILA 10

ILA 5*

A Hermitian matrix $\Pi \in \mathcal{L}(\mathbb{C}^d)$ is an orthogonal projection operator if $\Pi^2 = \Pi$.

Π has only eigenvalues 0 and 1.

$$\lambda \in \{0, 1\}$$

<u>Proof</u>.

spectral decomposition

$$\Pi \text{ is Hermitian} \longrightarrow \Pi = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$\begin{aligned}\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| = \Pi = \Pi^2 \\ &= \left(\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|\right)\left(\sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|\right) \\ &= \sum_i \lambda_i^2 |\lambda_i\rangle\langle\lambda_i|\end{aligned}$$

[$\{|\lambda_i\rangle\}$ forms an orthonormal basis]

$|\lambda_i\rangle$ are orthogonal

$\Longrightarrow \lambda_i = \lambda_i^2$ for all i

$\therefore \lambda_i \in \{0, 1\}$.

Ex: Verify that I, $|0\rangle\langle 0|$, $|1\rangle\langle 1|$ are all projectors. Show that for arbitrary unit vector $|\psi\rangle \in \mathbb{C}^d$, $|\psi\rangle\langle\psi|$ is a projector.

Ans: 1. $I^\dagger = I$ and $I^2 = I$

2. $\left(|0\rangle\langle 0|\right)^\dagger = \left(\langle 0|\right)^\dagger\left(|0\rangle\right)^\dagger = |0\rangle\langle 0|$

$\left(|0\rangle\langle 0|\right)^2 = |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0|$

3. For an arbitrary unit vector $|\psi\rangle \in \mathbb{C}^d$

$\langle\psi|\psi\rangle = 1$ and $\left(|\psi\rangle\right)^\dagger = \langle\psi|$.

$\left(|\psi\rangle\langle\psi|\right)^\dagger = \left(\langle\psi|\right)^\dagger\left(|\psi\rangle\right)^\dagger = |\psi\rangle\langle\psi|$

$\left(|\psi\rangle\langle\psi|\right)^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$.

$\Rightarrow$ For an arbitrary unit vector $|\psi\rangle \in \mathbb{C}^d$,

$P := |\psi\rangle\langle\psi|$ is a projection operator.

Since a projector Π's eigenvalues are all 0 or 1, its spectral decomposition must take the form:

Projection operator

$$\boxed{\Pi = \sum_i |\psi\rangle\langle\psi| \quad \text{where, } \{|\psi_i\rangle\} \text{ are an orthonormal set}}$$

* Let $\{|\psi_i\rangle\}$ be an orthonormal set; Prove that $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ is a projector.

$$\boxed{\langle\psi_i|\psi_j\rangle = \delta_{ij} \rightarrow \Pi = \sum_i |\psi_i\rangle\langle\psi_i| \text{ is a projection operator.}}$$

Proof

$$\Pi^\dagger = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)^\dagger = \sum_i \left(|\psi_i\rangle\langle\psi_i|\right)^\dagger$$

$$= \sum_i |\psi_i\rangle\langle\psi_i|$$

$$\Pi^2 = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)\left(\sum_j |\psi_j\rangle\langle\psi_j|\right)$$

$$= \sum_{i,j} |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|$$

$$= \sum_i |\psi_i\rangle\langle\psi_i| \quad \left[\langle\psi_i|\psi_j\rangle = \delta_{ij}\right.$$

$$= \Pi$$

* A projector Π has rank 1 if and only if $\Pi = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathbb{C}^d$. since the rank of Π equals the # of non-zero eigenvalues of Π, and here $\Pi = |\psi\rangle\langle\psi|$ is a spectral decomposition of Π.

For any projector $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ and state $|\phi\rangle$ to be measured, we have

$$\Pi|\phi\rangle = \left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\phi\rangle$$

$$= \sum_i |\psi_i\rangle\left(\langle\psi_i|\phi\rangle\right)$$

$$= \sum_i \left(\langle\psi_i|\phi\rangle\right)|\psi_i\rangle \in \text{span}(\{|\psi_i\rangle\})$$

where $\langle\psi_i|\phi\rangle \in \mathbb{C}$.

