Introduction to Linear Algebra
- Gilbert Strang

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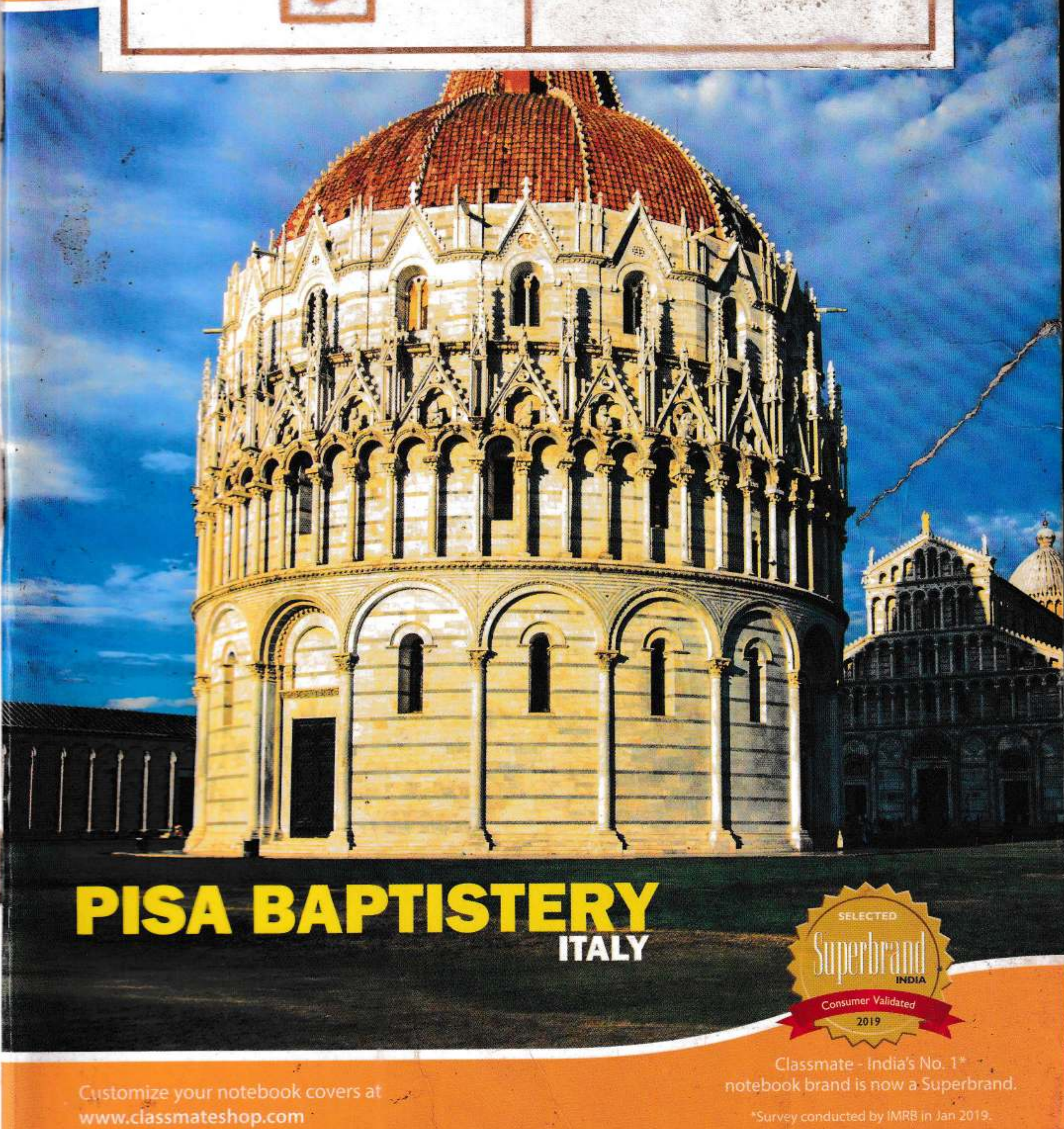
Eigenvalues & Eigenvectors
↓
Singular Value Decomposition



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		– Gilbert Strang, MIT		
		(5th Edition)		

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$\boxed{6.2}$

3. $A = X \Lambda X^{-1}$.
What is the eigenvalue matrix for $A + 2I$?
What is the eigenvector matrix ?

Ans. $A + 2I = X(\Lambda + 2I)X^{-1}$

7. Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Ans: $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \; X^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$A = X \Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$= \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times \frac{1}{2} = \frac{1}{2}\begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$$

9. Suppose, G_{k+2} is the average of the 2 previous numbers G_{k+1} and G_k:

$$\left.\begin{aligned} G_{k+2} &= \frac{1}{2} G_{k+1} + \frac{1}{2} G_k \\ G_{k+1} &= G_{k+1} \end{aligned}\right\} \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

(a) Find the eigenvalues & eigenvectors of A

Ans: $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 1 & -\lambda \end{bmatrix}$$

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0 \implies 2\lambda^2 - \lambda - 1 = 0$$

$$\Delta = 1 + 8 = 9 \implies \lambda = \frac{1 \pm 3}{4} = 1 \text{ (or) } -\frac{1}{2}$$

$$\lambda_1 = -\frac{1}{2}: \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies \begin{aligned} a + \tfrac{1}{2} b &= 0 \\ 2a + b &= 0 \\ b &= -2a \end{aligned}$$

$$x_2 = a \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \cancel{\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

$\lambda_2 = 1$: $\begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow$ $-\frac{1}{2}a + \frac{1}{2}b = 0$

$a - b = 0$

$a = b$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Find the limit as $n \to \infty$ of the matrices $A^n = X \Lambda^n X^{-1}$

Ans: $A^n = X \Lambda^n X = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{bmatrix} \times \frac{-1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$A^\infty = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$u_\infty = A^\infty u_0 = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$

Gibonacci numbers approach 2/3.

True/False

11. If the eigenvalues of A are 2, 2, 5 then the matrix is certainly ________

ⓐ Invertible — TRUE

Ans: $\lambda = 0$ is not an eigenvalue $\Rightarrow$ A is invertible

&

ⓑ diagonalizable — FALSE

~~A diagonalizable~~

$\lambda = 2$ have only one line of eigenvectors

ⓒ Non-diagonalizable — FALSE

12. If the only eigenvectors of A are multiples of (1,4) then A has ________

ⓐ No inverse — False

don't know if $\lambda = 0$ or not

ⓑ a repeated eigenvalue — TRUE

an eigenvector is missing, which can only happen for a repeated eigenvalue

(c) No diagonalization $X \Lambda X^{-1}$ — TRUE

14. $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable because the rank of $A-3I$ is ________.
Change 1 entry to make A diagonalizable.

Ans: $\lambda_1 = \lambda_2 = 3$

$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\text{rank}(A - 3I) = 1$

$\dim[N(A-3I)] = 1 \Rightarrow$ only one eigenvector for $\lambda = 3$.

Changing any entry except $a_{12} = 1$ makes 'A' diagonalizable.

15. $A^k = X\Lambda^k X^{-1}$ approaches zero matrix as $k \to \infty$ iff every λ has absolute value less than 1.

Which of these has $A^k \to 0$?

$$A_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}$$

Ans: A_1 is a markov matrix.

$\lambda_{max} = 1$

$A_1^k \to A_1^\infty$

A_2 has $\lambda = 0.6 \pm 0.3$.

$A_2^k \to 0$.

2a/ $A = X \Lambda X^{-1}$

$$\det(A) = \det(X \Lambda X^{-1}) = \det(X)\det(\Lambda)\det(X^{-1})$$

$$= \det(\Lambda)\det(X)\det(X^{-1}) = \det(\Lambda)\det(X X^{-1})$$

$$= \det(\Lambda)\det(I) = \det(\Lambda) = \lambda_1 \lambda_2 \cdots \lambda_n$$

This proof works only when A is diagonalizable.

(23) If $A = X\Lambda X^{-1}$, diagonalize the block matrix $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$. Find its eigenvalue and eigenvector (block) matrices

Ans: $A = X\Lambda X^{-1}$ & $2A = X(2\Lambda)X^{-1}$

$$B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}\begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix}$$

24. Consider all 4×4 matrices A that are diagonalized by the same fixed eigenvector matrix X. Show that A's form a subspace. What's this subspace when $X = I$? What's its dimension?

Ans: If A_1 & A_2 are in the subspace

$$A_1 = X\Lambda_1 X^{-1} \quad \& \quad A_2 = X\Lambda_2 X^{-1}$$

$$A_1 + A_2 = X\Lambda_1 X^{-1} + X\Lambda_2 X^{-1} = X(\Lambda_1 + \Lambda_2)X^{-1}$$

$A_1 + A_2$ have the same X.

$\Longrightarrow$ $A_1 + A_2$ in the subspace

$$cA_1 = cX\Lambda_1X^{-1} = X(c\Lambda_1)X^{-1}$$

$\Longrightarrow$ cA_1 have the same X & thus in the subspace

$\therefore$ A's that are diagonalized by the same fixed eigenvector matrix X.

When $X = I$, the A's with those eigenvectors give the subspace of diagonal matrices.

The dimension of this matrix space is 4 since the matrices are 4×4.

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25. Suppose, $A^2 = A$. On the left side 'A' multiplies each column of A. Which of our 4 subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So a matrix with $A^2 = A$ can be diagonalized,

Ans

$$A^2 = A\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} Aa_1 & Aa_2 & \cdots & Aa_n \end{bmatrix} = A$$
$$= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$\Rightarrow$ $Aa_i = a_i$ for all $a_i \in C(A)$

$\therefore$ All vectors in $C(A)$ are eigenvectors with $\lambda = 1$.

$N(A)$ has $\lambda = 0$ always.

$\dim[C(A)] + \dim[N(A)] = n$ by the Fundamental theorem.

$\therefore$ n independent eigenvectors

$\therefore$ A is diagonalizable

A

26. Suppose, $Ax = \lambda x$. If $\lambda = 0$ then $x \in N(A)$.
If $\lambda \neq 0$ then $x \in C(A)$. Those spaces have dimensions $(n-r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

Ans: There may not be 'r' independent eigenvectors in the $C(A)$.

$N(A)$ & $C(A)$ can overlap

27. The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$$

Find a matrix square root of A from $R = X\sqrt{\Lambda}X^{-1}$. Why is there no real matrix square root of B?

Ans:

$$\sqrt{A} = R = X\sqrt{\Lambda}\,X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has $R^2 = A$.

$\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1} = i$

$tr(\sqrt{B}) = \sqrt{9} + \sqrt{-1} = \sqrt{9} + i$, is not real

$\therefore \sqrt{B}$ can't be real.

