Introduction to Linear Algebra
- Gilbert Strang

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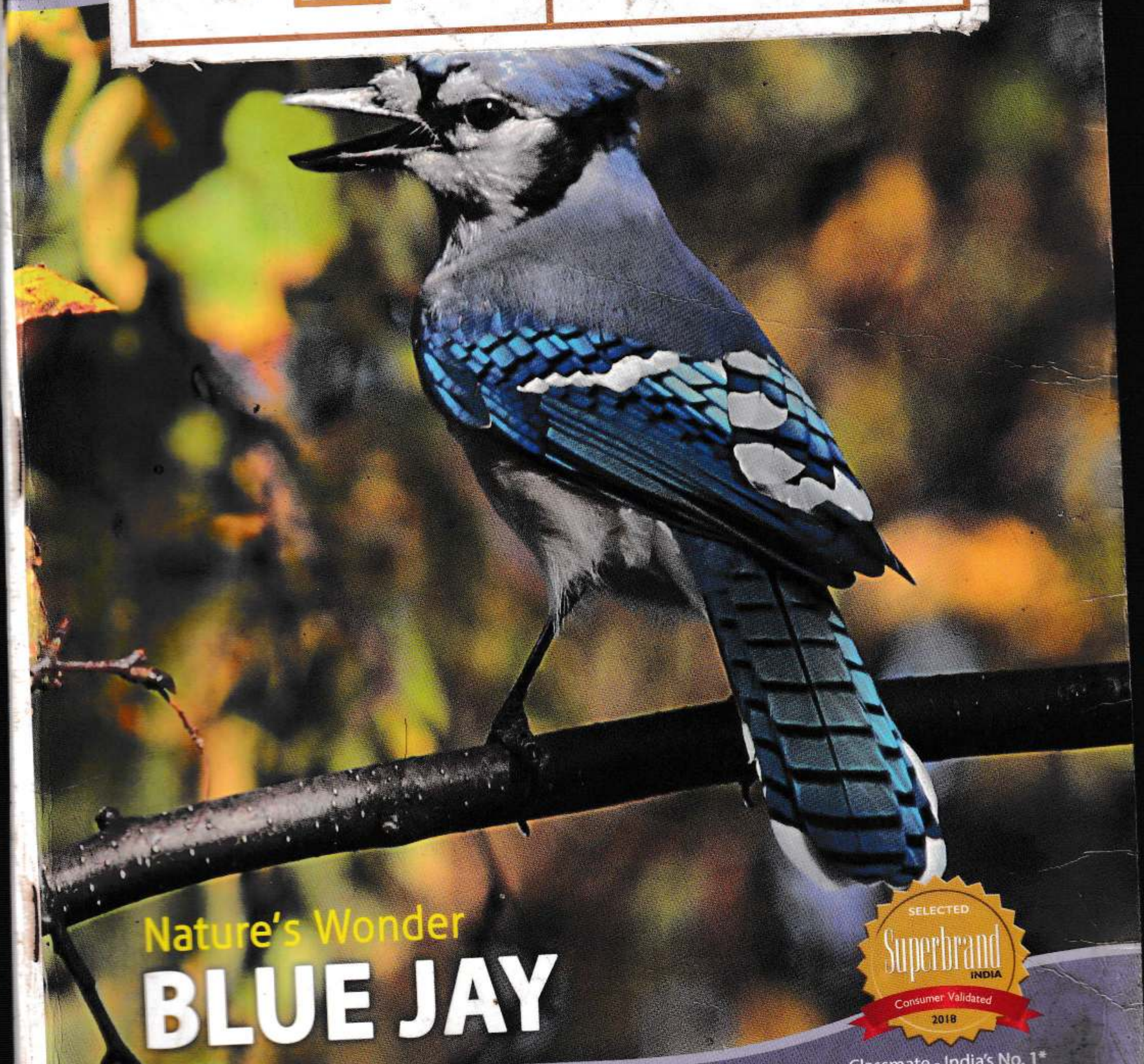
Eigenvalues & Eigenvectors



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		LINEAR ALGEBRA		
		- Gilbert Strang, MIT		
		(5th Edition)		

Differential to difference dynamics

$$\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x) - u(x-\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}$$

We need to change a differential equation to a matrix equation.

The continuous problem asks for $u(x)$ at every x, and a computer can not solve it exactly. It has to be approximated by a discrete problem.

The 1st derivative can be approximated by stopping $\frac{\Delta u}{\Delta x}$ at a finite step size, and not permitting Δx to approach zero.

The difference Δu can be forward, backward (or) centered :

$$\frac{du}{dx} \approx \frac{\Delta u}{\Delta x} = \frac{u(x+\Delta x) - u(x)}{\Delta x} \quad \text{(forward difference}$$

$$= \frac{u(x) - u(x-\Delta x)}{\Delta x} \quad \text{(backward difference}$$

$$= \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} \quad \text{(central difference}$$

The last is symmetric about x and it is the most accurate.

At each meshpoint $x = j\Delta x$,

$$\frac{\Delta u}{\Delta x} = \frac{u_{j+1} - u_j}{\Delta x} \quad \text{(forward difference)}$$

$$= \frac{u_j - u_{j-1}}{\Delta x} \quad \text{(backward difference)}$$

$$= \frac{u_{j+1} - u_{j-1}}{2\Delta x} \quad \text{(central difference}$$

for $j = 1, 2, \ldots\ldots, n$

For vectors $u = \{u(x_i)\}$ and $u' = \{u'(x_i)\}$:

$$\begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \end{bmatrix} \approx \frac{1}{\Delta x} \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{\text{forward difference matrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

$$\begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \end{bmatrix} \approx \frac{1}{\Delta x} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_{\text{Backward difference matrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \end{bmatrix} \approx \frac{1}{2\Delta x} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}}_{\text{central difference matrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

2nd derivative

Forward: $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{u'(x+\Delta x) - u'(x)}{\Delta x}$$

$$= \frac{\lim\limits_{\Delta x \to 0} \frac{u(x+2\Delta x) - u(x+\Delta x)}{\Delta x} - \lim\limits_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+2\Delta x) - 2u(x+\Delta x) + u(x)}{(\Delta x)^2}$$

$$\boxed{u''_j = \frac{u_{j+2} - 2u_{j+1} + u_j}{h^2}}, \text{ for } j = 1, 2, \cdots, n$$

Backward: $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \lim_{\Delta x \to 0} \frac{u'(x) - u'(x-\Delta x)}{\Delta x}$$

$$= \frac{\lim\limits_{\Delta x \to 0} \frac{u(x) - u(x-\Delta x)}{\Delta x} - \lim\limits_{\Delta x \to 0} \frac{u(x-\Delta x) - u(x-2\Delta x)}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x) - 2u(x-\Delta x) + u(x-2\Delta x)}{(\Delta x)^2}$$

$$\boxed{u''_j = \frac{u_j - 2u_{j-1} + u_{j-2}}{h^2}}, \text{ for } j = 1, 2, \cdots, n$$

2nd derivative,

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{du}{dx}\right) = \lim_{\Delta x \to 0} \frac{u'(x+\Delta x) - u'(x-\Delta x)}{2\Delta x}$$

$$= \left[\lim_{\Delta x \to 0} \frac{u(x+2\Delta x) - u(x)}{4(\Delta x)^2}\right] - \left[\lim_{\Delta x \to 0} \frac{u(x) - u(x-2\Delta x)}{4(\Delta x)^2}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x+2\Delta x) - 2u(x) - u(x-2\Delta x)}{\cancel{4}\,(2\Delta x)^2}\right]$$

$$\boxed{\frac{d^2u}{dx^2} = \lim_{2\Delta x = h \to 0} \frac{u(x+h) - 2u(x) - u(x-h)}{h^2}}$$

Second difference.

$$\frac{d^2u}{dx^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Symmetric about x.

At each mesh point $x = jh$,

$$\boxed{u''_j = \frac{\Delta^2 u}{\Delta x^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}}$$ for $j = 1, 2, \cdots, n$

$$\boxed{\frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} = \begin{cases} u''_{j-1} & \text{forward from } j-1 \\ u''_j & \text{centered at } j \\ u''_{j+1} & \text{backward from } j+1 \end{cases}}$$

$$\begin{bmatrix} u_1'' \\ u_2'' \\ u_3'' \\ u_4'' \\ u_5'' \end{bmatrix} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

Ex:- $\dfrac{d^2u}{dx^2} = f(x)$

Difference equation : $u_{j+1} - 2u_j + u_{j-1} = h^2 f(jh)$

for $j = 1, 2, \dots, n$

$u_0 = u_{n+1} = 0$ (Boundary conditions)

Matrix equation,

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

□ Difference equations (Optional)

Check Ex: 3.

motion around a circle with $y'' + y = 0$ and $y = \cos t$

$$y'' = -y \implies \frac{du}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

$\lambda_1 = i, \; \lambda_2 = -i$ [A is antisymmetric]

$x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \; x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{x_1 + x_2}{2}$.

$$u(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

To display a circle on a screen, replace $y'' = -y$ by a difference equation.

$$\frac{Y_{n+1} - 2Y_n + Y_{n-1}}{(\Delta t)^2} = \begin{cases} -Y_{n-1} & \text{Forward from } (n-1) \\ -Y_n & \text{Centered at time } n \\ -Y_{n+1} & \text{Backward from } (n+1) \end{cases}$$