$\Longrightarrow \Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ projects $|\phi\rangle$ down onto the span of the vectors $\{|\psi_i\rangle\}$.

Ex:- Consi

and Π

and ob

in the

Ans:

$\Pi|\phi\rangle = ($

Ex:- Consider 3D vector $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \in \mathbb{C}^3$ and $\Pi = |0\rangle\langle 0| + |1\rangle\langle 1|$. Compute $\Pi|\phi\rangle$ and observe that the latter indeed lies in the 2D space $\mathrm{span}(\{|0\rangle, |1\rangle\})$.

Ans:

$$\Pi|\phi\rangle = (|0\rangle\langle 0| + |1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle)$$

$$= \cancel{\alpha \langle 0|0\rangle\langle 0| +}$$

$$= (\alpha\langle 0|0\rangle)|0\rangle + (\beta\langle 0|1\rangle)|0\rangle + (\gamma\langle 0|2\rangle)|0\rangle$$

$$(\alpha\langle 1|0\rangle)|0\rangle + (\beta\langle 1|1\rangle)|1\rangle + (\gamma\langle 1|2\rangle)|1\rangle$$

$$= (\alpha\langle 0|0\rangle + \beta\langle 0|1\rangle + \gamma\langle 0|2\rangle)|0\rangle$$

$$+ (\alpha\langle 1|0\rangle + \beta\langle 1|1\rangle + \gamma\langle 1|2\rangle)|1\rangle$$

$$\in \mathrm{span}(\{|0\rangle, |1\rangle\})$$

$$= \alpha|0\rangle + \beta|1\rangle$$

Spectral decomposition

$$A = \sum_{i=1}^{d} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

where λ_i and $|\lambda_i\rangle$ are the eigenvalues and corresp. eigenvectors of A.

The spectral decomposition tells us how A acts on $\mathbb{C}^d$, because the eigenvectors $|\lambda_i\rangle \in \mathbb{C}^d$ form an orthonormal basis for $\mathbb{C}^d$.

$\therefore$ Any vector $|\psi\rangle \in \mathbb{C}^d$ can be written in terms of the eigenvectors of A, i.e., $|\psi\rangle = \sum_i \alpha_i |\lambda_i\rangle$ for some $\alpha_i \in \mathbb{C}$.

The spectral decomposition immediately reveals the rank(A).

rank(A) = # of non-zero eigenvalues of A.

$$Tr(A) = Tr\left(\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|\right)$$

$$\left[\begin{array}{l} Tr(A+B) = Tr(A) + Tr(B) \\ Tr(kA) = k\,Tr(A) \end{array}\right.$$

$$= \sum_i \lambda_i \, Tr\left(|\lambda_i\rangle\langle\lambda_i|\right)$$

$$\left[\, Tr(ba^T) = ab^T \right.$$

$$= \sum_i \lambda_i \langle\lambda_i|\lambda_i\rangle$$

$$\boxed{\begin{aligned} Tr(A) &= \sum_i \langle\lambda_i|A|\lambda_i\rangle \\ &= \sum_i \lambda_i \end{aligned}}$$

Projective measurements

A projective measurement is a set of projectors $B = \{\Pi_i\}_{i=1}^{m}$ such that $\sum_{i=1}^{m} \Pi_i = 1$. The later condition is called the completeness relation. and also satisfy $\Pi_i \Pi_j = \delta_{ij} \Pi_i$.

If each Π_i is rank one, i.e., $\Pi_i = |\psi_i\rangle\langle\psi_i|$ then we say that B models a measurement in basis $\{|\psi_i\rangle\}$.

Often, we shall measure in the computational basis, which is specified by $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ in the case of $\mathbb{C}^2$, and generalizes as $B = \{|i\rangle\langle i|\}_{i=0}^{d-1}$ for $\mathbb{C}^d$.