28. If A & B have the same λ's with the same independent eigenvectors, their factorizations into $X \Lambda X^{-1}$ are the same. So $A = B$.

32. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $AB = BA$, show that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also a diagonal matrix. B has the same eigen ______ as A but different eigen ______. These diagonal matrices B form a 2D subspace of matrix space. $AB - BA = 0$ gives 4 equations for the unknowns a, b, c, d - find the rank of the 4×4 matrix

Ans:- $AB = BA \Longrightarrow B$ has the same eigenvectors $(1,0)$ and $(0,1)$ as A

$\therefore X = I \Rightarrow B$ is diagonal.

$b = c = 0$

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix} - \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4 equations: $0 = 0, -b = 0, c = 0, 0 = 0$

These 4 equations have a 4×4 coefficient matrix with rank $= 4 - 2 = 2$

34. The n^{th} power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Prove by diagonalizing $A = X\Lambda X^{-1}$. The eigenvectors (columns of X) are $(1, i)$ and $(i, 1)$.

$\sin\theta + i\cos\theta = i[\cos\theta - i\sin\theta]$

Ans: ~~Eigenvalues~~

$$A\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{-i\theta} \\ i e^{-i\theta} \end{bmatrix} = e^{-i\theta}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$A\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i e^{i\theta} \\ e^{i\theta} \end{bmatrix} = e^{i\theta}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$A^n = X\Lambda^n X = \begin{bmatrix} 1 & i \\ +i & 1 \end{bmatrix}\begin{bmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \times \frac{1}{2} =$$

$$= \begin{bmatrix} e^{-in\theta} & i e^{in\theta} \\ i e^{-in\theta} & e^{in\theta} \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\frac{1}{2} = \begin{bmatrix} \dfrac{e^{-in\theta} + e^{in\theta}}{2} & -i\left(\dfrac{e^{-in\theta} - e^{in\theta}}{2}\right) \\ \boxed{i}\left(\dfrac{e^{-in\theta} - e^{in\theta}}{2}\right) & \dfrac{e^{-in\theta} + e^{in\theta}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

35. The transpose of $A = X \Lambda X^{-1}$ is $A^T = (X^{-1})^T \Lambda X^T$.

- The eigenvectors in $A^T y = \lambda y$ are the columns of that matrix $(X^{-1})^T$. They are often called left eigenvectors of A, because $y^T A = \lambda y^T$.

How do you multiply matrices to find this formula for A?

$$\boxed{\left.\begin{matrix}\text{Sum of rank-1} \\ \text{matrices}\end{matrix}\right\} \; A = X\Lambda X^{-1} = \lambda_1 x_1 y_1^T + \cdots + \lambda_n x_n y_n^T}$$

Ans: $A = X\Lambda X^{-1} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}$

$(X^{-1})^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$

$A = X\Lambda X^{-1} = \lambda_1 x_1 y_1^T + \cdots + \lambda_n x_n y_n^T$

$\text{rank}(A) = n \times 1 = n$

$\boxed{6.3}$

3.

(b) With $-ve$ diagonal & $+ve$ off diagonal adding to zero, $u' = Au$ will be a continuous Markov equation. Find the eigenvalues & eigenvectors, and the steady state as $t \to \infty$

Solve: $\frac{du}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} u$ with $u(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

What is $u(\infty)$?

Ans: $\lambda_1 = 0$: $\left.\begin{array}{l} -2a + 3b = 0 \\ \boxed{2a = 3b} \end{array}\right\} a = \frac{3}{2} b$ $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda_2 = tr(A) - 0 = -5$: $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a = -b.$

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = x_1 + x_2$

$u(t) = e^{\lambda_1 t} x_1 + e^{\lambda_2 t} x_2 = x_1 + e^{-5t} x_2$ has steady state $x_1 = (3, 2)$ since $e^{-5t} \to 0$

4. A door is opened b/w rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement b/w rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \& \quad \frac{dw}{dt} = v - w$$

Show that the total $v + w$ is constant (40 people).

What are v & w at $t = 1$ and $t = \infty$?

Ans: $\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$

$$\frac{d}{dt}(v + w) = \frac{dv}{dt} + \frac{dw}{dt} = (w - v) + (v - w) = 0$$

$$\Longrightarrow v + w \text{ is constant}$$

$|A| = 0. \Rightarrow$

$\lambda_1 = 0: \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2: \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a + b = 0$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix} = 20\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u(1) = 20\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10e^{-2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} v(1) \\ w(1) \end{bmatrix}$$

$u(\infty) = 20, \ w(\infty) = 20.$

5. Reverse the diffusion of people in problem 4 to $\frac{du}{dt} = -Au$:

$\frac{dv}{dt} = v - w$ & $\frac{dw}{dt} = w - v$

Ans. $\frac{du}{dt} = \frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix}$

$(1-\lambda)^2 - 1 = (2-\lambda)(-\lambda) = 0$

$\lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$

$|A| = 0$

$\lambda_1 = 0$: $a = b$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$: $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a = -b$

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\cancel{u(t)} = u(0) = 20x_1 + 10x_2$

$v(t) = 20 + 10e^{2t} \longrightarrow \infty$ as $t \to \infty$

6. A has real eigenvalues but B has complex eigenvalues

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}, \quad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \quad ; \; a, b \in \mathbb{R}.$$

Find the conditions on a and b so that all solutions of $\frac{du}{dt} = Au$ and $\frac{dv}{dt} = Bv$ approach zero as $t \to \infty$; $\mathrm{Re}(\lambda) < 0$ for all eigenvalues.

Ans: $A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$

$(a-\lambda)^2 - 1 = 0 \Rightarrow |a-\lambda| = 1$

$a - \lambda = 1$ (or) $a - \lambda = -1$

$\lambda_s = a-1, \; a+1$

$a-1 < 0$ & $a+1 < 0$

$a < 1$ & $a < -1 \Longrightarrow a < -1$

In this case, the solutions of $u' = Au$ approach zero.

$B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$: $(b-\lambda)^2 + 1 = (b-\lambda)^2 - (-1) = (b-\lambda)^2 - i^2 = 0.$

$|b-\lambda| = i \Rightarrow b - \lambda = i$ (or) $-i$

$\lambda = b+i, \; b-i$

$\mathrm{Re}(\lambda) = b < 0 \Rightarrow$ all solutions of $v' = Bv$ approach zero.

7. P: projection matrix onto the 45° line $y = x$ in $\mathbb{R}^2$. eigenvalues ?

$\frac{du}{dt} = -Pu$ find limit of $u(t)$ at $t = \infty$ starting from $u(0) = (3, 1)$?

Ans: $P = \frac{uu^T}{u^T u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \times \frac{1}{2} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\lambda_1 = 0$: $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ fills the $\perp$ subspace that P projects onto

$\lambda_2 = 1$; $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ fills the subspace that P projects onto.

$u(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u' = -pu$

$u(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ approaches $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as $t \to \infty$.

8. The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$

what are the populations at time t?

After a long time, what's the ratio of rabbits to wolves?

Ans:
$$\frac{d}{dt}\begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r \\ w \end{bmatrix}$$

$$(6-\lambda)(1-\lambda) + 4 = \lambda^2 - 7\lambda + 6 + 4 = \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 5)(\lambda - 2) = 0 \implies$$

$$\lambda_1 = 5: \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies a - 2b = 0 \implies a = 2b$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2: \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies 2a = b \implies$$

$$x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u(0) = \begin{bmatrix} 30 \\ 30 \end{bmatrix} = 10\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$2a + b = 30$$
$$a + 2b = 30$$
$$4a + 2b = 60$$
$$3a = 30$$
$$a = 10$$
$$b = 10$$

$$u(t) = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} = 10e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 10e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$r(t) = 20e^{5t} + 10e^{2t}$$

$$w(t) = 10e^{5t} + 20e^{2t}$$

$$\left.\frac{r(t)}{w(t)}\right|_{t\to\infty} = \frac{20e^{5t}}{10e^{5t}} = \frac{20}{10} = \frac{2}{1}$$ as e^{5t} dominates

10. Find A to change the Scalar equation $y'' = 5y' + 4y$ into a vector equation for $u = (y, y')$

$$\frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

What are the eigenvalues of A?

Find them by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$

Ans: $$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

$$(5-\lambda)(-\lambda) - 4 = \lambda^2 - 5\lambda - 4 = \cancel{0}\,(\lambda - 4)(\lambda - 1) = 0$$

$$\Delta = 25 + 16 = 41 \Longrightarrow \lambda = \frac{5 \pm \sqrt{41}}{2}$$

OR)

Substitute

$$\lambda^2 e^{\lambda t} = 5\lambda e^{\lambda t} + 4e^{\lambda t}$$

$$e^{\lambda t}\left[\lambda^2 - 5\lambda - 4\right] = 0$$

11. Solution to $y''=0$ is a st line $y = C + Dt$.

Convert to a matrix equation:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$ has the solution $$\begin{bmatrix} y \\ y' \end{bmatrix} = e^{At}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

A has eigenvalues $\lambda = 0, 0$ & it can't be diagonalized. Find A^2 and compute e^{At}

$e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots$

multiply e^{At} times $(y(0), y'(0))$ to check the st line $y(t) = y(0) + y'(0)t$.

Ans: $e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$u(t) = e^{At}u(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$$

$y(t) = \underline{y(0)} + y'(0)t$

A had only one eigenvector

Non diagonalizable

12. Substitute $e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble — we need a 2nd solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. Trouble here too. Show that the 2nd solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

Ans: $tr(A) = 6$, $|A| = 9 \Rightarrow \boxed{\lambda = 3, 3}$

$$y_1 = e^{3t}.$$

Let, $y_2 = v(t)\, e^{3t}$

$$y_2' = v'(t)e^{3t} + 3v(t)e^{3t}$$

$$y_2'' = v''(t)e^{3t} + 3v'(t)e^{3t} + 3v'(t)e^{3t} + 9v(t)e^{3t}$$

$$= \left[v''(t) + 6v'(t) + 9v(t)\right]e^{3t}$$

substitute: $\left[v''(t) + \cancel{6v'(t)} + 9v(t)\right]e^{3t} = \left[\cancel{6v'(t)} + 18v(t) - 9v(t)\right]e^{3t}$

$$\left[v''(t)\right]e^{3t} = 0 \implies v''(t) = 0$$

$$v(t) = k_1 t + k_2$$

$y_2(t) = (k_1 t + k_2) e^{3t}$

$y = (k_1 t + k_2) e^{3t} + k_3 e^{3t}$

$= k_1 t e^{3t} + (k_2 + k_3) e^{3t}$

$= C_1 t e^{3t} + C_2 e^{3t}.$

Particular solutions: te^{3t}, e^{3t}

14. $\frac{du}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} u \quad \left\{ \begin{array}{l} u_1' = cu_2 - bu_3 \\ u_2' = au_3 - cu_1 \\ u_3' = bu_1 - au_2 \end{array} \right.$

$A^T = -A$

(a) The derivative of $|u(t)|^2 = u_1^2 + u_2^2 + u_3^2$ is $2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. Substitute u_1', u_2', u_3' to get zero. Then $|u(t)|^2$ stays equal to $|u(0)|^2$.