~~$\Delta Y_n \quad Y'_{n+1} =$~~

Forward

$$Y'_n = \frac{Y_{n+1} - Y_n}{\Delta t} \Rightarrow Y_{n+1} = Y_n + \Delta t\, Y'_n$$

$$Y'_n - Y'_{n+1} = \frac{Y_{n+1} - Y_n}{\Delta t} - \frac{Y_{n+2} + Y_{n+1}}{\Delta t}$$

$$= \frac{-[Y_n - 2Y_{n+1} + Y_{n+2}]}{\Delta t} = \cancel{-} \Delta t$$

$$= \frac{-[Y_{n+2} - 2Y_{n+1} + Y_n]}{\Delta t} = +\Delta t \times Y_n$$

$$\Rightarrow Y'_{n+1} = Y'_n + \Delta t\, Y_n$$

$$U_{n+1} = \begin{bmatrix} Y_{n+1} \\ Y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Y'_n \end{bmatrix} = A U_n$$

$$(1-\lambda)^2 + (\Delta t)^2 = 0 \implies$$

$$1 - 2\lambda + \lambda^2 + (\Delta t)^2 = 0 \qquad D = 4 - 4(1+(\Delta t)^2)$$

$$\lambda^2 - 2\lambda + 1 + (\Delta t)^2 = 0$$

$$(1-\lambda)^2 - (-(\Delta t)^2) = (1-\lambda)^2 - (i\,\Delta t)^2 = 0$$

$$(1-\lambda - i\Delta t)(1-\lambda + i\Delta t) = 0$$

$$1-\lambda - i\Delta t = 0 \quad \text{(or)} \quad 1-\lambda + i\Delta t = 0$$

$$\boxed{\lambda_1 = 1 + i\Delta t}, \quad \boxed{\lambda_2 = 1 - i\Delta t}$$

$$|\lambda| = 1 + (\Delta t)^2 > 1$$

$$\lambda_1 = 1 + i\Delta t : \quad \begin{bmatrix} -i\Delta t & \Delta t \\ -\Delta t & -i\Delta t \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \Delta t\,[-ia + b] = 0$$

$$a + ib = 0$$

$$ia - b = 0$$

$$b = ia$$

$$x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 = 1 - i\Delta t : \quad \begin{bmatrix} i\Delta t & \Delta t \\ -\Delta t & i\Delta t \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Delta t\,(ia + b) = 0 \implies x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$b = -ia$$

$$\lambda_1 = 1 + i\Delta t, \quad x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$|\lambda| > 1 \left\{ \begin{array}{l} \lambda_1 = 1 + i\Delta t, \quad x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \\ \lambda_2 = 1 - i\Delta t, \quad x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \end{array} \right.$$

$$U_0 = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}(x_1 + x_2)$$

$$U_k = A^k U_0 = c_1 (\lambda_1)^k x_1 + c_2 (\lambda_2)^k x_2$$

Set $\Delta t = \frac{2\pi}{32}$,

$$U_k = \frac{1}{2}\left(1 + \frac{i\pi}{16}\right)^k \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}\left(1 - i\frac{\pi}{16}\right)^k \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$re^{i\theta} = r(\cos\theta + i\sin\theta) = 1 + i\frac{\pi}{16}$$

$$r = \sqrt{1 + \left(\frac{\pi}{16}\right)^2} = 1.019094275$$

$$\theta = \tan^{-1}(\pi/16) = 11.10868057\,5$$

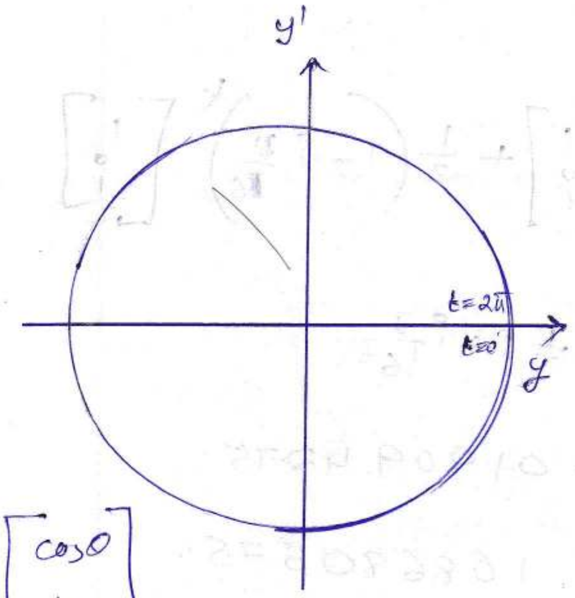
$$U_k = \begin{bmatrix} y_k \\ y'_k \end{bmatrix} = \begin{bmatrix} r^k \cos(k\theta) \\ -r^k \sin(k\theta) \end{bmatrix}$$

$$U_8 = \begin{bmatrix} y_8 \\ y'_8 \end{bmatrix} = \begin{bmatrix} 0.02295387\,1 \\ -1.1631355\,9 \end{bmatrix}$$

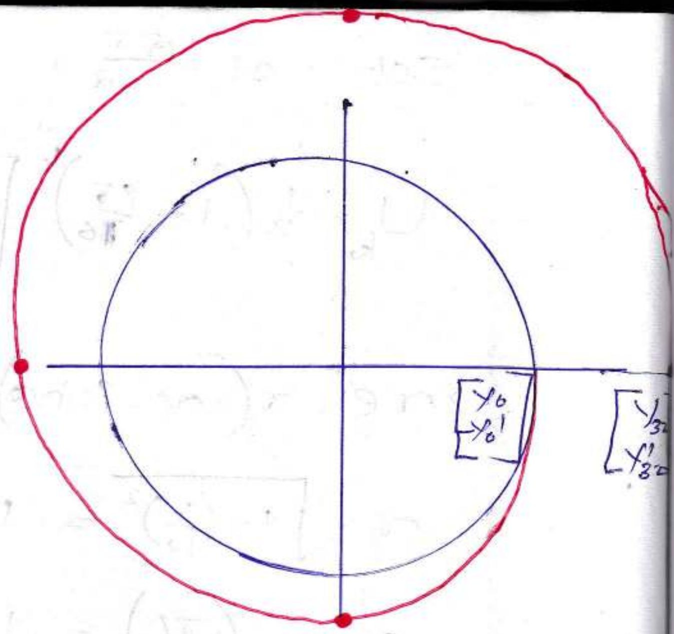
$$U_{16} = \begin{bmatrix} y_{16} \\ y'_{16} \end{bmatrix} = \begin{bmatrix} -1.35235722 \\ -0.053396928 \end{bmatrix}$$

$$U_{24} = \begin{bmatrix} y_{24} \\ y'_{24} \end{bmatrix} = \begin{bmatrix} -0.093149707 \\ 1.571749498 \end{bmatrix}$$

$$U_{32} = \begin{bmatrix} y_{32} \\ y'_{32} \end{bmatrix} = \begin{bmatrix} 1.826019634 \\ 0.14442347\,4 \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$



* Exact $u = (\cos t, -\sin t)$ on a circle

* Forward Euler spirals out (32 steps)

$|\lambda| > 1$

Backward

$$Y_{n+1} = Y_n + \Delta t\, Y'$$

$$Y'_{n+1} = \frac{Y_{n+1} - Y_n}{\Delta t} \Rightarrow \underline{Y_n = Y_{n+1} - \Delta t\, Y'_{n+1}}$$

$$Y'_n - Y'_{n+1} = \frac{Y_n - Y_{n-1}}{\Delta t} - \frac{Y_{n+1} - Y_n}{\Delta t}$$

$$= \frac{-\left[Y_{n+1} - 2Y_n + Y_{n-1}\right]}{\cancel{\Delta t}\ \Delta t} = \Delta t\, Y_{n+1}$$

$$\underline{Y'_n = \Delta t\, Y_{n+1} + Y'_{n+1}}$$

$$U_n = \begin{bmatrix} Y_n \\ Y'_n \end{bmatrix} = \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Y'_{n+1} \end{bmatrix} = A\, U_{n+1}$$

$$(1-\lambda)^2 + (\Delta t)^2 = 0 \Longrightarrow \boxed{\lambda_1 = 1 + i\Delta t\,,\ \lambda_2 = 1 - i\Delta t}$$

$\lambda_1 = 1+i\Delta t$: $\begin{bmatrix} -i\Delta t & -\Delta t \\ \Delta t & -i\Delta t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \Delta t \begin{bmatrix} -ia - b \end{bmatrix} = 0$

$b = ia$

$$x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$\lambda_2 = 1 - i\Delta t$: $x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$U_{n+1} = \begin{bmatrix} Y_{n+1} \\ Y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix}^{-1} \begin{bmatrix} Y_n \\ Y_n \end{bmatrix} = A^{-1} U_n$$

$$|\lambda| < 1 \left\{ \begin{array}{l} \lambda_1 = \dfrac{1}{1+i\Delta t} = \dfrac{1 - i\Delta t}{1+(\Delta t)^2}, \quad x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ \lambda_2 = \dfrac{1}{1-i\Delta t} = \dfrac{1+i\Delta t}{1+(\Delta t)^2}, \quad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \end{array} \right.$$

$$U_{n+1} = \begin{bmatrix} Y_{n+1} \\ Y'_{n+1} \end{bmatrix} = \frac{1}{1+(\Delta t)^2} \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Y'_n \end{bmatrix} = A^{-1} U_n = B U_n$$

$$x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$U_0 = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}(x_1 + x_2)$$

$$U_k = A^k U_0 = c_1 (\lambda_1)^k x_1 + c_2 (\lambda_2)^k x_2$$

$$= \frac{1}{2}\left(\frac{1 - i\Delta t}{1+(\Delta t)^2}\right)^k \begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{1}{2}\left(\frac{1+i\Delta t}{1+(\Delta t)^2}\right)^k \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$=$