* A projective measurement is described by an observable, Π, a Hermitian operator on the state space of the system being observed. The observable has a spectral decomposition

$$\Pi = \sum_i \lambda_i |\lambda_i \rangle\langle \lambda_i|$$

where $|\lambda_i\rangle\langle\lambda_i|$ is the projector onto the eigenspace of Π with eigenvalue λ_i

Projective measurement : $\mathcal{B} = \{\Pi_i\}_{i=1}^{m}$

How to use $\mathcal{B}$?

Suppose,
our quantum system is in state $|\psi\rangle \in \mathbb{C}^d$.

Then,
the probability of obtaining outcome $i \in \{0, \ldots, m\}$ when measuring $|\psi\rangle$ with $\mathcal{B}$ is given by :

$$\boxed{\begin{aligned} P_r(\text{outcome } i) &= \| \Pi_i |\psi\rangle \|^2 = \langle\psi|\Pi_i^2|\psi\rangle = \langle\psi|\Pi_i|\psi\rangle \\ &= Tr\left(\Pi_i |\psi\rangle\langle\psi| \Pi_i\right) \\ &= Tr\left(\Pi_i^2 |\psi\rangle\langle\psi|\right) = Tr\left(\Pi_i |\psi\rangle\langle\psi|\right) \end{aligned}}$$

$$\boxed{\begin{aligned} Tr(ABC) &= Tr(BCA) \\ &= Tr(CAB) \end{aligned}}$$

Proof

① $P_r(\text{outcome } i) = \| \Pi_i |\psi_i\rangle \|^2 = \langle\psi|\Pi_i^2|\psi\rangle$

$= \langle\psi|\Pi_i|\psi\rangle$

Let $\{|\lambda_i\rangle\}$ be the orthonormal eigenbasis of Π_i

$|\psi\rangle = \sum_i \langle\psi|\lambda_i\rangle |\lambda_i\rangle \Rightarrow \langle\psi| = \sum_i \langle\psi|\lambda_i\rangle\langle\lambda_i|$

$= \sum_i \underline{\langle\psi|\lambda_i\rangle}\langle\lambda_i|\Pi_i|\psi\rangle$

$= \sum_i \langle\lambda_i|\underline{\Pi_i|\psi\rangle\langle\psi}|\lambda_i\rangle$

$= T_r\left(\Pi_i|\psi\rangle\langle\psi|\right)$

② P_r (out

T_r (b

② $P_r(\text{outcome } i) = \| \Pi_i |\psi\rangle \|^2$

$$= \langle \psi | \Pi_i \Pi_i | \psi \rangle = \langle \psi | \Pi_i^2 | \psi \rangle \cancel{= \langle \psi}$$

$$= \langle \psi | \Pi_i | \psi \rangle$$

$$\mathrm{Tr}(b a^\dagger) = a^\dagger b$$

$$= \mathrm{Tr}\left(\Pi_i |\psi\rangle\langle\psi| \Pi_i \right)$$

$$= \mathrm{Tr}\left(\Pi_i |\psi\rangle\langle\psi| \right)$$

Ex:- Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$.

Show that if we measure in the computational basis, i.e., using $B = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ then the probabilities of obtaining outcomes 0 and 1 are $|\alpha|^2$ and $|\beta|^2$, respectively.

Ans:

$$P_r(\text{outcome } i{=}0) = Tr(\Pi_i |\psi\rangle\langle\psi|)$$

$$= Tr\left(|0\rangle\langle 0|\psi\rangle\langle\psi|\right)$$

$$= Tr\left(|0\rangle\langle 0|\left(\alpha|0\rangle + \beta|1\rangle\right)\left(\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|\right)\right)$$

$$= Tr\left(|0\rangle\langle 0|\left(|\alpha|^2|0\rangle\langle 0| + \cancel{|\beta|^2|1\rangle\langle 1|} + \cancel{\beta\bar{\alpha}|1\rangle\langle 0|} + \alpha\bar{\beta}|0\rangle\langle 1|\right)\right)$$