(b) When A is skew-symmetric, $Q = e^{At}$ is orthogonal. Prove $Q^T = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^TQ = I$

Ans:

Ans:

ⓐ $\frac{d}{dt}|u(t)|^2 = 0 = Au$ if $A^T = -A$

$u(t) = e^{At}u(0)$

$A^T = -A \Longrightarrow e^{At}$ is orthogonal
which preserves the norm squared

ⓑ $Q^T Q = e$

$Q = e^{At}$

$Q^T = \left[I + At + \frac{(At)^2}{2} + \cdots\right]^T = I + A^T t + \frac{(A^T t)^2}{2} + \cdots$

$= I - At + \left(\frac{-At}{2}\right)^2 + \cdots$

$= e^{-At}$

(15) A particular solution to $\frac{du}{dt} = Au - b$ is $u_p = A^{-1}b$, if A is invertible. The usual solutions to $\frac{du}{dt} = Au$ give u_n. Find the complete solution $u = u_p + u_n$:

(a) ~~$\frac{du}{dt} = u - 4$~~

Ans: $\frac{du}{dt} = Au - b \implies \frac{du}{dt} - Au = -b$

$\frac{du}{dt} = 0 \Rightarrow Au = b$

$u = A^{-1}b$

Particular solution: $u_p = A^{-1}b$

Nullspace solution: is given by $\frac{du}{dt} - Au = 0$

$\implies \frac{du}{dt} = Au$

(a) $\frac{du}{dt} = u - 4$

Ans: $\frac{du}{dt} - u = -4$

$u_p = 4$

u_n: $\frac{du}{dt} = u \implies ce^t$

$u(t) = ce^t + 4$

(b)

ILA (7)

Ans:

u_n

Assume,

(b) $\dfrac{du}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

ILA(7)

Ans: $\dfrac{du}{dt} = 0 \Rightarrow \left\{ u_p = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right.$

$$u_p = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{T} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

u_n: $\dfrac{du}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u$

$\lambda = 1, 1$

$$\left.\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a + 0 = 0 \right\} x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_1(t) = e^{t} x_1 = e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assume, $u_2(t) = t e^{t} x_1 + e^{t} x_2$

$$u_2'(t) = e^{t} x_1 + \underline{t e^{t} x_1} + e^{t} x_2 = A\left(\underline{t e^{t} x_1} + e^{t} x_2\right)$$

$A x_1 = x_1 \longrightarrow x_1$ is an eigenvector of A

$(A - I) x_1 = 0$

$x_1 + x_2 = A x_2 \longrightarrow (A - I) x_2 = x_1$

$(A - I)^2 x_2 = 0$

$$(A-I)^2 x_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_2 =$$

$$(A-I)\, x_2 = x_1$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \underline{a = 1}$$

$$x_2 = \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_h(t) = k_1 e^t x_1 + k_2 \left(t e^t x_1 + e^t x_2 \right)$$

$$= k_1 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + k_2 \left(t e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= k_1 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + k_2 \left(e^t \begin{bmatrix} 0 \\ t \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= k_1 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + k_2 e^t \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$$u(t) = u_h(t) + u_p(t)$$

$$= k_1 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + k_2 e^t \begin{bmatrix} 1 \\ t \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

16. If c is not an eigenvalue of A, substitute $u = e^{ct} v$ and find a particular solution to $\frac{du}{dt} = Au - e^{ct} b$. How does it break down when c is an eigenvalue of A? The "nullspace" of $\frac{du}{dt} = Au$ contains the usual solutions $e^{\lambda_i t} x_i$.

Ans: $$c e^{ct} v = A e^{ct} v - e^{ct} b$$

$$(A - cI) v = b \quad \text{(or)} \quad v = (A - cI)^{-1} b$$

since c is not an eigenvalue.

$$\cancel{\frac{du}{dt} - Au = - e^{ct} b}$$

$$\cancel{\frac{du}{dt} = 0 \Rightarrow Au = e^{ct} b.}$$

If c is an eigenvalue, $(A - cI)$ is not invertible.

17. Find a matrix A to illustrate each of the unstable regions. Fig 6.5.

(a) $\lambda_1 < 0, \lambda_2 > 0$.

(b) $\lambda_1 > 0, \lambda_2 > 0$

(c) $\lambda = a \pm ib$ with $a > 0$.

the

22. If $A^2 = A$, show that the infinite series produces

$$e^{At} = I + (e^t - 1)A \,, \quad A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

Ans. $$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots$$

$$= I + At + \frac{At^2}{2} + \frac{At^3}{6} + \cdots$$

$$= I + A\left[t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right]$$

$$= I + A\left[e^t - 1\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}(e^t - 1)$$

$$= \begin{bmatrix} 1 + e^t - 1 & 4(e^t - 1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 4e^t - 4 \\ 0 & 1 \end{bmatrix}$$

26 Give 2 reasons why the matrix exponential e^{At} is never singular:

(a) $(e^{At})^{-1} = e^{-At}$

(b) ~~The~~ The eigenvalues of e^{At} are non-zero.

$$e^{At}x = \left[I + At + \frac{A^2t^2}{2} + \cdots\right]x = \left(1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \cdots\right)x = e^{\lambda t}x$$

$$e^{\lambda t} \neq 0.$$

27. Find a solution $x(t), y(t)$ that gets large as $t \longrightarrow \infty$. To avoid this instability a scientist exchanged the 2 equations:

$$\frac{dx}{dt} = 0x - 4y \qquad\qquad \frac{dy}{dt} = -2x + 2y$$

$$\Longrightarrow$$

$$\frac{dy}{dt} = -2x + 2y \qquad\qquad \frac{dx}{dt} = 0x - 4y$$

Now the matrix $\begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$ is stable. It has <u>-ve eigenvalues</u>. How can this be?

Ans: $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$(2-\lambda)(-\lambda) - 8 = \lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda-4)(\lambda+2) = 0 \Longrightarrow \lambda = -2 \text{ (or) } 4$$

$$\lambda_1 = -2: \begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 : \left.\begin{array}{l} 2a - 4b = 0 \\ a = 2b \end{array}\right\} x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 4: \begin{bmatrix} -4 & -4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 : \left. a + b = 0 \right\} x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

unstable.

Exchange

$$\frac{d}{dt}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix}$$

$$(2-\lambda)(-\lambda) - 8 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0 \Rightarrow (\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = -2, 4$$

It does have the same eigenvalues as the original matrix.

31. The cosine of a matrix is defined like e^A;

$$\cos t = \frac{e^t + e^{-t}}{2} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$$

$$\boxed{\cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots\cdots}$$

(a) If $Ax = \lambda x$,

eigenvalues of $\cos A$.

Ans: $$(\cos A)\,x = \left(I - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots\right) x$$
$$= \left(1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} - \cdots\right) x = (\cos \lambda)\, x$$

(b) Eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors $(1,1)$ and $(1,-1)$.

Find the eigenvalues & eigenvectors of $\cos A$, find that matrix $C = \cos A$.

Ans: $|A| = 0 \Longrightarrow \lambda_1 = 0,\ \lambda_2 = 2\pi$ for A

For $\cos A$: $\underbrace{\lambda_1 = \cos \lambda_1 = 1,\ \lambda_2 = \cos \lambda_2 = 1}$

$$\cos A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) 2nd derivative of $\cos(At)$ is $-A^2\cos(At)$

$u(t) = \cos(At)u(0)$ solves $\frac{d^2u}{dt^2} = -A^2u$ starting from $u'(0) = 0$

Ans:

$$\frac{d^2u}{dt^2} = -A^2u$$

(1) Expand $u(0) = c_1x_1 + c_2x_2$ in the eigenvectors.

(2) Multiply each eigenvector x_i by $\cos(\lambda_i t)$

(3) Add up the solution

$$\frac{d^2u}{dt^2} = -A^2u : u(t) = c_1\cos(\lambda_1 t)x_1 + c_2\cos(\lambda_2 t)x_2$$

$\boxed{6.4}$

8. Find all orthogonal matrices that diagonalize

$$S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

Ans: $|S| = 144 - 144 = 0$

$\lambda_1 = 0, \ \lambda_2 = 25$

$$\left\{ \begin{array}{l} 9a + 12b = 0 \Rightarrow 3a + 4b = 0 \\ 12a + 16b = 0 \quad 3a + 8b = 0 \end{array} \right\} \quad b = -\frac{3}{4}a \qquad \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \Big/ 5$$

$\cancel{6a+8b}$

$$\lambda_1 = 0, \quad x_1 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} \quad \text{———} \quad (1)$$

$$\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Longrightarrow \begin{array}{l} 16a + 12b = 0 \\ 12a - 9b = 0 \\ 4a = 3b \end{array} \quad \begin{array}{l} 4a + 3b = 0 \\ 4a + 3b = 0 \\ \hline a = 0, b = 0. \end{array}$$

$$\begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Big/ 5$$

$$\lambda_2 = 25, \quad x_2 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \quad (\text{or}) \quad \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

(or)

$$Q = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad (\text{or}) \quad \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

10. If $A^3 = 0$, then the eigenvalues of A must be ____. Ex:- that has $A \neq 0$. But, if A is symmetric, diagonalize it to prove that A must be a zero matrix.

Ans: $A^3 x = \lambda^3 x = 0 \implies \lambda^3 = 0$ since $x \neq 0$

$\therefore$ all $\lambda = 0$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

If A is symmetric, $A^3 = Q \Lambda^3 Q^T = 0$ requires $\Lambda^3 = 0 \implies \Lambda = 0$.

$\therefore$ The only symmetric A is $Q 0 Q^T = 0$.

11. If $\lambda = a+ib$ is an eigenvalue of a real matrix A then its conjugate $\bar{\lambda} = a-ib$ is also an eigenvalue. Explain why every real 3x3 matrix has at least one real eigenvalue.

Ans: $A x = \lambda x \implies A\bar{x} = \bar{\lambda}\bar{x}$

For a 3x3 matrix,

$\lambda + \bar{\lambda} = $ real #.

$tr(A)$ is real, so λ_3 must be real.