Set $\Delta t = \frac{2\pi}{32}$,

$$U_k = \frac{1}{2}\left(\frac{1 - i\frac{\pi}{16}}{1+\left(\frac{\pi}{16}\right)^2}\right)^k \begin{bmatrix} 1 \\ -i \end{bmatrix} + \frac{1}{2}\left(\frac{1+i\frac{\pi}{16}}{1+\left(\frac{\pi}{16}\right)^2}\right)^k \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$re^{i\theta} = r(\cos\theta + i\sin\theta) = \frac{1}{1+\left(\frac{\pi}{16}\right)^2} + i\,\frac{\pi/16}{1+(\pi/16)^2}$$

$$\theta = \tan^{-1}\frac{\pi}{16} = 11.108680575$$

$$r = \sqrt{\left(\frac{1}{1+\left(\frac{\pi}{16}\right)^2}\right)^2 + \left(\frac{(\pi/16)^2}{1+(\pi/16)^2}\right)^2} = \sqrt{\frac{1+(\pi/16)^4}{\left[1+(\pi/16)^2\right]^2}}$$

$$= 0.963593345$$

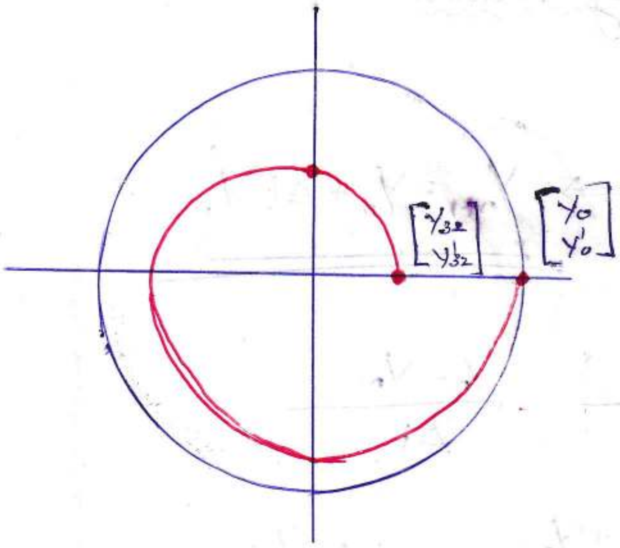
$$U_k = \begin{bmatrix} y_k \\ y'_k \end{bmatrix} = \begin{bmatrix} r^k \cos(k\theta) \\ -r^k \sin(k\theta) \end{bmatrix}$$

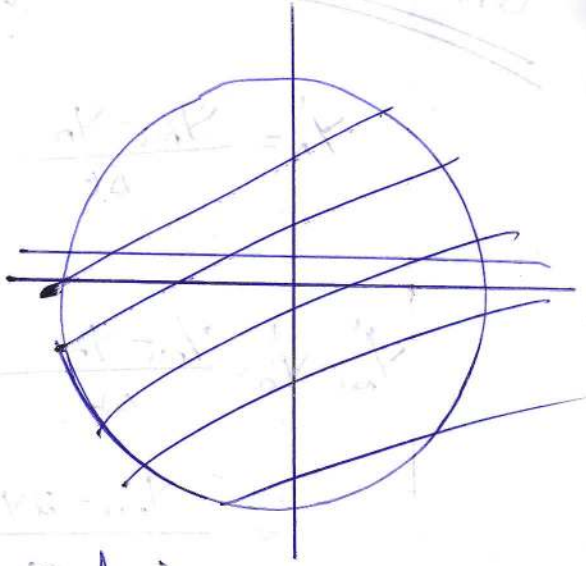
$$U_8 = \begin{bmatrix} y_8 \\ y'_8 \end{bmatrix} = \begin{bmatrix} 0.014665314 \\ -0.743131694 \end{bmatrix}$$

$$U_{16} = \begin{bmatrix} y_{16} \\ y'_{16} \end{bmatrix} = \begin{bmatrix} -0.552029643 \\ -0.021796519 \end{bmatrix}$$

$$U_{24} = \begin{bmatrix} y_{24} \\ y'_{24} \end{bmatrix} = \begin{bmatrix} -0.024293372 \\ 0.409911071 \end{bmatrix}$$

$$U_{32} = \begin{bmatrix} y_{32} \\ y'_{32} \end{bmatrix} = \begin{bmatrix} 0.304261638 \\ 0.02406465 \end{bmatrix}$$





Backward differences spiral in .

$|\lambda|<1$.

Centered

Leapfrog method

$$y_n' = \frac{y_{n+1} - y_n}{\Delta t} \Longrightarrow y_{n+1} = y_n + \Delta t \, y_n'$$

$$y_{n+1}' - y_n' = \frac{y_{n+1} - y_n}{\Delta t} - \frac{y_n - y_{n-1}}{\Delta t}$$

$$= \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t} = -(\Delta t)\, y_n$$

$$y_{n+1}' = y_n' - (\Delta t)\, y_n$$

$$\frac{Y_{n+1} - Y_n}{\Delta t} = Y_n' \Longrightarrow \underline{Y_{n+1} = Y_n + \Delta t\, Y_n'}$$

$$Y_n' - Y_{n+1}' = \frac{Y_n - Y_{n-1}}{\Delta t} - \frac{Y_{n+1} - Y_n}{\Delta t}$$

$$= \frac{-\left[Y_{n+1} - 2Y_n + Y_{n-1}\right]}{\Delta t} = \Delta t\, Y_{n+1}$$

$$\Longrightarrow \underline{Y_n' = \Delta t\, Y_{n+1} + Y_{n+1}'}$$

$$Y_{n+1} = Y_n + \Delta t\, Y_n'$$

$$\Delta t \cdot Y_{n+1} + Y_{n+1}' = Y_n'$$

$$\begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Y_{n+1}' \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Y_n' \end{bmatrix}$$

$$\begin{bmatrix} Y_{n+1} \\ Y_{n+1}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Y_n' \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Y_n' \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1-(\Delta t)^2 \end{bmatrix} \begin{bmatrix} Y_n \\ Y_n' \end{bmatrix}$$

$$U_{n+1} = \begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1-(\Delta t)^2 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} = AU_n$$

$$(1-\lambda)\left(1-\lambda-(\Delta t)^2\right) + (\Delta t)^2 = 0$$

$$1-\lambda - \cancel{(\Delta t)^2} - \lambda + \lambda^2 + \lambda(\Delta t)^2 + \cancel{(\Delta t)^2} = 0$$

$$\lambda^2 - \left(2-(\Delta t)^2\right)\lambda + 1 = 0$$

$$\delta = 4 + (\Delta t)^4 - 4(\Delta t)^2 - 4$$

$$= (\Delta t)^2\left((\Delta t)^2 - 4\right)$$

$$\lambda = \frac{2-(\Delta t)^2 \pm \Delta t\sqrt{(\Delta t)^2 - 4}}{2}$$

$$\cancel{\lambda_1 =}$$

$$\lambda_1 = 1 - \frac{(\Delta t)^2}{2} - \frac{\Delta t\sqrt{(\Delta t)^2-4}}{2}$$

$$\lambda_2 = 1 - \frac{(\Delta t)^2}{2} + \frac{\Delta t\sqrt{(\Delta t)^2-4}}{2}$$

$$x_1 = \begin{bmatrix} -\dfrac{2}{a+\sqrt{a^2-4}} \\ 1 \end{bmatrix} t$$

$$x_2 = \begin{bmatrix} \dfrac{-2}{a-\sqrt{a^2-4}} \\ 1 \end{bmatrix} t$$

$$\cancel{\text{Take, } \Delta t = \frac{1}{10}}$$

$$\cancel{A = \begin{bmatrix} 1 & \frac{1}{10} \\ -\frac{1}{10} & \frac{99}{100} \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 100 & 10 \\ -10 & 99 \end{bmatrix}}$$

Take $\Delta t = 1$,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \lambda_1 = \frac{1}{2} - \frac{i\sqrt{3}}{2} = e^{-i\pi/3}.$$

$|\lambda| = 1$

$$\lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\pi/3}$$

$$\cancel{\begin{bmatrix} 1 + \frac{i\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$x_1 = \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix} t = \begin{bmatrix} e^{i2\pi/3} \\ 1 \end{bmatrix} t$$

$$x_2 = \begin{bmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix} t = \begin{bmatrix} e^{4\pi/3} \\ 1 \end{bmatrix} t$$

$$U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{i\sqrt{3}} \left[x_1 - x_2 \right] = \frac{-i}{\sqrt{3}} \left[x_1 - x_3 \right]$$

$$\boxed{U_k = \frac{-i}{\sqrt{3}} e^{-ik\pi/3} \begin{bmatrix} e^{i2\pi/3} \\ 1 \end{bmatrix} + \frac{i}{\sqrt{3}} e^{ik\pi/3} \begin{bmatrix} e^{i4\pi/3} \\ 1 \end{bmatrix} = \begin{bmatrix} y_k \\ y'_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix} = A^k U_0}$$

$$= \frac{-i}{\sqrt{3}} \begin{bmatrix} e^{\frac{i\pi}{3}(2-k)} \\ e^{-ik\pi/3} \end{bmatrix} + \frac{i}{\sqrt{3}} \begin{bmatrix} e^{\frac{i\pi}{3}(4+k)} \\ e^{ik\pi/3} \end{bmatrix}$$

$$U_6 = \frac{-i}{\sqrt{3}} \cdot 1 \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix} + \frac{i}{\sqrt{3}} \cdot 1 \begin{bmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$$A^6 U_0 = \begin{bmatrix} y_6 \\ y'_6 \end{bmatrix} = \frac{i}{\sqrt{3}} \begin{bmatrix} -i\sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = U_0 .$$

$$U_{12} = U_0 .$$

$$U_3 = U_{15} = \begin{bmatrix} y_{15} \\ y'_{15} \end{bmatrix} = A^{15} U_0 = \frac{-i}{\sqrt{3}} \cdot -1 \begin{bmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix} + \frac{i}{\sqrt{3}} \cdot -1 \begin{bmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$$= \frac{i}{\sqrt{3}} \begin{bmatrix} i\sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} =$$

$\Delta t = \sqrt{2}$: $A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix}$

$\lambda_1 = -i$: $x_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$

$\lambda_2 = i$:

$x_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-i\pi/4} \\ 1 \end{bmatrix}$

$U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}\, i}(x_1 - x_2) = \frac{-i}{\sqrt{2}}(x_1 - x_2)$

$U_k = \begin{bmatrix} y_k \\ y'_k \end{bmatrix} = A^k U_0 = \frac{-i}{\sqrt{2}}(-i)^k \begin{bmatrix} \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \\ 1 \end{bmatrix} + \frac{i}{\sqrt{2}}(i)^k \begin{bmatrix} \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \\ 1 \end{bmatrix}$

$U_2 = \begin{bmatrix} y_2 \\ y'_2 \end{bmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} i\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$U_4 = \begin{bmatrix} y_4 \\ y'_4 \end{bmatrix} = \frac{-i}{\sqrt{2}} \begin{bmatrix} i\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = U_0.$

$\Delta t = 2$: $A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$

$\lambda_1 = \lambda_2 = -1$

$x_1 = x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Any time step $\Delta t > 2$ will lead to $|\lambda| > 1$, and the powers in $U_n = A^n U_0$ will explode.