$$= Tr\left(|\alpha|^2|0\rangle\langle 0| + \alpha\bar{\beta}|0\rangle\langle 1|\right)$$

$$= |\alpha|^2 Tr(|0\rangle\langle 0|) + \alpha\bar{\beta}\, Tr(|0\rangle\langle 1|)$$

$$= |\alpha|^2 \langle 0|0\rangle + \alpha\bar{\beta}\langle 0|1\rangle$$

$$= |\alpha|^2$$

Pr (outcom

Requirin
a unit
that whe
distributio
probability
i.e., the
outcomes s
Measuremen

$$Pr(\text{outcome } i=1) = Tr\left(|1\rangle\langle 1|\psi\rangle\langle\psi|\right)$$

$$= Tr\left(|1\rangle\langle 1|\left(\alpha|0\rangle+\beta|1\rangle\right)\left(\bar{\alpha}\langle 0|+\bar{\beta}\langle 1|\right)\right)$$

$$= Tr\Big(|1\rangle\langle 1|\big(|\alpha|^2|0\rangle\langle 0|+|\beta|^2|1\rangle\langle 1| + \alpha\bar{\beta}|0\rangle\langle 1|+\beta\bar{\alpha}|1\rangle\langle 0|\big)\Big)$$

$$= Tr\left(|\beta|^2|1\rangle\langle 1|+\beta\bar{\alpha}\,|1\rangle\langle 0|\right)$$

$$= |\beta|^2\, Tr\left(|1\rangle\langle 1|\right)+\beta\bar{\alpha}\, Tr\left(|1\rangle\langle 0|\right)$$

$$= |\beta|^2\langle 1|1\rangle+\beta\bar{\alpha}\,\langle 1|0\rangle$$

$$= |\beta|^2$$

→ Requiring a quantum state $|\psi\rangle$ to be a unit vector (i.e., $|\alpha|^2+|\beta|^2=1$) ensures that when measuring $|\psi\rangle$, the distribution over the outcomes is a valid probability distribution.

i.e., the probabilities for all possible outcomes sum to 1.

Measurements in Q.M are inherently probabilistic.

* The very act of measuring a quantum state disturbs the system.

Let's formalize:

A projector projects a vector $|\psi\rangle$ down into a smaller subspace.

Upon obtaining outcome Π_i when measuring B, the state of the system "collapses" to

$$\frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\mathrm{Tr}(\Pi_i|\psi\rangle\langle\psi|\Pi_i)} = \frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\mathrm{Tr}(\Pi_i|\psi\rangle\langle\psi|)}$$

where,

$$\Pr(\text{outcome } i) = \mathrm{Tr}(\Pi_i|\psi\rangle\langle\psi|)$$

Numerator

The o/p state is indeed a vector, namely $\Pi_i|\psi\rangle$. However, there is a more general formalism called the density operator formalism, in which quantum states are written as matrices, not vectors. The density matrix representing vector $|\psi\rangle$ would be written as matrix $|\psi\rangle\langle\psi|$.

Denominator

Since we projected out part of $|\psi\rangle$ during the measurement, the output $\Pi_i|\psi\rangle$ may not necessarily be normalized. To renormalize $\Pi_i|\psi\rangle$, we simply divide by its Euclidean norm to obtain

$|\psi'\rangle =$

The st
state of
obtained

The den

$|\psi'\rangle\langle\psi'|$

$$|\psi'\rangle = \frac{\Pi_i|\psi\rangle}{\|\Pi_i|\psi\rangle\|} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i\Pi_i|\psi\rangle}} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}$$

The state $|\psi'\rangle$ describes the post-measurement state of our system, assuming we have obtained outcome i.