12. False ~~Proof~~ - eigenvalues of every real matrix A are real

$\cancel{A}x = \lambda x \implies x^T A x = \lambda x^T x \implies \lambda = \frac{x^T A x}{x^T x} = \frac{\text{real}}{\text{real}}$

test for 90° rotation matrix $[0, -1; 1, 0]$

$[0 \ -1 ; 1 \ 0]$ with $\lambda = i$ and $x = (i, 1)$.

Ans: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $(-\lambda)^2 + 1 = 0 \implies \lambda^2 = -1 \implies |\lambda| = i$

$\lambda = \pm i$

$\lambda_1 = i : \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies -ia - b = 0 \implies x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$a - ib = 0$

$\lambda_2 = -i : \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies ia - b = 0 \implies x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$x^T A x = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = 0$$

$$x^T A x = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i - i = 0$$

$$\cancel{\begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} -1 \\ i \end{bmatrix} = -i + i = 0.}$$

$$\lambda = \frac{x^T A x}{x^T x} = \frac{0}{0} \quad \text{Not defined}$$

13. Write S & B in the form $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$ of the spectral theorem $Q\Lambda Q^T$:

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad \left(\text{keep } |x_1| = |x_2| = 1 \right)$$

Ans: (a) $(3-\lambda)^2 - 1 = 0$

$(2-\lambda)(4-\lambda) = 0$

$\lambda_1 = 2 : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a+b=0 : \quad x_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 4 : \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a-b=0 : \quad x_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} \times \frac{1}{2} + 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} \times \frac{1}{2}$$

$$= 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

14. Every 2×2 symmetric matrix is $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = x_1 x_1^T + x_2 x_2^T = I$ from columns times rows of Q. Why is $P_1 P_2 = 0$?

Ans: $Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$$P_1 + P_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} = x_1 x_1^T + x_2 x_2^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$P_1 P_2 = x_1 x_1^T x_2 x_2^T = x_1 (x_1^T x_2) x_2^T = 0.$$

(or)

$$P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0.$$

$$P_1^2 = P_1$$

15. What are the eigenvalues of $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$?

Create a 4×4 antisymmetric matrix $(A^T = A)$ & verify that all its eigenvalues are imaginary.

Ans: $(-\lambda)^2 + b^2 = \lambda^2 + b^2 = 0$

$|\lambda| = i|b| \Longrightarrow \underline{\lambda = \pm ib}$

$$\cancel{B_{4\times4}} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$$

$$B^T_{4\times4} = -B \Longrightarrow B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right] \text{ (or) } \left[\begin{array}{c|c} 0 & A \\ \hline A & 0 \end{array}\right]$$

$$B^T = \left[\begin{array}{c|c} -A & 0 \\ \hline 0 & -A \end{array}\right] \quad \text{(or)} \quad \left[\begin{array}{c|c} 0 & -A \\ \hline -A & 0 \end{array}\right]$$

16.

Ans:

16. $M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$ can only have

Eigenvalues from $tr(M)$?

Ans: M is antisymmetric & also orthogonal.

$M^T = -M$ $\qquad$ $M^T = M^{-1}$

$(M^T M = I)$

λ is purely imaginary $\qquad$ $|\lambda| = 1$

$\bar{\lambda} + \lambda = 0$

$\lambda = \pm i$

$tr(M) = 0$

$\lambda : i, i, -i, -i$

17. Show that this A (symmetric but complex) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$$ is not even diagonalizable:

$$\lambda = 0, 0$$

* $A^T = A$ is not such a special property for complex matrices.

Ans. The good property is $\bar{A}^T = A$, then all λ's are real & the eigenvectors are orthogonal.

Ans. $A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$, $A^T = -A$

$$(i-\lambda)(-i-\lambda) - 1 = 0.$$

$$(\lambda - i)(\lambda + i) = \lambda^2 + 1 = 1 \Rightarrow \lambda = 0, 0.$$

$$\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow \begin{matrix} ia + b = 0 \\ a - ib = 0 \\ a = ib \end{matrix} \quad ; \quad x = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

only one independent eigenvector.

18. Even if 'A' is rectangular, the block matrix

$$S = \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right] \text{ is symmetric:}$$

$Sx = \lambda x$ is $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}$ which is

$Az = \lambda y$

$A^T y = \lambda z$.

$$S^T = \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right]^T = \left[\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right] = S$$

(a) Show that $-\lambda$ is also an eigenvalue, with the eigenvector $(y, -z)$

Ans: $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ -z \end{bmatrix} = \begin{bmatrix} -Az \\ A^T y \end{bmatrix} = \begin{bmatrix} -\lambda y \\ \lambda z \end{bmatrix} = -\lambda \begin{bmatrix} y \\ -z \end{bmatrix}$

(b) Show that $A^T A z = \lambda^2 z$, so that λ^2 is an eigenvalue of $A^T A$

Ans: $A^T A z = A^T(\lambda y) = \lambda(A^T y) = \lambda \cdot \lambda z = \lambda^2 z$.

(c) If $A = I_{2\times 2}$, find all 4 eigenvalues & eigenvectors of S

Ans: $S = \begin{bmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{bmatrix}$

Symmetric & orthogonal

$\lambda \in \mathbb{R}$ $\quad |\lambda| = 1$

$tr(S) = 0$

$\Big\}$ $1, 1, -1, -1$

$\lambda = 1$: $\begin{bmatrix} -I & I \\ I & -I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0$

$X = Y$ & $\Rightarrow x_1 = y_1$ & $x_2 = y_2$

$-x_1 + y_1 = 0 \Rightarrow y_1 = x_1$

$-x_2 + y_2 = 0 \Rightarrow y_2 = x_2$

$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} I & I \\ I & I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0$

$X + Y = 0 \Longrightarrow Y = -X$

$x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

20. Proof — Eigenvectors are $\perp$ when $S = S^T$

Ans: Suppose,

$$Sx = \lambda x \quad \& \quad Sy = 0y \quad \& \quad \lambda \neq 0.$$

$x \perp y$ because $x \in C(A) = C(A^T)$ & $y \in N(A)$

since $S^T = S$ $\quad C(A^T) \perp N(A)$

If $Sx = \lambda x$ and $Sy = \beta y$, then shift S by βI

$$(S - \beta I)x = (\lambda - \beta)x \quad \text{and} \quad (S - \beta I)y = 0$$

$$\therefore \underline{\underline{x \perp y}}$$

25. Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Which of these factorizations are possible for A & B: LU, QR, $X\Lambda X^{-1}$, $Q\Lambda Q^T$?

Ans: A: invertible, orthogonal, permutation, diagonalizable, Markov

QR, $X\Lambda X^{-1}$, $Q\Lambda Q^T$

B: projection, diagonalizable, Markov

$X\Lambda X^{-1}$, $Q\Lambda Q^T$

Q1 Find all 2×2 matrices that are orthogonal & also symmetric. Which 2 numbers can be eigenvalues of these 2 matrices?

Ans: orthogonal & Symmetric

$|\lambda| = 1$ $\qquad \lambda \in \mathbb{R}$

x_i is real for real sym matrix

$\Longrightarrow \lambda = \pm 1$ & $x_i \cdot x_j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

$$S = Q\Lambda Q^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$= Ref(\theta)$$

$$S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$S = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = -I$$

31. Normal matrices : $\bar{N}^T N = N \bar{N}^T$

For real matrices,

$$N^T N = N N^T$$

Normal matrices : symmetric, skew-symmetric, orthogonal

(with real λ, imaginary λ, $|\lambda| = 1$)

* Normal matrices have n orthonormal eigenvectors.

Those eigenvectors x_i probably will have complex components. In the complex case, orthogonality means $\bar{x}_i^T x_j = 0$. Inner products $x^T y$ becomes $\bar{x}^T y$.

$$Q^T Q = I \longrightarrow \bar{Q}^T Q = I$$

* N has n orthonormal eigenvectors ($N = Q \Lambda \bar{Q}^T$) iff N is normal.

(a) Start from $N = Q\Lambda\bar{Q}^T$ with $\bar{Q}^T Q = I$.
Show that $\bar{N}^T N = N\bar{N}^T$: N is normal.

Ans: $N = Q\Lambda\bar{Q}^T$

$\bar{N}^T = Q\bar{\Lambda}^T\bar{Q}^T$

$N\bar{N}^T = Q\Lambda\bar{Q}^T Q\bar{\Lambda}^T\bar{Q}^T = Q\Lambda I\bar{\Lambda}^T\bar{Q}^T$

$= \cancel{Q\bar{Q}^T}\, Q\Lambda\bar{\Lambda}^T\bar{Q}^T = Q\bar{\Lambda}^T\Lambda\bar{Q}^T$

$= Q\bar{\Lambda}^T\bar{Q}^T Q\Lambda\bar{Q}^T = \bar{N}^T N$

(b) ~~Schur~~

ILA(8) Schur's theorem

$A = QT\bar{Q}^T$ for every matrix A, with a triangular T.

For normal matrices $A = N$,
this triangular matrix T will be diagonal.

Then $T = \Lambda$

Proof $N = QT\bar{Q}^T$

$\bar{N}^T N = N\bar{N}^T \Rightarrow Q\bar{T}^T\bar{Q}^T Q T\bar{Q}^T = QT\bar{Q}^T Q\bar{T}^T\bar{Q}^T$

$\bar{T}^T T = T\bar{T}^T$

Let,

$$T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$$\begin{bmatrix} \bar{a} & 0 \\ \bar{b} & \bar{d} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} \bar{a} & 0 \\ \bar{b} & \bar{d} \end{bmatrix}$$

$$\begin{bmatrix} |a|^2 & \bar{a}b \\ a\bar{b} & |b|^2+|d|^2 \end{bmatrix} = \begin{bmatrix} |a|^2+|b|^2 & b\bar{d} \\ \bar{b}d & |d|^2 \end{bmatrix}$$

$$|a|^2 = |a|^2 + |b|^2 \Longrightarrow |b|^2 = 0$$

$$\therefore\ b = 0$$

$$T = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$\therefore\ \underline{T = \Lambda}$$

32. If λ_{max} is the largest eigenvalue of a symmetric matrix S, no diagonal entry can be larger than λ_{max}.
What's the 1st entry a_{11} of $S = Q\Lambda Q^T$?
Show ~~why~~ why $a_{11} \le \lambda_{max}$.