* You might say that nobody would compute with $\Delta t > 2$. But if an atom vibrates with $y'' = -10^6 y$ then $\Delta t > 0.0002$ will give instability. Leapfrog has a very strict stability limit.
$Y_{n+1} = Y_n + 3Z_n$ and $Z_{n+1} = Z_n - 3Y_{n+1}$ will explode because $\Delta t = 3$ is too large. The matrix has $|\lambda| > 1$

>1,
de.

Trapezoidal method. (half forward/half back)

$$\begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix}$$

$$U_{n+1} = \begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \frac{1}{1+\left(\Delta t/2\right)^2} \begin{bmatrix} 1-\left(\Delta t/2\right)^2 & \Delta t \\ -\Delta t & 1-\left(\Delta t/2\right)^2 \end{bmatrix}$$

Orthogonal matrix.

$\boxed{6.3(c)}$ Solve 4 equations $\frac{da}{dt}=0$, $\frac{db}{dt}=a$, $\frac{dc}{dt}=2b$, $\frac{dz}{dt}=3c$ in that order starting from $u(0)=(a(0), b(0), c(0), z(0))$. Solve the same equations by the matrix exponential in $u(t)=e^{At}u(0)$

$$\frac{d}{dt}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix}$$

$$\frac{du}{dt}=Au$$

First find A^2, A^3, A^4 and $e^{At}=I+At+\frac{1}{2}(At)^2+\frac{1}{6}(At)^3$

Why does the series stop?

Why is it true that $(e^A)(e^A)=e^{2A}$?

Always $\boxed{e^{As}e^{At}=e^{A(s+t)}}$

Ans: Integrate

$a(t)=a(0)$

$b(t)=t\,a(0)+b(0)$

$c(t)=$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{3} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix} + \frac{t^3}{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2 & 2t & 1 & 0 \\ t^3 & 3t^2 & 3t & 0 \end{bmatrix}$$

Symmetric Matrices

Spectral theorem

Every symmetric matrix has the factorization $S = Q \Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q:

Symmetric diagonalization: $S = Q \Lambda Q^T = Q \Lambda Q^{-1}$

$S^T = S$ with $Q^{-1} = Q^T$

Ex:1 Find the λ's and x's when $S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

and $S - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

Ans: $\lambda^2 - 5\lambda = 0 \Rightarrow \lambda(\lambda - 5) = 0$

$\underline{\lambda_1 = 0 \text{ (or) } \lambda_2 = 5}$

$|S| = 4 - 4 = 0 \Longrightarrow S$ is singular

S^{-1} does not exists

$Sx_1 = 0 = 0x_1 \not\Longrightarrow x_1 = 0$

$\therefore \lambda_1 = 0$

$Sx_2 = 5x_2$

$x_1 \in N(S)$ and $x_2 \in C(S)$

$S^T = S \Longrightarrow C(S) = C(S^T)$

$N(S) \perp C(S^T) \Longrightarrow N(S) \perp C(S)$

$x_1 \perp x_2$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \perp \begin{bmatrix} +1 \\ 2 \end{bmatrix}$

$Q^{-1}SQ = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda$

~~Prisma~~=

* All eigenvalues of a symmetric matrix are real.

$$\boxed{S^T = S \Longrightarrow \lambda \in \mathbb{R} \ \& \ x\text{'s are real}}$$

Proof

$\pm 2(4)$ $Sx = \lambda x$

λ might be a complex # i.e., $\lambda = a+ib$.

the components of x may be complex #

S is real.

$$Sx = \lambda x \longrightarrow \overline{Sx} = \overline{\lambda x} \Longrightarrow S\bar{x} = \bar{\lambda}\bar{x}$$

$$\longrightarrow \bar{x}^T S^T = \bar{x}^T \bar{\lambda}$$

$$\Longrightarrow \bar{x}^T S x = \bar{x}^T \bar{\lambda} x \quad \text{——— (1)}$$

$$\bar{x}^T S x = \bar{x}^T \lambda x \quad \text{——— (2)}$$

$$\bar{x}^T x = |x_1|^2 + |x_2|^2 + \cdots \neq 0.$$

$$\left.\bar{x}^T x (\bar{\lambda} - \lambda) = 0. \right\} \ \bar{\lambda} = \lambda$$

Eigenvectors come from solving the real equation $(S - \lambda I)x = 0$. $\therefore$ x's are also real.

* Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are always $\perp$.

Proof $\rightarrow$ $Sx_1 = \lambda_1 x_1$ & $Sx_2 = \lambda_2 y$

Assume $\lambda_1 \neq \lambda_2$,

$S^T = S$: $(\lambda_1 x_1)^T y = (Sx)^T y = x^T S^T y = x^T S y = x^T \lambda_2 y$

$$\underline{\underline{x^T \lambda_1 y = x^T \lambda_2 y}}$$

$\lambda_1 \neq \lambda_2 \implies x^T y = 0$

$$\underline{\underline{x \perp y}}$$

*

$$S = Q \Lambda Q^T$$

$$= \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T = \sum \lambda_i q_i q_i^T$$

$$= \sum \lambda_i q_i \otimes q_i$$

$$S q_i = \left(\lambda_1 q_1 q_1^T + \cdots\cdots + \lambda_n q_n q_n^T \right) q_i = \lambda_i q_i$$

For real matrices, complex λ's and x's come in "conjugate pairs".

$$Ax = \lambda x \Rightarrow A\bar{x} = \bar{\lambda}\bar{x} \quad \left.\right\} \quad \begin{aligned} \lambda &= a+ib \\ \bar{\lambda} &= a-ib \end{aligned}$$

☐ Eigenvalues v/s Pivots

* For symmetric matrices the pivots and the eigenvalues have the same signs:

The # of +ve eigenvalues of $S = S^T$ equals the # of +ve pivots

ie., S has all $\lambda_i > 0$ iff all pivots are +ve.

□ All symmetric matrices are diagonalizable

When no eigenvalues of A are repeated, the eigenvectors are sure to be independent. Then A can be diagonalized. But a repeated eigenvalue can produce a shortage of eigenvectors. This sometimes happen for non-symmetric matrices. It never happens for symmetric matrices.

$\Rightarrow$ There are always enough eigenvectors to diagonalize $S = S^T$.

ie., all symmetric matrices are diagonalizable.

Proof

Schur's theorem: All square matrices are similar to an upper triangular matrix.

$$S = QTQ^{-1} \Longrightarrow S = QTQ^T$$

$$T = Q^TSQ \Rightarrow T^T = Q^TS^TQ = Q^TSQ = T$$

$\therefore$ T must be diagonal.

$$T = \Lambda$$

$$\Rightarrow S = Q\Lambda Q^{-1}$$

$\therefore$ The symmetric S has n orthonormal eigenvectors in Q.

* Schur's theorem

Every square matrix A factors into QTQ^{-1} where T is upper triangular and $\bar{Q}^T = Q^{-1}$.

If A has real eigenvalues then Q and T can be chosen real : $Q^T Q = I$.

$$\Longrightarrow A = QTQ^{-1}$$

(OR)

If A is a square real matrix with real eigenvalues, then there ~~exists~~ is an orthogonal matrix Q and an upper triangular matrix T such that $A = QTQ^T$.

(OR)

Every matrix is unitarily similar to an upper triangular matrix.

* Every square matrix can be "triangularized" by $A = QTQ^{-1}$.

Proof - Induction

Every 1 by 1 matrix is similar to an upper ~~triangu~~ triangular matrix.

Assume every $(n-1)$ by $(n-1)$ matrix is similar to an upper triangular matrix

Let,

A be an $n \times n$ matrix and let λ_1 be an eigenvalue, with eigenvector u_1.

Then put u_1 in the 1^{st} column of S, and pick the other columns of S to complete a basis for $\mathbb{C}^n$. (or)

Assume that $\|u_1\| = 1$ and use it to form an orthonormal basis $(u_1, u_2, \ldots, u_n)$. and put as the columns of S.

i.e., $U = [u_1 \; u_2 \; \cdots \; u_n]$

then $AS = SB$ (or) $A = SBS^{-1}$

ie.,

The matrix 'A' is equivalent to the matrix B of the linear map relative to the basis $(u_1, u_2, \ldots, u_n)$.

where, B has the form

$$B = \left[\begin{array}{c|c} \lambda_1 & W^T \\ \hline 0 & B_1 \end{array}\right]$$

Since B_1 is an $(n-1)$ by $(n-1)$ matrix,
$B_1 = P_1 T_1 P_1^{-1}$ where T_1 is an upper-triangular $(n-1)$ by $(n-1)$ matrix.