The density matrix,

$$|\psi'\rangle\langle\psi'| = \frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\langle\psi|\Pi_i|\psi\rangle} = \frac{\Pi_i|\psi\rangle\langle\psi|\Pi_i}{\mathrm{Tr}(\Pi_i|\psi\rangle\langle\psi|)}$$

Ex:-

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Show that if we measure in the computational basis, i.e., using $\mathcal{B} = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and obtain outcome $i \in \{0,1\}$ then the post-measurement state is $|i\rangle$ (or $|i\rangle\langle i|$ in density matrix form).

Ans: $$|\psi'\rangle = \frac{\Pi_i|\psi\rangle}{\|\Pi_i|\psi\rangle\|} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}$$

For $\Pi_i = |0\rangle\langle 0|$

$$\Pi_i|\psi\rangle = |0\rangle\langle 0|(\alpha|0\rangle + \beta|1\rangle)$$
$$= \alpha|0\rangle$$

$$\|\Pi_i|\alpha\rangle\| = |\alpha|^2$$

$$|\psi'\rangle = \frac{\alpha|0\rangle}{|\alpha|}$$

$$|\psi'\rangle = |0\rangle$$ [since $\left|\frac{\alpha}{|\alpha|}\right| = 1$, can be ignored in the post measurement state.

Consequence of the fact that a projector Π_i satisfies $\Pi_i^2 = \Pi_i$?

If you observe a quantum system now, and then again 5 minutes from now, and if the system has not been subjected to any gates or noise in b/w the measurements, then the two measurements results you obtain should agree. 1

To model this,

suppose we measure using $B = \{\Pi_i\}$ and obtain ~~obtain~~ results i and j in measurements 1 and 2, respectively. Then:

$$\boxed{\begin{aligned} \Pr(\text{outcome } j \mid \text{outcome } i) &= \mathrm{Tr}\left(\Pi_j \frac{\Pi_i |\psi\rangle\langle\psi| \Pi_i}{\mathrm{Tr}(\Pi_i |\psi\rangle\langle\psi|)} \Pi_j\right) \\ &= \frac{\mathrm{Tr}(\Pi_j \Pi_i |\psi\rangle\langle\psi| \Pi_i \Pi_j)}{\mathrm{Tr}(\Pi_i |\psi\rangle\langle\psi|)} . \end{aligned}}$$

If completeness relation holds for projectors $\{\Pi_i\}$, ie., $\sum_i \Pi_i = 1$.

$$\Longrightarrow \Pi_j \Pi_i = \delta_{ij} \Pi_i$$

If $i \neq j$,

$$P_r(\text{outcome } j \mid \text{outcome } i) = 0$$

If $i=j$,

$$Pr(\text{outcome } j | \text{outcome } i) = \frac{Tr(\Pi_j \Pi_i |\psi\rangle\langle\psi| \Pi_i \Pi_j)}{Tr(\Pi_i |\psi\rangle\langle\psi|)}$$

$$Pr(\text{outcome } i | \text{outcome } i) = \frac{Tr(\Pi_i^2 |\psi\rangle\langle\psi| \Pi_i^2)}{Tr(\Pi_i |\psi\rangle\langle\psi|)}$$

$$= \frac{Tr(\Pi_i |\psi\rangle\langle\psi| \Pi_i)}{Tr(\Pi_i |\psi\rangle\langle\psi|)}$$

$$= \frac{Tr(\Pi_i |\psi\rangle\langle\psi|)}{Tr(\Pi_i |\psi\rangle\langle\psi|)} \quad \left[\begin{aligned} Tr(ABC) &= Tr(BCA) \\ &= Tr(CAB) \end{aligned} \right.$$

$$= 1$$

⟹ Measuring the state a second time again yields outcome i with probability 1.

Although observing a state for the 1st time disturbs it, subsequent measurements will consistently return the same measurement result.

Ex:- Suppose we measure $|0\rangle$ in basis $B = \{|+\rangle\langle+|, |-\rangle\langle-|\}$. What are the probabilities of outcomes $+$ and $-$, respectively? What are the post-measurement states if one obtains outcome $+$ or $-$, resp.?