Ans: $S = Q\Lambda Q^T = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{q}_1^T \\ \bar{q}_2^T \\ \vdots \\ \bar{q}_n^T \end{bmatrix}$

$$= \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 \bar{q}_1^T \\ \lambda_2 \bar{q}_2^T \\ \vdots \\ \lambda_n \bar{q}_n^T \end{bmatrix} = \lambda_1 q_1 \bar{q}_1^T + \cdots + \lambda_n q_n \bar{q}_n^T$$

$\cancel{a_{11} = \lambda + \lambda_1 |q_{n}|^2}$

$\cancel{a_{11} = \lambda_1 (|q_{11}|^2 + \cdots + |q_{1n}|^2)}$

$$a_{11} = \lambda_1 |q_{11}|^2 + \lambda_2 |q_{12}|^2 + \cdots + \lambda_n |q_{1n}|^2 \le \lambda_{max} (|q_{11}|^2 + \cdots + |q_{1n}|^2)$$
$$\le \lambda$$

?

33. If $A^T = -A$. Explain

(a) $x^T A x = 0$ for every real vector x

Ans: $x^T A x$ is a scalar.

$$x^T(Ax) = (x^T(Ax))^T = (Ax)^T x = x^T A^T x = -x^T(Ax)$$

$$2(x^T A x) = 0 \implies \underline{x^T A x = 0}$$

(b) Eigenvalues of A are purely imaginary

Ans: $A x = \lambda x \implies x^T A^T = \bar{\lambda} x^T$

$x^T A = -\bar{\lambda} x^T$

$x^T A x = \lambda |x|^2$

$x^T A x = -\bar{\lambda} |x|^2$

$(\lambda + \bar{\lambda}) |x|^2 = 0 \implies \bar{\lambda} = -\lambda$

λ is purely imaginary

(c) $\det(A)$ is +ve (or) 0, not −ve

Ans: $\det(A) = \lambda_1 \cdots \lambda_n = ia \times ib \times \cdots \geq 0$

34. If S is symmetric & all its eigenvalues are $\lambda = 2$, how do you know that $S = 2I$?

Ans: $S^T = S$ (S is symmetric)

$$S = Q \Lambda Q^T = 2 Q I Q^T = 2 Q Q^T = 2I.$$

$\cancel{S = 0}$

36. If S is symmetric, show that $A^T S A$ is also symmetric. Here, A is $m \times n$ and S is $m \times m$.

Are ~~eigenvalues~~ of

$$(A^T S A)^T = A^T S^T A = A^T S A$$

* If A is square & invertible, $A^T S A$ is called congruent to S. They have the same # of +ve, -ve and zero eigenvalues : Law of Inertia.

$\boxed{\text{Sylvester's law of Inertia.}}$

37. Way to show that 'a' is in b/w the eigenvalues λ_1 and λ_2 of S:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \; : \; \cancel{\det(S-\lambda)}$$

Ans: $\det(S - \lambda I) = (a - \lambda)(c - \lambda) - b^2 = \lambda^2 - \lambda(a+c) - b^2 + ac = 0$

$$\det(S - \lambda I)\Big|_{\lambda = a} = \cancel{a^2} - \cancel{a^2} - \cancel{ac} - b^2 + \cancel{ac} < 0.$$

$y = x^2$.

$\therefore$

$\boxed{6.5}$

4. What is the function $f = ax^2 + 2bxy + cy^2$ for each of these ~~fu~~ matrices?
Complete the square to write each f as a sum of one or ~~two~~ squares $f = d_1(\ )^2 + d_2(\ )^2$.

$$S_1 = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}, \quad f = [x\ y]\,[S]\begin{bmatrix} x \\ y \end{bmatrix}.$$

Ans:- $x^T S_1 x = [x\ y]\begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x\ y]\begin{bmatrix} x+2y \\ 2x+9y \end{bmatrix}$

$= x + 2xy + 2xy + 9y^2 = x + 4xy + 9y^2$

6. The function $f(x,y) = 2xy$ certainly has a saddle point and not a min. at $(0,0)$. What symmetric matrix S produces this f? What are its eigenvalues?

Ans: $x^T S x = x^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} x = ax^2 + 2bxy + by^2 = 2xy$

$a = b = 0$ & $b = 1$

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ has } \lambda = 1, -1$$

S is indefinite

8. The function $f(x,y) = 3(x+2y)^2 + 4y^2$ is +ve except at $(0,0)$. What is the matrix in $f = [x, y]\, S \begin{bmatrix} x \\ y \end{bmatrix}$? Check that the pivots of A are 3 & 4.

Ans: $f(x,y) = 3x^2 + 12xy + 16y^2 = [x\,y] \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$S = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Pivots are: 3, 4

$x^T S x = 3(x+2y)^2 + 4y^2$

9. Find the 3x3 matrix S and its pivots, rank, eigenvalues & determinant

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

Ans: $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 + cx_3 \\ bx_1 + e x_2 + f x_3 \\ cx_1 + f x_2 + i x_3 \end{bmatrix}$

$$= ax_1^2 + ex_2^2 + ix_3^2 + x_1x_2(b+b) + x_2x_3(f+f) + x_3x_1(c+c)$$

$$= 4x_1^2 + 4x_2^2 + 16x_3^2 - 8x_1x_2 + 16x_1x_3 - 16x_2x_3$$

$\Rightarrow a = 4, e = 4, i = 16.$

$b = -4, \; c = 8, \; f = -8$

$$S = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16. \end{bmatrix}$$

only 1 pivot

rank(S) = 1

$\lambda = 24, 0, 0$

det(S) = 0.

6. The function $f(x,y) = 2xy$ certainly has a saddle point and not a min. at $(0,0)$. What symmetric matrix S produces this f? What are its eigenvalues?

Ans: $x^T S x = x^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} x = ax^2 + 2bxy + by^2 = 2xy$

$a = b = 0$ & $b = 1$

$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\lambda = 1, -1$

S is indefinite

8. The function $f(x,y) = 3(x+2y)^2 + 4y^2$ is +ve except at $(0,0)$. What is the matrix in $f = [x\ y]\, S \begin{bmatrix} x \\ y \end{bmatrix}$? Check that the pivots of A are 3 & 4.

Ans: $f(x,y) = 3x^2 + 12xy + 16y^2 = [x\ y] \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$S = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Pivots are: 3, 4

$x^T S x = 3(x+2y)^2 + 4y^2$

9. Find the 3x3 matrix S and its pivots, rank, eigenvalues & determinant

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2$$

Ans:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 + cx_3 \\ bx_1 + ex_2 + fx_3 \\ cx_1 + fx_2 + ix_3 \end{bmatrix}$$

$$= ax_1^2 + ex_2^2 + ix_3^2 + x_1x_2(b+b) + x_2x_3(f+f)$$

$$+ x_3x_1(c+c)$$

$$= 4\{x_1^2 + 4x_2^2 + 16x_3^2 - 8x_1x_2 + 16x_1x_3 - 16x_2x_3$$

$\Rightarrow a = 4, e = 4, i = 16.$

$b = -4, c = 8, f = -8$

$$S = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$$

only 1 pivot

rank(S) = 1

$\lambda = 24, 0, 0$

det(S) = 0.

11. Compute the 3 upper left determinants of S to establish +ve definiteness. Verify that their ratios give the 2^{nd} & 3^{rd} pivots.

$$\boxed{\text{Pivots} = \text{ratio of determinants}}$$

$$S = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$

Ans: $|S_1| = 2$, $|S_2| = 6$, $|S_3| = 30$

$$\rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow$$

Pivots: $2,\ \frac{6}{2} = 3,\ \frac{30}{6} = 5$

13. Find a matrix with ~~coit~~ $a>0$ and $c>0$ and $a+c \geqslant 2b$ that has a -ve eigenvalue.

Ans: $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}$

$$ac-b^2<0 \Longrightarrow ac<b^2 \ \&\ a>0, c>0$$

$$a+c>2b$$

$$S = \begin{bmatrix} 10 & 4 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

14. If S is +ve definite then S^{-1} is positive definite

Ans: The eigenvalues of S^{-1} are +ve because, because they are $\frac{1}{\lambda(S)}$.

(OR) $\qquad\qquad\qquad$ $S^T = S$

$$x^T S^{-1} x = \cancel{(x^T (S^{-1})^T S)^{-1}}$$
$$= x^T (S^T)^{-1} S^T S^{-1} x$$
$$= x^T (S^{-1})^T S S^{-1} x$$
$$= (S^{-1}x)^T S (S^{-1}x) > 0 \quad \text{for all } x \neq 0$$

15. If S and T are +ve definite, their sum $S+T$ is +ve definite.

Ans: $x^T(S+T)x = x^TSx + x^TTx > 0$ for all $x \neq 0$

(OR)

$S = A^TA$ and $T = B^TB$

Independent columns in A & B.

$$S+T = A^TA + B^TB = \begin{bmatrix} A^T & B^T \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$= \begin{bmatrix} A \\ B \end{bmatrix}^T \begin{bmatrix} A \\ B \end{bmatrix}$$

17. A diagonal entry s_{jj} of a symmetric matrix can't be smaller than all the λ's.

Ans: If a_{jj} were smaller than all λ's,

$S - a_{jj}I$ would have all eigenvalues > 0

$\longrightarrow$ +ve definite.

But, $S - a_{jj}I$ has a 0 is in the (j,j) position.

19. If all $\lambda > 0$ then $x^T S x > 0$

must be true for all $x \neq 0$.

Ans: $S = Q \Lambda Q^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$

$$= \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T$$

$$x^T S x = (c_1 x_1 + \cdots + c_n x_n)^T (\lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T) \times (c_1 x_1 + \cdots + c_n x_n)$$

$$= (\lambda_1 c_1 x_1^T x_1 x_1^T + \cdots + \lambda_n c_n x_n^T x_n x_n^T)(c_1 x_1 + \cdots + c_n x_n)$$

$$\left[x_i x_j = 0 \right]$$

$$= c_1^2 \lambda_1 x_1^T x_1 + \cdots + c_n^2 \lambda_n x_n^T x_n > 0$$

+ve eigenvalues	$\Longrightarrow$	+ve energy
$\lambda > 0$		$x^T S x > 0$

21. For which s & t do S and T have all $\lambda > 0$ ($\therefore$ +ve definite) ?

$$S = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \& \quad T = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}$$

Ans: $S = \begin{bmatrix} s+4 & 0 & 0 \\ 0 & s+4 & 0 \\ 0 & 0 & s+4 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= (s+4) I -$

$S = -4 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + (s+4) I$

$(s+4)$

$(1-\lambda)^2 = 0$

$\boxed{\lambda = 1}$

λ : s, s, s

$\lambda > 0$

22. From $S = Q\Lambda Q^T$, compute +ve definite symmetric square root $Q\sqrt{\Lambda}Q^T$ of each matrix. Check that this square root gives $A^TA = S$:

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \& \quad S = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Ans.
(a) $(5-\lambda)^2 - 16 = 0$ $\quad (5-\lambda)^2 - 4^2 = 0$.