Then it can be verified that

$$B = \left[\begin{array}{c|c} \lambda_1 & W^T \\ \hline 0 & B_1 \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & W^T P_1 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_1^{-1} \end{bmatrix} = PTP^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & W^T \\ 0 & T_1 P_1^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & W^T \\ 0 & P_1 T_1 P_1^{-1} \end{bmatrix}$$

$$A = SBS^{-1} = S(PTP^{-1})S^{-1} = (SP)T(SP)^{-1}$$

$\longrightarrow$ A is similar to an upper triangular matrix

$\boxed{6.4(A)}$: What matrix 'A' has eigenvalues $\lambda = 1, -1$ and eigenvectors $x_1 = (\cos\theta, \sin\theta)$ and $x_2 = (-\sin\theta, \cos\theta)$? Which of these properties can be predicted in advance?

$A = A^T$, $A^2 = I$, $|A| = -1$

pivots are + and − | $A^{-1} = A$.

Ans: real eigenvalues & $\lambda = 1, -1$ and orthonormal x_1, x_2 (eigenvectors)

$\Rightarrow A = Q\Lambda Q^T$ must be symmetric.

The matrix A will be a reflection.

Vectors in the direction of x_1 are unchanged by A. (since $\lambda = 1$).

Vectors in the $\perp$ direction are reversed ($\lambda = -1$).

$c = \cos\theta$, $s = \sin\theta$

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

The reflection $A = Q\Lambda Q^T$ is across the "θ-line".

$\boxed{6.4(B)}$ Find the eigenvalues and eigenvectors (discrete sines and cosines) of A_3 and B_4.

$$A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Ans $-1, 2, -1$ pattern in the matrices is a 2^{nd} difference $\Longrightarrow$ like a 2^{nd} derivative

$\therefore$ $Ax = \lambda x$ and $Bx = \lambda x$ are like $\frac{d^2x}{dx^2} = \lambda x$.

? This has eigenvectors $x = \sin kt$ and $x = \cos kt$. that are the bases for Fourier series

A_n and B_n lead to "discrete sines" and "discrete cosines" that are the bases for the Discrete Fourier Transform (DFT)

$\boxed{A_3}$ $\lambda = 2-\sqrt{2},\ 2,\ 2+\sqrt{2}$

The eigenvector matrix gives the "Discrete Sine Transform" and the eigenvectors fall into the sine curves.

Sine matrix = Eigenvectors of A_3

$$= \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

Cosine matrix = Eigenvectors of B_4

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$\boxed{B_4}$ $\lambda_1 = 2-\sqrt{2}, 2, 2+\sqrt{2}, 0$

The eigenvector matrix gives the 4-point Discrete Cosine Transform.

Positive Definite Matrices

(LA 6)

Symmetric matrices that have +ve eigenvalues are called positive definite.

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

* The eigenvalues of S are +ve iff $a > 0$ & $ac - b^2 > 0$

* The eigenvalues of S are +ve iff the pivots are +ve:

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & c - \frac{b^2}{a} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & \frac{ac - b^2}{a} \end{bmatrix}$$

- Each pivot is a ratio of upper left determinants.

$$\boxed{+\text{ve eigenvalues} \Longleftrightarrow +\text{ve pivots}}$$

Ex:-

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$\frac{9}{6} - \frac{1}{6} = \frac{8}{6} = \frac{4}{3}$$

$\lambda = 1, 1, 4$

determinants $= 2, 3, 4$

pivots $= 2, \frac{3}{2}, \frac{4}{3}$

$\Bigg\}$ S is +ve definite

$\boxed{S - I}$ eigenvalues : $0, 0, 3$

$\Longrightarrow$ $S - I$ is semidefinite.

$\boxed{S - 2I}$ eigen values : $-1, -1, 2$

$\Longrightarrow$ $S - 2I$ is indefinite

Energy-based definition

$$Sx = \lambda x \implies x^T S x = \lambda x^T x$$

$$\lambda > 0 \Rightarrow \underline{x^T S x > 0} \text{ for any eigenvector } x$$

$\boxed{\begin{array}{l} * \; x^T S x > 0 \text{ for all non-zero vectors } x, \\ \text{not just the eigenvectors.} \\ x^T S x = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + by^2 > 0 \end{array}}$

In many applications, this number $x^T S x$ (or $\frac{1}{2}x^T S x$) is the energy in the system!

$\boxed{\Rightarrow \text{If } S \text{ and } T \text{ are symmetric positive definite, so is } \underline{S+T}.}$

$\boxed{A < B \Rightarrow B - A \text{ is positive definite}}$

GC(12)

- The pivots & eigenvalues are not easy to follow when matrices are added, but the energies just add.

* If S & T are symmetric +ve definite, so is ST

Proof

$$ST x = \lambda x \longrightarrow (Tx)^T S (Tx) = (Tx)^T \lambda x$$

$$\lambda = \frac{(Tx)^T S (Tx) > 0}{x^T T x > 0} > 0$$

* If C is +ve definite & A has independent columns, then $S = A^T C A$ is +ve definite.

Proof

$$x^T (A^T C A) x = (Ax)^T C (Ax) > 0 \quad \text{for } Ax \neq 0$$

$$x^T (A^T C A) x = 0 \quad \text{iff } Ax = 0 \text{, ie., } x = 0$$

$$x^T (A^T C A) x > 0 \quad \text{for all } x \neq 0.$$

If $\{v_1, v_2, \ldots, v_i, \ldots, v_n\}$ are linearly dependent vectors, then

$v_i = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \Longrightarrow v_i + c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$

for $c_j \neq 0$ for $j = 1, \ldots, n$.

* If the columns of 'A' are independent, then $S = A^TA$ is +ve definite.

$$x^TSx = x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2$$

i.e. $\|Ax\| \neq 0 \Leftarrow N(A) = \phi$

i.e. $Ax \neq 0$ when $x \neq 0$

$\Longrightarrow$ x^TSx is the +ve number $\|Ax\|^2$. and the matrix S is +ve definite.

---

* $N(A) = \{0\}$ iff A has independent columns

Proof

$$\underset{m \times n}{A} = [\vec{v}_1 \cdots \vec{v}_n] \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$Ax = \vec{v}_1 x_1 + \cdots + \vec{v}_n x_1 = 0$$

has a non-trivial solution iff the columns of A are linearly dependent.

If A has linearly independent columns, then
$Ax = 0 \Longrightarrow x = 0$

When a symmetric matrix S has one of these 5 properties, it has them all:

1. All n eigenvalues of S are +ve

2. All n pivots of S are +ve

3. All upper left determinants are +ve

4. $x^T S x > 0$ except at $x = 0$. This is the energy-based definition

5. $S = A^T A$ for a matrix A with independent columns. (there can be many such A)

Square root of a matrix

A matrix B is said to be a square root of A if the matrix products BB is equal to A.

* Let 'A' be a positive semi-definite matrix (real or complex). Then there is exactly one positive semi-definite matrix B such that $A = B^{+}B = BB$.

(Note: There can be many such square roots, but precisely one sq. root that is positive semi-definite)

* The principal square root of a positive definite matrix is positive definite.

ie., the rank of the principal square root of A is the same as the rank of A.

Existence of a Square Root

A is ~~Hermi~~ Positive definite $\Longrightarrow$ Hermitian

$\swarrow$

A is diagonalizable by a unitary matrix S

$$A = SDS^{\dagger}$$

where, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$

and $S^{-1} = S^{\dagger}$.

$$A = SDS^{\dagger} = S\sqrt{D}\sqrt{D}S^{\dagger} = S\sqrt{D}(S^{\dagger}S)\sqrt{D}S^{\dagger}$$

$$= \left(S\sqrt{D}S^{\dagger}\right)\left(S\sqrt{D}S^{\dagger}\right) = BB$$

where, $B = S\sqrt{D}S^{\dagger}$

$$\sqrt{D} = \mathrm{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}\right)$$

Ex:1 Test for +ve definiteness

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \& \quad T = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

$2 - \frac{2}{3}$

Ans:- $S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$

Pivots of S : $2, \frac{3}{2}, \frac{4}{3}$ all +ve.

Upper left determinants : 2, 3, 4

λ : $2-\sqrt{2}, 2, 2+\sqrt{2}$ all +ve

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A_1^T A_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The 3 columns of A_1 are independent

$\Longrightarrow$ S is +ve definite.

$S = LDL^T$

$$LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (L\sqrt{D})(L\sqrt{D})^T = A_2^T A_2$$

A_2 is the Cholesky factor of S.

Eigenvalues give the symmetric choice $A_3 = Q\sqrt{\Lambda}Q^T$.

$$A_3^T A_3 = Q\Lambda Q^T = S$$

$$x^T S x = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2x - y \\ -x + 2y - z \\ -y + 2z \end{bmatrix}$$

$$= 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz$$

$$|A_1 x|^2 = x^2 + (y - x)^2 + (z - y)^2 + z^2$$

$$|A_2 x|^2 = 2\left(x - \tfrac{1}{2}y\right)^2 + \tfrac{3}{2}\left(y - \tfrac{2}{3}z\right)^2 + \tfrac{4}{3}z^2$$

$$|A_3 x|^2 = \lambda_1 (q_1^T x)^2 + \lambda_2 (q_2^T x)^2 + \lambda_3 (q_3^T x)^2$$

$$T = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

$$|T| = -2b^2 + 2b + 4 = 2(-b^2 + b + 2) = -2(b^2 - b - 2)$$

$$= -2(b+1)(b-2) > 0$$

$\Delta = 1 + 8 = 9.$

$b = \dfrac{-1 \pm 3}{-2} = 2 \text{ (or) } -1$

$b+1>0$ & $b-2<0$	$b+1<0$ & $b-2>0$
$b>-1$ & $b<2$	$b<-1$ & $b>2$
$b \in (-1, 2)$ (or)	

$\Downarrow$

T is +ve definite.