Ans.

$$Pr(\text{outcome } i) = \cancel{\| \Pi_i |\psi\rangle \|} = \| \langle\psi|\Pi_i|\psi\rangle \|$$

$$= Tr\left(\Pi_i |\psi\rangle\langle\psi|\right)$$

$$Pr(\text{outcome } +) = Tr\left(|+\rangle\langle+|0\rangle\langle0|\right)$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |-\rangle)$, $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$$= Tr\left(|+\rangle\langle+|\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{\langle+| + \langle-|}{\sqrt{2}}\right)\right)$$

$$= Tr\left(\frac{|+\rangle}{\sqrt{2}} \frac{(\langle+| + \langle-|)}{\sqrt{2}}\right)$$

$$= Tr\left(\frac{|+\rangle\langle+|}{2} + \frac{|+\rangle\langle-|}{2}\right)$$

$$= \frac{1}{2}\langle+|+\rangle + \frac{1}{2}\langle+|-\rangle = \frac{1}{2}$$

Similarly,

$$Pr(\text{outcome } -) = \frac{1}{2}$$

$$|\psi^+\rangle = \frac{\Pi_i|\psi\rangle}{\| \Pi_i|\psi\rangle \|} = \frac{\Pi_i|\psi\rangle}{\sqrt{\langle\psi|\Pi_i|\psi\rangle}}$$

$$= \frac{\Pi_i|\psi\rangle}{\sqrt{\mathrm{Tr}\left(\Pi_i|\psi\rangle\langle\psi|\right)}}$$

$$= \frac{|+\rangle\langle+|0\rangle}{1/\sqrt{2}}$$

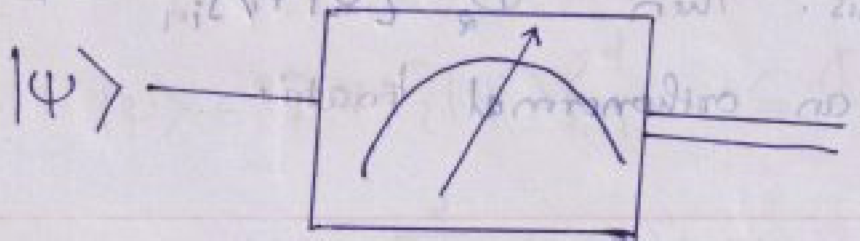
$$= \sqrt{2}\,|+\rangle\langle+|\left(\frac{|+\rangle+|-\rangle}{\sqrt{2}}\right)$$

$$= |+\rangle$$

$$|\psi^-\rangle = |-\rangle$$

□ Simulating arbitrary measurements via standard basis measurements

The circuit symbol which denotes a measurement of a qubit $|\psi\rangle \in \mathbb{C}^2$ in the computational basis $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$:



The double-wire on the right side indicate that the o/p of the measurement is a classical string (indicating which measurement outcome was obtained).

Projective measurements allow measuring in an arbitrary basis $B = \{|\psi_1\rangle, |\psi_2\rangle\} \subseteq \mathbb{C}^2$

Without loss of generality we can restrict ourselves to measurements in the standard basis

* Let $B_1 = \{|\psi_i\rangle\}_{i=1}^{d} \subseteq \mathbb{C}^d$ be an orthonormal basis. Then $B_2 = \{U|\psi_i\rangle\}_{i=1}^{d} \subseteq \mathbb{C}^d$ is an orthonormal basis.

* For any orthonormal bases $B_1 = \{|\psi_i\rangle\}_{i=1}^{d} \subseteq \mathbb{C}^d$ and $B_2 = \{|\phi_i\rangle\}_{i=1}^{d} \subseteq \mathbb{C}^d$,

there exists a unitary U mapping B_1 to B_2, given by

$$U = \sum_{i=1}^{d} |\phi_i\rangle\langle\psi_i|$$

such that $U|\psi_i\rangle = |\phi_i\rangle$

Proof

VVIQ $P_{B_2 \leftarrow B_1} = B_2^{-1} B_1 = B_2^{T} B_1$

Stacks 21/2/21

$$= \left[|\phi_1\rangle \cdots\cdots |\phi_d\rangle\right]^{T} \left[|\psi_1\rangle \cdots\cdots |\psi_d\rangle\right]$$