$\lambda = -11$

$\longrightarrow (1-\lambda)(9-\lambda) = 0$

$\therefore \lambda = 1$ (or) 9

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \qquad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

$a + b = 0 \qquad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A = Q\sqrt{\Lambda}Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ +1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

23. You may have seen the equation for an ellipse as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. What are a & b when the equation is written $\lambda_1 x^2+\lambda_2 y^2=1$? The ellipse $9x^2+4y^2=1$ has axes with half lengths a= ______ and b= ______

Ans: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=x^T S x$ when $S=\text{diag}\left(\frac{1}{a^2}, \frac{1}{b^2}\right)$

$\lambda_1=\frac{1}{a^2}, \lambda_2=\frac{1}{b^2} \Longrightarrow a=\frac{1}{\sqrt{\lambda_1}}; b=\frac{1}{\sqrt{\lambda_2}}$

24. Draw the tilted ellipse $x^2+xy+y^2=1$ & find the half lengths of its axes from the eigenvalues of the corresp. matrix S.

Ans: $4AC=4 \times 1 = B^2$: ellipse

$ax^2+2bxy+by^2=1$

$S=\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \rightarrow (1-\lambda)^2-\left(\frac{1}{2}\right)^2=0$

$\left(\frac{1}{2}-\lambda\right)\left(\frac{3}{2}-\lambda\right)=0$

$\lambda=\frac{1}{2}, \frac{3}{2}$

$\lambda_1 = \frac{1}{2}$: $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a + b = 0$

$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = \frac{3}{2}$: $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Rightarrow a = b$

$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x^T S x = [x \;\; y] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x+y}{\sqrt{2}} & \frac{x-y}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{x-y}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{3}{2} \left(\frac{x+y}{\sqrt{2}} \right)^2 + \frac{1}{2} \left(\frac{x-y}{\sqrt{2}} \right)^2 = 1$$

Axes with half lengths $\sqrt{2}$ & $\sqrt{2/3}$

25. With +ve pivots in D, the factorization $S = LDL^T$ becomes $S = L\sqrt{D}\sqrt{D}L^T$.

Thus $C = \sqrt{D}L^T$ yields the Cholesky factorization $A = CC^T$ which is "symmetrized LU".

(a) From $C = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, find S

Ans: $S = C^T C = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$

(b) From $S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix}$ find C

Ans: $S = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & \\ & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = L\sqrt{D}\sqrt{D}L^T$

$C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

26 In the Cholesky factorization $S = C^T C$, with $C = \sqrt{D} L^T$, the square roots of the pivots are on the diagonal of C. Find C for:

(a) $S = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$

Ans: $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = LDL^T$
$= (L\sqrt{D})(L\sqrt{D})^T$

$$C = (L\sqrt{D})^T = \sqrt{D}L^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

---

27. The symmetric factorization $S = LDL^T$ means that $x^T S x = x^T LDL^T x$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Ans:

$$= \begin{bmatrix} x + \frac{b}{a}y & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} x + \frac{b}{a}y \\ y \end{bmatrix}$$

$$= \begin{bmatrix} a\left(x + \frac{b}{a}y\right) & y\left(\frac{ac-b^2}{a}\right) \end{bmatrix} \begin{bmatrix} x + \frac{b}{a}y \\ y \end{bmatrix}$$

$$ax^2 + 2bxy + by^2 = a\left(x + \frac{b}{a}y\right)^2 + \left(ac - \frac{b^2}{a}\right)y^2$$

29. For $F_1(x,y) = \frac{1}{4}x^4 + x^2y + y^2$ and $F_2(x,y) = x^3 + xy - x$ find the 2^{nd} derivative matrices S_1 & S_2:

Test for min: $S = \begin{bmatrix} \partial^2F/\partial x^2 & \partial^2F/\partial x\,\partial y \\ \partial^2F/\partial y\,\partial x & \partial^2F/\partial y^2 \end{bmatrix}$ is +ve definite.

Find a min. point of F_1.

Find the saddle point of F_2.

Ans:

(a) $S_1 = \Big[$

(b) $S_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$

$\frac{\partial F_2}{\partial x} = 3x^2 + y - 1 = 0 \Rightarrow y = 1 - 3x^2$

$\frac{\partial F_2}{\partial y} = x = 0$

$(0,1)$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite at $(0,1)$

$(0,1)$ is a saddle point of the function $F_2(x,y)$.

(30) The graph of $z = x^2 + y^2$ is a bowl opening upward. The graph of $z = x^2 - y^2$ is a saddle. The graph of $z = -x^2 - y^2$ is a bowl opening downward. What is a test on a, b, c for $z = ax^2 + 2bxy + cy^2$ to have a saddle point at $(x, y) = (0, 0)$?

Ans: $f(x, y) = ax^2 + 2bxy + cy^2$

$$S = \begin{bmatrix} 2a & 2b \\ 2b & 2c \end{bmatrix} =$$

$f(x, y) = ax^2 + 2bxy + cy^2$ has a saddle point if

$ac - b^2 < 0 \Rightarrow ac < 0 \; (|S| < 0)$.

(31.) Which value of 'c' give a bowl & which c give a saddle point for the graph of $z = 4x^2 + 12xy + cy^2$? Describe this graph at the borderline value of c.

The min. of a function $F(x, y, z)$

What tests you expect for a min. point ?

1st derivatives are zero : $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ at the min. point

2nd derivative matrix S : ~~[illegible]~~ is +ve definite : $S = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}$

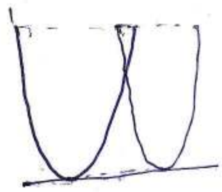
$$S = \begin{bmatrix} 8 & 12 \\ 12 & 2c \end{bmatrix}$$

$|S| = 16c - 144 = 8(2c-18) = 16(c-9)$

$c-9>0 \Rightarrow c>9$: graph of z is a bowl

$c-9<0 \Longrightarrow c<9$: " is a saddle point

$c-9=0 \Rightarrow c=9$: graph of $z=(2x+3y)^2$ is a trough staying at 0 along the line $2x+3y=0$.



32. A group of non-singular matrices include AB & A^{-1} if it includes A and B. "Products and inverses stay in the group". Which of these are groups?
Invent a subgroup of two of these groups

(a) +ve definite symmetric matrices S
(b) Orthogonal matrices Q
(c) All exponentials e^{tA} of a fixed matrix A
(d) Matrices P with +ve eigenvalue
(e) Matrices D with determinant 1.

Ans: Orthogonal matrices, exponentials e^{At}, matrices with det $=1$ are groups.

Examples of Subgroups: orthogonal matrices with det $= 1$, exponentials e^{An} for integer n.

Another subgroup: lower triangular elimination matrices E with diagonal 1's.

33. When S & T are symmetric +ve definite, ST might not even be symmetric. But its eigenvalues are still +ve. ~~Start~~

Prove $\lambda > 0$

Ans:

$$ST x = \lambda x$$

$$(Tx)^T S(Tx) = (Tx)^T \lambda x \cancel{>0}.$$

$$\Longrightarrow \lambda = \frac{x^T T^T S T x}{x^T T x} = \frac{(Tx)^T S(Tx)^{>0}}{x^T T x > 0.} > 0$$

$\therefore$

35. Suppose, C is +ve definite ($y^T C y > 0$ whenever $y \neq 0$) and A has independent columns ($Ax = 0$ iff $x = 0$)

Apply the energy test to $x^T A^T C A x$ to show that $S = A^T C A$ is +ve definite

Ans: $x^T (A^T C A) x = (Ax)^T C (Ax) > 0$ for $Ax \neq 0$

$x^T (A^T C A) x = 0$ only when $Ax = 0$.

i.e., $x = 0$

36.

(a)

Ans:

(b) C

Ans:

(c)

Ans:

36. Suppose S is +ve definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) Is $\lambda_1 I - S$ +ve definite

Ans: $\lambda : 0, \lambda_1 - \lambda_2, \ldots, \lambda_1 - \lambda_n$

$(\lambda_1 I - S)$ is semidefinite

(b) How does it follow that $\lambda_1 x^T x \geq x^T S x$ for every x?

Ans: $x^T(\lambda_1 I - S)x \geq 0$ for all $x \neq 0$

$$\lambda_1 x^T x \geq x^T S x$$

(c) The max. value of $\dfrac{x^T S x}{x^T x}$ is ______

Ans: $\dfrac{x^T S x}{x^T x} \leq \lambda_1$

mms

$\neq 0$

7 THE SINGULAR VALUE DECOMPOSITION (SVD)

* The singular value theorem for A is the eigenvalue theorem for A^TA and AA^T.

* A is often rectangular, but A^TA and AA^T are square, symmetric, and +ve definite.

* The SVD separates any matrix into simple pieces:

  Each piece is a column vector times a row vector.

  An $m \times n$ matrix has mn entries (big # when the matrix represents an image)
  But a column & a row only have $(m+n)$ components (far less than mn)

Think of an image as a large rectangular matrix. The entries a_{ij} tell the grayscales of all the pixels in the image. Think of a pixel as a small square

Its grayscale is a # (often a whole # in the range $0 \le a_{ij} < 256 = 2^8$).

An all white pixel has $a_{ij} = 255 = 11111111$

An image that has mn pixels, with each pixel using 8 bits (0 or 1) for its grayscale, becomes an $m \times n$ matrix with 256 possible values for each entry a_{ij}.

ie., an image is a large matrix. To copy it perfectly, we need $8mn$ bits of information. HD tv typically has $m = 1080$ and $n = 1920$.

Often there are 24 frames per second, and you probably like to watch in color (3 color scales). This requires transmitting $3 \times 8 \times 48\,470400$ bits per second. That is too expensive & it is not done. The transmitter can't keep up with the show.

When compression is well done, you can't see the difference from the original. Edges in the image (sudden changes in the grayscale) are the hard parts to compress. Major success in compression will be impossible if every a_{ij} is an independent random #. We totally depend on the fact that nearby pixels generally have similar grayscales.

For a video, the #'s a_{ij} don't change much b/w frames. We only transmit the small changes.

□ Low Rank Images (Examples)

Easiest images to compress are all black or all white or all a constant grayscale g.

The matrix A has the same # g in every entry : $a_{ij} = g$

Ex:1 When $g = 1$ and $m = n = 6$,

Don't send
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Send
$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Ex: 2 French flag A

Italian flag A

German flag A^T

Don't send: $A = \begin{bmatrix} a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \\ a & a & c & c & e & e \end{bmatrix}$

Send: $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} a & a & c & c & e & e \end{bmatrix}$

This flag has 3 colors, but it still has rank 1.