~~Maai~~

Positive Semidefinite Matrices

When the determinant is zero, the smallest eigenvalue is zero. The energy in its eigenvector is $x^T S x = x^T 0\, x = 0$. These matrices on the edge of +ve definiteness are called positive semidefinite.

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \text{ are +ve semidefinite}$$

- This matrix S factors into $A^T A$ with dependent columns in A:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = A^T A$$

If 4 is increased by any small #, the matrix S will become +ve definite.

* Positive semidefinite matrices have all $\lambda \geq 0$ and all $x^T S x \geq 0$.

Those weak inequalities include positive definite S and also the singular matrices at the edge.

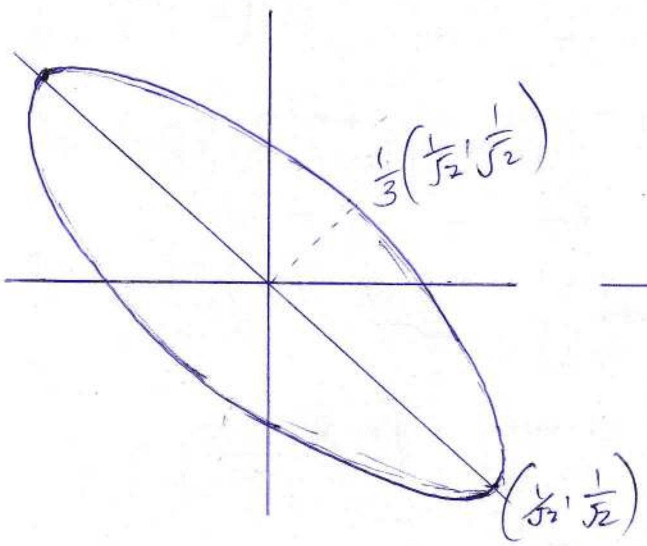
□ The <u>Ellipse $ax^2+2bxy+cy^2=1$</u>

$$\boldsymbol{x}^T S \boldsymbol{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2+2bxy+\cancel{b}y^2$$

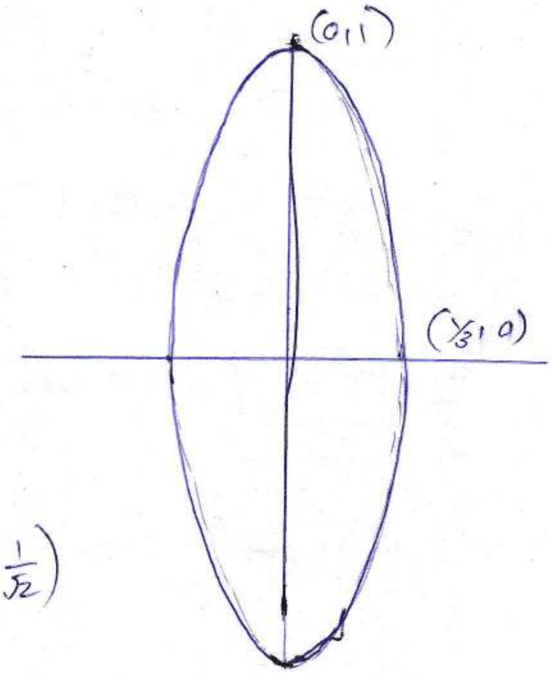
Think of a tilted ellipse $x^TSx=1$. Its center is $(0,0)$. Turn it to line up with the coordinate axes (X and Y axes).

$$x^TSx = x^TQ\Lambda Q^Tx = \begin{bmatrix} x & y \end{bmatrix} Q \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} Q^T \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X^T\Lambda X = \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2 = 1$$



$x^TSx=1$

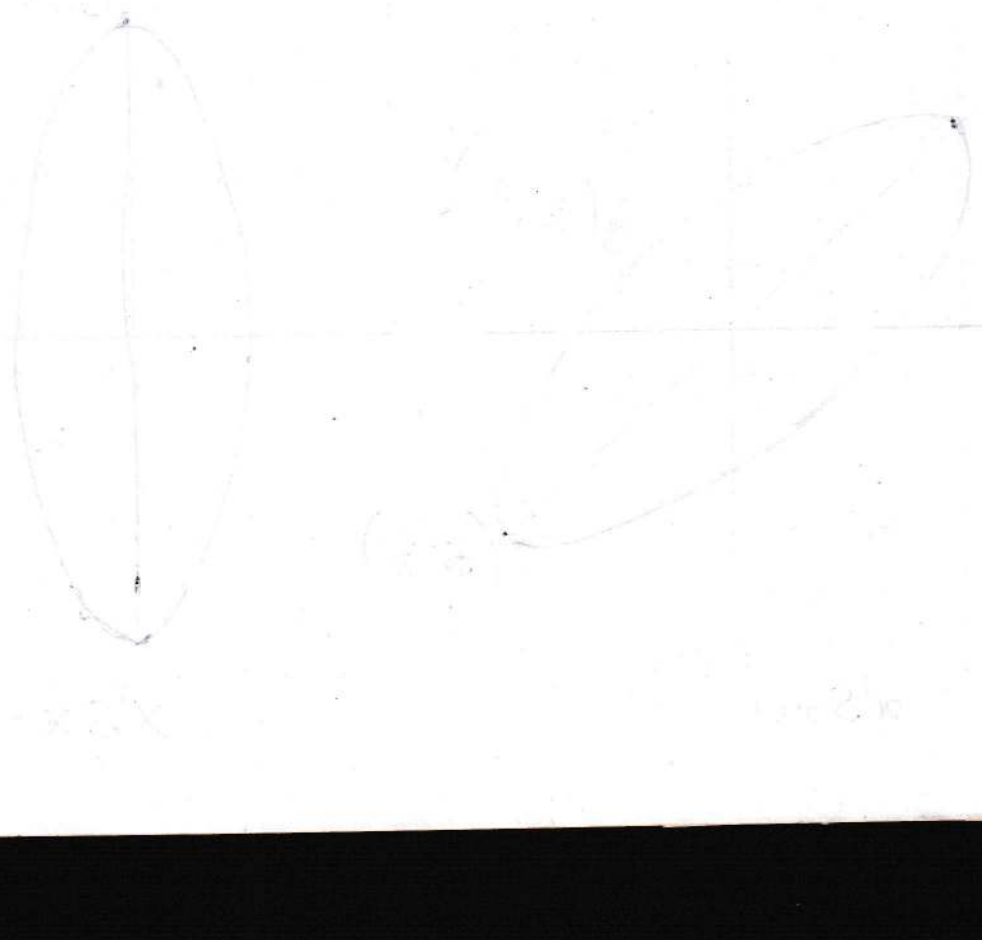


$X^TSX=1$

1. The tilted ellipse is associated with S. Its equation is $x^T S x = 1$

2. The lined-up ellipse is associated with Λ. Its equation is $X^T \Lambda X = 1$

3. The rotation matrix that lined up the ellipse is the eigenvector matrix Q.



Ex: 2. Find the axes of this tilted ellipse
$5x^2+8xy+5y^2=1$.

Ans:

$$[x\ \ y]\begin{bmatrix}5&4\\4&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=1 \Longrightarrow S=\begin{bmatrix}5&4\\4&5\end{bmatrix}$$

$$(5-\lambda)^2-16=0 \Longrightarrow (1-\lambda)(9-\lambda)=0$$

$$(1-\lambda)(9-\lambda)=0 \Longrightarrow \lambda=1,\ 9.$$

eigenvectors : $\begin{bmatrix}1\\-1\end{bmatrix}$, $\begin{bmatrix}1\\+1\end{bmatrix}$

$$S=\begin{bmatrix}5&4\\4&5\end{bmatrix}=\frac{1}{\sqrt2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}9&0\\0&1\end{bmatrix}\frac{1}{\sqrt2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}=Q\Lambda Q^T$$

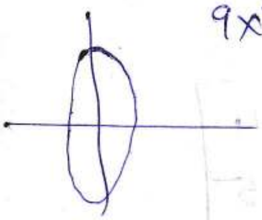
$$[x\ \ y]\,Q\Lambda Q^T\begin{bmatrix}x\\y\end{bmatrix}=[X\ \ Y]\begin{bmatrix}\lambda_1&\\&\lambda_2\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}=[X\ \ Y]\begin{bmatrix}9&0\\0&1\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}$$

$$[x\ \ y]\,S\begin{bmatrix}x\\y\end{bmatrix}=[x\ \ y]\,Q\Lambda Q^T\begin{bmatrix}x\\y\end{bmatrix}=[x\ \ y]\frac{1}{\sqrt2}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}9&0\\0&1\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\frac{1}{\sqrt2}\begin{bmatrix}x\\y\end{bmatrix}$$

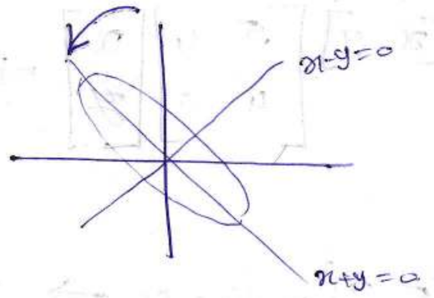
$$=\begin{bmatrix}\frac{x+y}{\sqrt2}&\frac{x-y}{\sqrt2}\end{bmatrix}\begin{bmatrix}9&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{x+y}{\sqrt2}\\\frac{x-y}{\sqrt2}\end{bmatrix}=9\left(\frac{x+y}{\sqrt2}\right)^2+1\left(\frac{x-y}{\sqrt2}\right)^2=1$$

$4AC = 1\cancel{20} \geq 64 = B^2 \longrightarrow$ ellipse

~~ABC~~



$9x^2 + 1\cdot y^2 = 1$: x - distance from $y=0$
y - distance from $x=0$



$$9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2 = 1$$

$\frac{x+y}{\sqrt{2}}$: distance from $\frac{x+y}{\cancel{\sqrt{2}}} = 0$ [rotated y-axis CCW 45°]

$\frac{x-y}{\sqrt{2}}$: distance from $x-y=0$

$$x^T S x = 5x^2 + 8xy + 5y^2 = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2 = 9X^2 + Y^2 = 1$$

The coefficients are the eigenvalues 9 and 1 from Λ. Inside the squares are the eigenvectors, $q_1 = \frac{(1,1)}{\sqrt{2}}$ and $q_2 = \frac{(1,-1)}{\sqrt{2}}$

The axes of the tilted ellipse point along those eigenvectors.