Its different

$$U|\psi_i\rangle = |\phi_i\rangle \quad \text{for all } i = 1, \dots, d$$

For $|\psi\rangle \in B_1 = \{|\psi_i\rangle\}_{i=1}^{d} \subseteq \mathbb{C}^d$

$$|\psi\rangle = \sum_{i=1}^{d} \langle\psi|\psi_i\rangle|\psi_i\rangle$$

$$U|\psi\rangle = \sum_{i=1}^{d} \langle\psi|\psi_i\rangle\, U|\psi_i\rangle = \sum_{i=1}^{d} \langle\psi|\psi_i\rangle|\phi_i\rangle$$

$$= \sum_{i=1}^{d} \langle\psi_i|\psi\rangle|\phi_i\rangle = \sum_{i=1}^{d} |\phi_i\rangle\langle\psi_i|\psi\rangle$$

$$= \left(\sum_{i=1}^{d} |\phi_i\rangle\langle\psi_i|\right)|\psi\rangle$$

Simulating a measurement in an arbitrary basis $B=\{|\psi_1\rangle, |\psi_2\rangle\}$ on $\mathbb{C}^2$ with standard basis measurements.

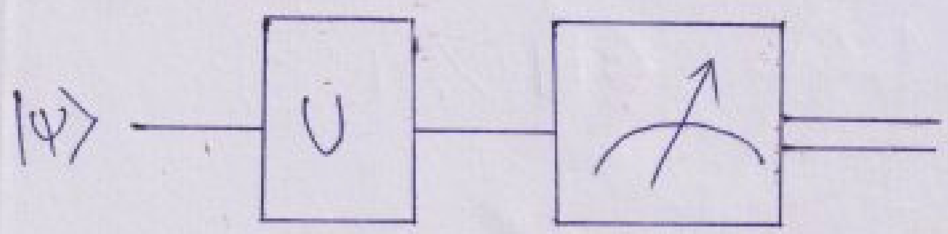
There exists a unitary mapping $U=\sum_{i=1}^{d}|\phi_i\rangle\langle\psi_i|$ from B to the standard basis,

i.e., $U|\psi_1\rangle=|0\rangle$ and $U|\psi_2\rangle=|1\rangle$

Then,

instead of measuring a state $|\psi\rangle\in\mathbb{C}^2$ in basis B, one can equivalently measure $U|\psi\rangle$ in the standard basis.

i.e., we can simulate a measurement in B via the circuit diagram:



$$\Pr(\text{outcome } |\psi_1\rangle) = \mathrm{Tr}\left(\Pi_1 |\psi\rangle\langle\psi|\right) = \langle\psi|\Pi_1|\psi\rangle$$

$$= \mathrm{Tr}\left(|\psi_1\rangle\langle\psi_1| \cdot |\psi\rangle\langle\psi|\right)$$

$$= \mathrm{Tr}\left(\left(U^{+}|0\rangle\langle 0|U\right)|\psi\rangle\langle\psi|\right)$$

$$= \mathrm{Tr}\left(|0\rangle\langle 0|\left(\underline{U|\psi\rangle\langle\psi|U^{+}}\right)\right)$$

$$\left[\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)\right]$$

⇒ One might as well measure state $U|\psi\rangle$ against operator $|0\rangle\langle 0|$; in order to make measurement for the state $|\psi\rangle$ with operator $|\psi_1\rangle\langle\psi_1|$.

Ex:- Give a quantum circuit for measuring a qubit in the $\{|+\rangle, |-\rangle\}$ basis.

Ans: $U|+\rangle = |0\rangle$ & $U|-\rangle = |1\rangle$

$$\Rightarrow U = \sum_{i=1}^{2} |\phi_i\rangle\langle\psi_i| = |0\rangle\langle+| + |1\rangle\langle-|$$