Ex: 3 Embedded square

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} =$$

$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not orthogonal to $u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ is not orthogonal to $\begin{bmatrix} 0 & 1 \end{bmatrix}$.

Theory $\Rightarrow$ Orthogonality will produce a smaller 2nd piece $c_2 u_2 v_2^T$.

Eigenvectors for the SVD

Eigenvectors of most images are not orthogonal. The eigenvectors x_1, x_2 give only one set of vectors. and we want 2 sets (u's and v's).

SVD : Use the eigenvectors u of AA^T and the eigenvectors v of A^TA.

$u \in C(A)$
$v \in C(A^T)$

AA^T and A^TA are symmetric — the u's will be one orthogonal set and the eigenvectors v will be another orthogonal set.

$|u_i| = 1$ & $|v_i| = 1$

Our rank 2 matrix will be,

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

The size of those numbers σ_1 and σ_2 will decide whether they can be ignored in compression. We keep larger σ's, we discard small σ's.

The u's from the SVD are called left singular vectors (unit eigenvectors of AA^T). The v's are right singular vectors (unit eigenvectors of A^TA). The σ's are singular values, square roots of the equal eigenvalues of AA^T and A^TA:

Choices from the SVD : $AA^T u_i = \sigma_i^2 u_i$, $A^TAv_i = \sigma_i^2 v_i$

$$\boxed{Av_i = \sigma_i u_i}$$

(*) AA^T & A^TA have the same eigenvalues.

ILA(8) * A^TA & AA^T are +ve (semi) definite [unless A has independent columns]

* $\sigma_i > 0$ is purely conventional.

If $\sigma_i > 0$, then there is a unique way to write $A = U\Sigma V^T$.

OM(23) * $\text{rank}(AA^T) = \text{rank}(A^TA) = \text{rank}(A)$.

* AA^T & A^TA are symmetric, eigenvectors corresp. to distinct eigenvalues are orthogonal.

In Example 3 (the embedded square)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad ; \quad \lambda_2 = \frac{3-\sqrt{5}}{2}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{\frac{3+\sqrt{5}}{2}}$$

$$= \sqrt{\frac{6+2\sqrt{5}}{4}} = \sqrt{\frac{(\sqrt{5})^2 + 2\sqrt{5} + 1}{4}} \quad , \quad \sigma_2 = \frac{\sqrt{5}-1}{2}$$

$$= \frac{\sqrt{5}+1}{2}$$

The nearest rank 1 matrix to A will be $\sigma_1 u_1 v_1^T$.

The error is only $\sigma_2 \approx 0.6$ = best possible.

The orthonormal eigenvectors of AA^T and $A^T A$ are,

$u_1 = \begin{bmatrix} 1 \\ \sigma_1 \end{bmatrix}$, $u_2 = \begin{bmatrix} \sigma_1 \\ -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} \sigma_1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -\sigma_1 \end{bmatrix}$ all divided by $\sqrt{1+\sigma_1^2}$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

$$= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

(OR)

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 \end{bmatrix}$$

* The key point is not that images tend to have low rank. No!

Images mostly have full rank. But they do have low effective rank.

ie.,

Many singular values are small and can be set to zero.

⇒ We transmit a low rank approximation.

Ex: 4 Suppose the flag has 2 triangles of different colors.

Image matrix when $n=4$. It has full rank $r=4$ so it is invertible:

Triangular flag matrix: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

With full rank, A has a full set of 'n' singular values σ (all +ve). The SVD will produce n pieces $\sigma_i u_i v_i^T$ of rank 1. Perfect production needs all n pieces.

In compression small σ's can be discarded with no serious loss in image quality.
We want to understand and plot the σ's for $n=4$ and also for large n.

$$AA^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$(AA^T)^{-1} = (A^{-1})^T (A^{-1}) =$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Bases & Matrices in the SVD

$A_{m \times n}$ is square or rectangular, with rank r.

We'll diagonalize this A, but not by $X^{-1}AX$.

The eigenvectors in X have 3 big problems

- they are usually not orthogonal
- there are not always enough eigenvectors
- $Ax = \lambda x$ requires A to be a square matrix.

The singular vectors of A solve all these problems in a perfect ways.

$u_i \in \mathbb{R}^m$: $C(A)$

$v_i \in \mathbb{R}^n$: $C(A^T)$

The u's and v's give bases for the 4 fundamental subspaces :

$u_1, \ldots, u_r$	: orthonormal basis for $C(A)$
$u_{r+1}, \ldots, u_m$	: orthogonal basis for $N(A^T)$
$v_1, \ldots, v_r$	: ,, $C(A^T)$
$v_{r+1}, \ldots, v_n$	: ,, $N(A)$

* More than just orthogonality,
these basis vectors diagonalize the matrix A.

'A' is diagonalized : $Av_1 = \sigma_1 u_1$; $Av_2 = \sigma_2 u_2$; $\cdots$; $Av_r = \sigma_r u_r$

- The singular values, $\sigma_i > 0$.

σ_i : is the length of Av_i

$$\underset{(m\times n)}{A}\ \underset{(n\times r)}{V_r} = \underset{(m\times r)}{U_r}\ \underset{(r\times r)}{\Sigma_r}$$

$$A\begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

Reduced SVD

* Those v's and u's account for the row space & column space of A. We have $n-r$ more v's and $m-r$ more u's, from the $N(A)$ and $N(A^T)$. They are automatically orthogonal to the 1st v's and u's $\left(N(A) \perp C(A^T)\right)$

$$\underset{(m\times n)}{A}\ \underset{(n\times n)}{V} = \underset{(m\times m)}{U}\ \underset{(m\times n)}{\Sigma}$$

$$A\begin{bmatrix} v_1 \cdots v_r \cdots v_n \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_r \cdots u_m \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

Full SVD with nullspaces included.

* Both full & reduced SVD split up A into the

- When we put the singular values in the descending order, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$,

the splitting in the eqⁿ (SVD) gives the r-rank-one pieces of A in order of importance.

$$\boxed{\begin{aligned} \text{SVD} : \quad A &= U \Sigma V^T \\ &= u_1 \sigma_1 v_1^T + \cdots + u_r \sigma_r v_r^T \end{aligned}}$$

* For any orthogonal matrices Q_1 and Q_2, $Q_1 A Q_2^T$ has the same σ's as A.

Ex:1 When $A = U\Sigma V^T$ (singular values) the same as $X\Lambda X^{-1}$ (eigenvalues) ?

Ans: 'A' needs orthonormal eigenvectors. to allow $X = U = V$

'A' also needs eigenvalues $\lambda \geq 0$ if $\Lambda = \Sigma$

$\Longrightarrow$ 'A' must be a +ve definite symmetric matrix.

Only then,

$$A = X\Lambda X^{-1} = Q\Lambda Q^T = U\Sigma V^T$$

Ex: 2. If $A = xy^T$ (rank 1) with unit vectors x & y what is the SVD of A?

Ans:

Proof - SVD

$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^T U^T U \Sigma V^T$$
$$= V\Sigma^T\Sigma V^T$$

$\Sigma^T\Sigma$: eigenvalue matrix of A^TA : Each σ^2 is $\lambda(A^TA)$

V : eigenvector matrix for the symmetric +ve (semi) definite matrix A^TA

Now, $Av_i = \sigma_i u_i$ tells us the unit vectors u_1 to u_r

The whole reason that the SVD succeeds is that those unit vectors u_1 to u_r are automatically orthogonal to each other (because V's are orthogonal) :

$$i \neq j : \quad u_i^T u_j = \left(\frac{Av_i}{\sigma_i}\right)^T\left(\frac{Av_j}{\sigma_j}\right) = \frac{v_i^T A^T A v_j}{\sigma_i \sigma_j} = \frac{\sigma_j^2}{\sigma_i\sigma_j} v_i^T v_j = 0$$

The v's are eigenvector of A^TA (symmetric). They are orthogonal & now the u's are also orthogonal. Actually those u's will be eigenvectors of AA^T.

Example of SVD

Find the matrices U, Σ, V for $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.

The rank is $r = 2$

Ans: A has the singular values σ_1 & σ_2.

We'll see that σ_1 is larger than $\lambda_{max} = 5$ and σ_2 is smaller than $\lambda_{min} = 3$.

$$A^T A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}, \quad A A^T = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$$

Same eigenvalues $\sigma_1^2 = 45$ & $\sigma_2^2 = 5$

$\sigma_1 = \sqrt{45} = 3\sqrt{5}$ and $\sigma_2 = \sqrt{5}$ $\Big\}$ $\sigma_1 \sigma_2 = 3 \times 5 = 15 = |A|$

$$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 45 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Right singular vectors $\rightarrow$ $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Left singular vectors: $u_i = \frac{Av_i}{\sigma_i}$ $\rightarrow$

$$Av_1 = \frac{3}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \sqrt{45} \times \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \sigma_1 u_1$$

$$Av_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \sqrt{5} \times \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \sigma_2 u_2$$

$$U = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{45} & \\ & \sqrt{5} \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\underline{\underline{\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \frac{\sqrt{45}}{\sqrt{20}}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} + \frac{\sqrt{5}}{\sqrt{20}}\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix}}} = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = A$$

An Extreme matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\lambda : 0, 0, 0, 0$

only one eigenvector : $(1, 0, 0, 0)$

singular values, $\sigma = 3, 2, 1$

singular vectors are columns of I

$$A^T A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}, \quad A A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3 & & & \\ & 2 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A = U \Sigma V^T = 3 u_1 v_1^T + 2 u_2 v_2^T + 1 u_3 v_3^T$$

Removing the zero row of A (now 3×4) just removes the last row of Σ & also the last row & columns of U.

Then. $\underset{(3\times4)}{A} = \underset{(3\times3)}{U}\underset{(3\times4)}{\Sigma}\underset{(4\times4)}{V^T}$

$\Rightarrow$ SVD is totally adapted to rectangular matrices.

* If we have the grades for all courses, there would be a column for each student and a row for each course: The entry a_{ij} would be a grade. Then,

$\sigma_1 u_1 v_1^T$ could have u_1 = combination course &

v_1 = combination student.

And σ_1 would be the grade for these combinations: the highest grade

* The matrix A could count the frequency of key words in a journal: A different article for each column of A and a different word for each row. The whole journal is indexed by the matrix A and the most important information is in $\sigma_1 u_1 v_1^T$. Then σ_1 is the largest frequency for a hyperword (the word combination u_1) in the hyperarticle v_1.