$\Longrightarrow$ $S = Q\Lambda Q^T$ is called principal axis theorem.

It displays the axes. Not only the axis directions (from the eigenvectors) but also the axis lengths (from eigenvalues).

The bigger eigenvalue λ_1 gives the shorter axis, of half length $= \frac{1}{\sqrt{\lambda_1}} = \frac{1}{3}$

The smaller eigenvalue $\lambda_2 = 1$ gives the greater length $\frac{1}{\sqrt{\lambda_2}} = 1$

⇒ In the xy-system, the axes are along the eigenvectors of S.

In the XY system, the axes are along the eigenvectors of Λ - the coordinate axes.

All comes from $S = Q\Lambda Q^T$.

$S = Q\Lambda Q^T$ is positive definite when all $\lambda_i > 0$.

The graph of $x^T S x = 1$ is an ellipse.

Ellipse: $\begin{bmatrix} x & y \end{bmatrix} Q\Lambda Q^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X & Y \end{bmatrix} \Lambda \begin{bmatrix} X \\ Y \end{bmatrix} = \lambda_1 X^2 + \lambda_2 Y^2 = 1$

The axes point along eigenvectors of S.

The half lengths are $\frac{1}{\sqrt{\lambda_1}}$ and $\frac{1}{\sqrt{\lambda_2}}$.

$S = I$ gives the circle $x^2+y^2=1$

If one eigenvalue is -ve, the ellipse changes to a hyperbola. The sum of squares becomes a difference of squares: $9X^2 - Y^2 = 1$.

If $S = -I$, both λ's are -ve.

$-x^2-y^2=1$ has no points at all.

If S is $n \times n$,

$x^T S x = 1$ is an ellipsoid in $\mathbb{R}^n$.

Its axes are the eigenvectors of S.

$$x^T S x = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

$$= x^T L D L^T x = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x + \frac{b}{a} y & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} x + \frac{b}{a} y \\ y \end{bmatrix}$$

$$X^T S X = \begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= a\left(x + \frac{b}{a} y\right)^2 + \frac{ac-b^2}{a} y^2 = d_1 X^2 + d_2 Y^2$$

Test for a minimum

Does $F(x,y)$ have a minimum if $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$ at the point $(x,y) = (0,0)$?

For $f(x)$, the test for minimum comes from calculus : $\frac{df}{dx} = 0$ and $\frac{d^2 f}{dx^2} > 0$.

Two variables in $F(x,y)$ produce a symmetric matrix S. It contains 4 2nd derivatives.

+ve $\frac{d^2 f}{dx^2}$ $\xrightarrow{\text{changes to}}$ +ve definite S

$$S = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix}$$

$F \approx \frac{1}{2} x^T S x > 0$

$F(x,y)$ has a minimum if $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$ and

S is +ve definite

The min. of a Function $F(x,y,z)$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0 \quad \text{at the min. point}$$

$$S = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix} \quad \text{is +ve definite}$$

6.5(B) When is the symmetric block matrix $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ positive definite?

Ans: $\left[\begin{array}{c|c} I & 0 \\ \hline -B^T A^{-1} & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right] = \left[\begin{array}{c|c} I & B \\ \hline -B^T A^{-1} & C - B^T A^{-1} B \end{array}\right] = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$

These two blocks A & S must be +ve definite.

6.5(C) Find the eigenvalues of the −1, 2, −1 tridiagonal $n \times n$ matrix S.

Ans:

The 2nd difference matrix S is like a second derivative.

$$\boxed{\frac{d^2 y}{dx^2} = \lambda y(x)} \quad \text{with } y(0) = 0, \; y(1) = 0$$

Eigenvalues : $\lambda_1, \lambda_2, \ldots$

Eigenvectors : $y_1, y_2, \ldots$

Guess

Try $y = \sin cx$,

$y'' = -c^2 \sin cx = \lambda \sin cx$

$\Longrightarrow \lambda = -c^2$ [Eigenvalue]

provided $y = \sin cx$ satisfies the end point conditions $y(0) = 0 = y(1)$.

$\sin 0 = 0$

At $x = 1$, we need $y(1) = \sin c = 0$.

$\Longrightarrow c = k\pi \quad , k = 0, 1, \dots$

$\therefore \quad \lambda = -k^2\pi^2$

$$\boxed{\begin{array}{ll} \text{Eigenvalues: } \lambda = -k^2\pi^2 & \multirow{2}{*}{} \\ \text{Eigenfunctions: } y = \sin k\pi x & \end{array} \Bigg\} \; \frac{d^2}{dx^2} \sin k\pi x = -k^2\pi^2 \sin k\pi x}$$

$\underline{S}$

Guess it's eigenvectors.

The eigenvectors of S come from $\sin k\pi x$ at n points $x = h, 2h, \ldots, nh$, equally spaced b/w 0 and 1.

The spacing $\Delta x = \cancel{h}$ is $h = \frac{1}{n+1}$

$\therefore$ $(n+1)^{st}$ point has $(n+1)h = 1$

$\boxed{6.1}$

1. ~~The~~ ~~example~~

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \ A^2 = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}, \ldots, A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

Find the eigenvalues of these.

~~All~~ powers have the same eigenvectors.

(a) Show from A how a row exchange can produce different eigenvalues?

(b) Why is a zero eigenvalue not changed by the steps of elimination?

Sol: A : $\lambda = 1$ & 0.5

A^2 : $\lambda = 1$ & 0.25

A^∞ : $\lambda = 1$ & 0

(a) $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix} = B$

$tr(A) = \cancel{\quad} 1.5$ $\qquad tr(B) = 0.\cancel{55}$

$\lambda_A = 1, \ 0.5$ $\qquad \lambda_B = 1$ and -0.5

(b). $A\vec{x} = 0\vec{x}$ & $B\vec{x} = 0\vec{x}$

$|A| \neq 0 \Rightarrow |B| \neq 0$.

Singular matrices remain singular during elimination & $N(A) = N(B)$

$\Rightarrow \lambda = 0$ does not change

3. Eigenvalues & eigenvectors of A & A^{-1}

Ans: $Ax = \lambda x \implies A^{-1}Ax = Ix = \lambda A^{-1}x$

$$A^{-1}x = \frac{1}{\lambda}x$$

5. Find the eigenvalues of A & B and $A+B$

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \text{ and } A+B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Ans: $\left.\begin{array}{l} \lambda_A = 3, 1 \\ \lambda_B = 1, 3 \end{array}\right\}$ $(4-\lambda)^2 - 1 = (3-\lambda)(5-\lambda) = 0$

$\implies \lambda_{A+B} = 3, 5$

$\implies$ Eigenvalues of $A+B$ are not equal to eigenvalues of A + eigenvalues of B.

6. Find the eigenvalues of A & B and AB & BA

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

@ Ans:

10. m w A

Ans A^{∞} A'

Ans: (a) A & B : $\lambda_1, \lambda_2 = 1$

AB & BA : $(1-\lambda)(3-\lambda) - 2 = 0$

$$\lambda^2 - 4\lambda + 1 = 0$$

$$\underline{\lambda = 2 \pm \sqrt{3}}$$

Eigenvalues of $AB \neq$ eigenvalues of A times eigenvalues of B.

Eigenvalues of AB & BA are equal.

10. Find eigenvalues & eigenvectors for these Markov matrices A & A^{∞}. Explain from those ans. why A ~~are~~ is close to A^{∞}:

$$A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}, \quad A^{\infty} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

Ans A : $\lambda_1 = 1$, $\lambda_2 = 0.4 \implies x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

A^{∞} : $\lambda_1 = 1$, $\lambda_2 = 0 \implies$ " "

A^{100} : $A^{100}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $A^{100}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (0.4)^{100}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(0.4)^{100} \sim 0$.

(11.) For 2×2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$. The columns of $(A - \lambda_1 I)$ are multiples of the eigenvector x_2. Why?

Ans: $f(\lambda) = 0 \Rightarrow (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$

Cayley-Hamilton theorem:

OM(23)

$$(A - \lambda_1 I)(A - \lambda_2 I) = 0$$

$$(A - \lambda_2 I)(A - \lambda_1 I) = (A - \lambda_2 I)(\vec{a}_1 \ \ \vec{a}_2) = 0$$

$$(A - \lambda_2 I)\vec{a}_1 = 0 \quad \& \quad (A - \lambda_2 I)\vec{a}_2 = 0$$

$$\vec{a}_1, \vec{a}_2 \in N(A - \lambda_2 I)$$

$\therefore$ $\vec{a}_1$ & $\vec{a}_2$, the columns of $(A - \lambda_1 I)$ are eigenvectors.