Singular Value Stability v/s Eigenvalue Stability

Instability of eigenvalues

$$a_{4,1} : 0 \longrightarrow \frac{1}{60,000}$$

New rank = 4

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ \frac{1}{60000} & 0 & 0 & 0 \end{bmatrix}$$

Change by $\frac{1}{60,000}$ produces a much bigger jump in the eigenvalues of A

$$\lambda = 0,0,0,0 \longrightarrow \lambda = \frac{1}{10}, \frac{i}{10}, \frac{-1}{10}, \frac{-i}{10}$$

The 4 eigenvalues moved from 0 onto a circle around zero, of radius $\frac{1}{10}$. when the new entry is only $\frac{1}{60,000}$. This shows serious instability of eigenvalues when AA^T is far from A^TA.

If $A^TA = AA^T$ (a normal matrix), the eigenvectors of A are orthogonal & the eigenvalues of A are totally stable.

By constrast,

the singular values of any matrix are stable. They don't change more than they change in A.

New singular values are : $3, 2, 1, \frac{1}{60{,}000}$

The matrices U and V stay the same.

The new 4th piece of A is $\sigma_4 u_4 v_4^T$ with

$$\sigma_4 = \frac{1}{60{,}000}$$

☐ Singular Vectors of A & Eigenvectors of $S = A^T A$

The singular vector v_i are the eigenvectors q_i of $S = A^T A$. The eigenvalues λ_i of S are the same as σ_i^2 for A. The rank of S = rank of A.

OM (Q3) → $\boxed{\text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^T)}$

The expansion in eigenvectors & singular vectors are perfectly parallel.

Symmetric S : $S = Q \Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_r q_r q_r^T$

Any matrix A : $A = U \Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$

The q's are orthonormal, the u's are orthonormal, the v's are orthonormal. Beautiful.

Look Again

→ To fix a weak point in the eigenvalue part, where Ch. 6, was not complete. If λ is a double eigenvalue of S, we can and must find 2 orthonormal eigenvectors.

→ To see how the SVD picks off the largest term $\sigma_1 u_1 v_1^T$ before $\sigma_2 u_2 v_2^T$. We want to understand the eigenvalues λ of S and the singular values σ (of A) one at a time instead of all at once.

Largest eigenvalue λ_1 of S

$\lambda_1 = \max.$ ratio $\dfrac{x^T S x}{x^T x}$.

The winning vector is $x = q_1$ with $Sq_1 = \lambda q_1$

Largest singular value σ_1 of A

$A^T A v_i = \sigma_i^2 v_i$

$A v_i = \sigma_i u_i$

$\sigma_1 = \max.$ ratio $\dfrac{\|Ax\|}{\|x\|}$.

The winning vector is $x = v_1$ with $Av_1 = \sigma_1 u_1$

$\lambda_2 = \max.$ ratio $\dfrac{x^T S x}{x^T x}$ among all x's with $q_1^T x = 0$.

$x = q_2$ will win

$\sigma_2 = \max.$ ratio $\dfrac{\|Ax\|}{\|x\|}$ among all x's with $v_1^T x = 0$

$x = v_2$ will win

When $S = A^T A$ we find $\lambda_1 = \sigma_1^2$ and $\lambda_2 = \sigma_2^2$,

Why does this approach succeed ?

Start with the ratio, $\boxed{r(x) = \frac{x^T S x}{x^T x}}$.

This is called the Rayleigh quotient

$\boxed{\text{Maximize } r(x)}$

1. Multivariable calculus approach

The Rayleigh quotient is scaling invariant

$$r(tx) = \frac{(tx)^T S (tx)}{(tx)^T (tx)} = \frac{t^2 x^T S x}{t^2 x^T x} = \frac{x^T S x}{x^T x} = r(x)$$

$\therefore$ It is sufficient to study the special case $|x|^2 = x^T x = 1$.

The problem is then to find the critical points of the function, $r(x) = x^T S x$ subject to the constraint $|x|^2 = x^T x = 1$.

ie., It is to find the critical points of the Lagrangian function

$$L(x, \lambda) = x^T S x - \lambda (x^T x - 1)$$

where,

λ : Lagrange Multipliers.

We need to find the partial derivatives

$$\frac{\partial L}{\partial x} = \left(\frac{\partial L}{\partial x_1}, \dots, \frac{\partial L}{\partial x_n}\right)^T = \begin{bmatrix} \frac{\partial L}{\partial x_1} \\ \vdots \\ \frac{\partial L}{\partial x_n} \end{bmatrix}, \quad \frac{\partial L}{\partial \lambda}$$

and set them equal to zero, in order to find its critical points.

We need to ~~find~~ know how to differentiate functions like $x^T A x$, $x^T x = |x|^2$ w.r.t the vector valued variable x.

For any fixed symmetric matrix $A \in \mathbb{R}^{n \times n}$, fixed rectangular matrix $B \in \mathbb{R}^{m \times n}$ and fixed vector $a \in \mathbb{R}^n$,

$\frac{\partial}{\partial x}(a^T x) = a$	$\frac{\partial}{\partial x} \lvert x \rvert^2 = 2x$
$\frac{\partial}{\partial x}(x^T A x) = 2Ax$	$\frac{\partial}{\partial x} \lvert Bx \rvert^2 = 2B^T B x$

Proof

For each $1 \le k \le n$:

$$\frac{\partial}{\partial x_k}(a^T x) = \frac{\partial}{\partial x_k}\left(\sum a_i x_i\right) = a_k$$

$$\Rightarrow \frac{\partial}{\partial x}(a^T x) = \begin{bmatrix} \frac{\partial}{\partial x_1}(a^T x) \\ \vdots \\ \frac{\partial}{\partial x_k}(a^T x) \\ \vdots \\ \frac{\partial}{\partial x_n}(a^T x) \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{bmatrix}$$

$$\frac{\partial}{\partial x_k}\left(|x|^2\right) = \frac{\partial}{\partial x_k}\left(\sum x_i^2\right) = 2x_k$$

$$\frac{\partial}{\partial x_k}\left(x^T A x\right) = \frac{\partial}{\partial x_k}\left[\left(\sum_i x_i a_{ij}\right)_{n} x\right]$$

$$= \frac{\partial}{\partial x_k}\left[\sum_j \left(\sum_i x_i a_{ij}\right)_j x_j\right]$$

$$= \frac{\partial}{\partial x_k}\left[\sum_i \sum_j a_{ij} x_i x_j\right]$$

$$= \frac{\partial}{\partial x_k}\left[\sum_{\substack{j \neq k \\ i = k}} a_{kj} x_k x_j + \sum_{\substack{i \neq k \\ j = k}} a_{ik} x_i x_k + a_{kk} x_k^2\right]$$

$$= \sum_{j \neq k} a_{kj} x_j + \sum_{i \neq k} a_{ik} x_i + 2a_{kk} x_k$$

$$= \sum_j a_{kj} x_j + \sum_i a_{ik} x_i$$

$$= (Ax)_{(k,-)} + (x^T A)_{(-,k)}$$

(k^{th} row of Ax) (k^{th} column of $x^T A$)

$\parallel$

(k^{th} row of Ax) since $A^T = A$

$$= (2Ax)_{(k,-)}$$

$$\frac{\partial}{\partial x}\left(x^T A x\right) = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(x^T A x\right) \\ \vdots \\ \vdots \\ \frac{\partial}{\partial x_n}\left(x^T A x\right) \end{bmatrix} = \begin{bmatrix} 2Ax_{(1,-)} \\ \vdots \\ \vdots \\ 2Ax_{(n,-)} \end{bmatrix} = 2Ax$$

$$|Bx|^2 = (Bx)^T (Bx) = x^T \left(B^T B\right) x$$

$$\frac{\partial}{\partial x}\left(|Bx|^2\right) = 2B^T B x$$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(x^T S x\right) - \lambda \frac{\partial}{\partial x} |x|^2$$

$$= 2Sx - \lambda(2x) = 0$$

$$\Longrightarrow \boxed{Sx = \lambda x}$$

$$\frac{\partial L}{\partial \lambda} = |x|^2 - 1 = 0 \Longrightarrow \boxed{|x|^2 = 1}$$

$\therefore$ x, λ must be an eigenpair of A.

ie.,

the eigenvectors $x_1, x_2, \ldots, x_n$ are the critical points of the Rayleigh quotient, $r(x)$.

$$r(x_i) = x_i^T S x_i = x_i^T \lambda x_i = \lambda_i |x_i|^2 = \lambda_i$$

$\therefore$ r_{max} when $x_i = x_1$

The max ratio $r(x)$ is the largest eigenvalue λ_1 of S.

$S = A^T A$

Maximizing $\boxed{\frac{\|Ax\|}{\|x\|}}$ also maximizes $\left(\frac{\|Ax\|}{\|x\|}\right)^2 = \frac{x^T A^T A x}{x^T x}$

$$= \frac{x^T S x}{x^T x}$$

So the winning $x = v_1$ is the same as the top eigenvector q_1 of $S = A^T A$.

(2) S is positive semidefinite $\longrightarrow$ eigenvectors form an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ such that $e_i^\dagger e_j = \delta_{ij}$

QC 11.2

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where,

$$A\hat{e}_i = \lambda_i \hat{e}_i \quad \text{with } \lambda_i \geq 0 \text{ for all } i$$

such that $A = \sum_i \lambda_i \hat{e}_i e_i^\dagger$

A general unit vector x can be written as a linear combination of the eigenvector basis, $x = \sum_i c_i \hat{e}_i$ such that $\|x\|^2 = \sum_i |c_i|^2 = 1$.

We have,

$$x^\dagger S x = \left(\sum_i c_i^* \hat{e}_i^\dagger\right)\left(\sum_j \lambda_j \hat{e}_j \hat{e}_j^\dagger\right)\left(\sum_k c_k \hat{e}_k\right)$$

$$= \left(\sum_i c_i^* \hat{e}_i^\dagger\right)\left(\sum_{j,k} \lambda_j c_k \hat{e}_j (\hat{e}_j^\dagger \hat{e}_k)\right)$$

$$= \left(\sum_i c_i^* \hat{e}_i^\dagger\right)\left(\sum_j \lambda_j c_j e_j\right)$$

$$= \sum_{i,j} c_i^* c_j \lambda_j \hat{e}_i^\dagger e_j$$

$$= \sum_i |c_i|^2 \lambda_i$$

$\max_{x^\dagger x} x^\dagger S x$ is the maximum value of $\sum_i |c_i|^2 \lambda_i$ subject to the constraint $\sum_i |c_i|^2 = 1$.

Numbering the λ_i's and $\hat{e}_i$ so that λ_1 is the maximum value of $\sum_i |c_i|^2 \lambda_i$ is achieved when $c_1 = 1$ and $c_2 = \cdots = c_n = 0$. The maximum value achieved is λ_1.

$\therefore$

$$\max_{x^\dagger x} x^\dagger S x = \lambda_{max}$$