19. A 3×3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find 3 of these (give ans. where possible)

(a) rank (B)

Ans: $\lambda_1 = 0 \implies |B| = 0$

$Bx = 0x$ $\quad \left.\begin{array}{l} \dim[N(B)] = 1 \\ \dim[C(B^T)] = 3-1 = 2 \end{array}\right\}$

$\underline{\text{rank}(B) = 2}$

(b). $\det(B^T B)$

Ans: ~~det~~ $= 2\det(B) = 2.0 = 0$

(c). Eigenvalues of $B^T B$

Ans: Nope

(d) Eigenvalues of $(B^2+I)^{-1}$

Ans: $\dfrac{1}{\lambda^2+1}$: $1, \frac{1}{2}, \frac{1}{5}$

20. Choose the last rows of ~~A~~ & C to give eigenvalues F 1, 2, 3 :

Companion matrices :
$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}$$

Ans: $tr(C) = \sum_{i=1}^{3} a_{ii} = \sum_{i=1}^{3} \lambda_i \Rightarrow \underline{a_{33} = 6}$

$\det(C) = -[0 - *] = * = \prod \lambda_i = \underline{6} = a_{31}$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & * & 6 \end{bmatrix}$$

$$\det\left[C - \lambda I\right] = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & * & 6-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 6\lambda - a_{32}) - [-6] = 0$$

$$\lambda(\lambda^2 - 6\lambda - a_{32}) = 6$$

$$1 - 6 - a_{32} = 6 \Longrightarrow \underline{a_{32} = -11}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

21. The eigenvalues of A equals the eigenvalues of A^T.

Eigenvectors of A and A^T are not the same.

Ans: $P_{A^T}(\lambda) = \det(A^T - \lambda I) = \det(A^T - \lambda I^T)$

$= \det((A - \lambda I)^T) = \det(A - \lambda I) = P_A(\lambda)$

$\therefore$ Eigenvalues of A & A^T are the same.

$A = X D_A X^{-1}$

$A^T = (X^T)^{-1} D_A X^T$

$(X^T)^{-1} = X$ iff $X^{-1} = X^T \Rightarrow XX^T = I$

X is orthogonal.

22. Construct any 3×3 Markov matrix M.

Show that $M^T(1,1,1) = (1,1,1)$

$\lambda = 1$ is an eigenvalue of M

Q) 3×3 singular Markov matrix with trace $\frac{1}{2}$ has what λ's ?

Ans:-

23.

Ans:

Ca

Ex

23. Find 3 2x2 matrices that have $\lambda_1 = \lambda_2 = 0$.

- The trace is zero and the determinant is zero. 'A' might not be the zero matrix but check $A^2 = 0$.

ace $\frac{1}{2}$

Ans: ~~f(λ)~~ $\lambda^2 - \underbrace{tr(A)}_{=0}\lambda + \underbrace{\det(A)}_{=0} = 0$

Cayley-Hamilton theorem

$A^2 - \underbrace{tr(A)\lambda + \det(A)}_{=0} = 0$

$\therefore A^2 = 0$

Ex:-

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

24. This matrix is singular with rank 1.
Find 3 λ's & 3 eigenvectors

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$

Ans: Singular $\Longrightarrow$ $Ax = 0 = 0x$

$$\underline{\underline{\lambda = 0}}$$

$x_1, x_2 \in N(A)$

$\dim(N(A)) = 2$ $\qquad$ $x_3 \in C(A) = C(A^T)$

$$\therefore \lambda : \boxed{0, 0, 6}$$

$A = uv^T$

$Ax = uv^T x = u(v^T x) = 0$

$v \perp x_1, x_2$ & $x_3 \in C(A)$

$$\Downarrow$$

$$x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\cancel{x_1' = t\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}, \quad \cancel{x_2' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$$x_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \; \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \Bigg[$$

$2x + y + 2z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\qquad x_1 \qquad\qquad x_2$$

---

(25). Suppose A & B have the same eigenvalues $\lambda_1, \dots, \lambda_n$ with the same independent eigenvectors $x_1, \dots, x_n$. Then $A = B$.

Proof

$V = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

$AV = c_1 \lambda_1 x_1 + \dots\dots + c_n \lambda_n x_n$

$BV = c_1 \lambda_1 x_1 + \dots\dots + c_n \lambda_n x_n$

for all vectors V.

$\Longrightarrow \underline{A = B}$

26. The block B has eigenvalues 1, 2 and C has eigenvalues 3, 4 and D has eigenvalues 5, 7. Find the eigenvalues of the 4x4 matrix A:

$$A = \left[\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right] = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Ans:

$$\det(A - \lambda I) = \det \begin{bmatrix} B - \lambda I_2 & C \\ 0 & D - \lambda I_2 \end{bmatrix}$$

$$= \det(B - \lambda I) \cdot \det(D - \lambda I)$$

$\lambda = 1, 2$ from B and

$\lambda = 5, 7$ from D.

27

(a)

Ans:

(b)

Ans:

27. Find the rank & the 4 eigenvalues of A & C:

(a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Ans: $r_A = 1 \implies \dim(N(A)) = 4 - 1 = 3.$

$\therefore \lambda = 0, 0, 0, 4$ $\quad [tr(A) = 4 = \sum \lambda_i]$

(b) $C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Ans: $r_C = 2$, $N(C) = 2 \implies \lambda = 0, 0.$

$C\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 2, 2$ $\quad [tr(C) = 4]$

31. If we exchange rows 1 and ~~row~~ 2, ~~and~~ columns 1 and 2, the eigenvalues won't change. Find the eigenvectors of A and B for $\lambda = 11$. Rank $= 1$ gives $\lambda_2 = \lambda_3 = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix}, \quad B = PAP^T = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}$$

Ans: $B = PAP^T = PAP^{-1} = PAP$

A & B are similar & have equal eigenvalues.

$$P_B(\lambda) = \det(B - \lambda I) = \det(PAP - \lambda I)$$
$$= \det(PAP - \lambda PIP) = \det(P(A - \lambda I)P)$$
$$= \det(P) \cdot \det(A - \lambda I) \cdot \det(P) = -1 \cdot \det(A - \lambda I) \cdot -1$$
$$= \det(A - \lambda I) = P_A(\lambda)$$

rank $= 1$, $|A| = 0$

$\lambda_2 = \lambda_3 = 0, \quad \lambda_1 = 11$

Eigenvector with $\lambda_1 = 11$ $\in C(A) = C(A^T)$

$\therefore x_1 = (1,3,4)$

for both A & B.

32. A has eigenvalues $0, 3, 5$ with independent eigenvectors u, v, w.

(b) Find a particular solution to $Ax = v + w$. Find all solutions

Ans: $A\left(\frac{v}{3} + \frac{w}{5}\right) = v + w$

$x = \frac{v}{3} + \frac{w}{5}$ is a particular solution to

$Ax = v + w$.

general solution, $x = cu + \left(\frac{v}{3} + \frac{w}{5}\right)$

34. Find the eigenvalues of this permutation matrix P from $\det(P-\lambda I)=0$. Which vectors are not changed by the permutation?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ans: $\det(P-\lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = \lambda$

$$= -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -\lambda(-\lambda^3) - 1 = 0 \implies \lambda^4 = +1$$

$P^4 = I$

$$\lambda = 1, i, -1, -i$$

$x_1 = (1,1,1,1)$, $x_2 = (1, i, i^2, i^3)$,

$x_3 = (1,-1,1,-1)$, $x_4 = (1, -i, (-i)^2, (-i)^3)$

36. Heisenberg's Uncertainty Principle

$AB - BA = I$ can happen for infinite matrices with $A = A^T$ and $B = -B^T$. Then

$$x^T x = x^T AB x - x^T BA x \le 2|Ax||Bx|$$

Explain the last step by using the Schwarz inequality $|u^T v| \le |u||v|$. Then Heisenberg's inequality says that $\frac{|Ax|}{|x|}$ times $\frac{|Bx|}{|x|}$ is atleast $\frac{1}{2}$. It is impossible to get the position error & momentum error both very small.

Ans : For $n \times n$ matrices

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0 \ne n = \operatorname{tr}(I)$$

$\therefore$ $AB - BA = I$ can happen only for infinite matrices.

If $A^T = A$ and $B^T = -B$ then

$$x^T x = x^T(AB - BA)x = x^T(A^T B + B^T A)x$$

$$= (Ax)^T(Bx) + (Bx)^T(Ax) \le |Ax||Bx| + |Bx||Ax|$$

$$\le 2|Ax||Bx|$$

$$\therefore \ |Ax||Bx| \ge \frac{1}{2}|x|^2 \ \ \& \ \left(\frac{|Ax|}{|x|}\right)\left(\frac{|Bx|}{|x|}\right) \ge \frac{1}{2}$$

$\Rightarrow$ It is impossible to get the position error and momentum error both very small.

38. $A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$. Markov matrix

Then A^n will approach what matrix A^∞ ?

Ans: $\lambda_1 = \lambda_2 = 1$

$\lambda = 1, \quad x_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = 1.2 - 1 = 0.2 \quad , x_2$

If $A^T = A$ and $B^T = -B$ then

$$x^T x = x^T(AB - BA)x = x^T(A^T B + B^T A)x$$

$$= (Ax)^T(Bx) + (Bx)^T(Ax) \le |Ax||Bx| + |Bx||Ax|$$

$$\le 2|Ax||Bx|$$

$$\therefore \; |Ax||Bx| \ge \frac{1}{2}|x|^2 \;\; \& \;\; \left(\frac{|Ax|}{|x|}\right)\left(\frac{|Bx|}{|x|}\right) \ge \frac{1}{2}$$

$\Rightarrow$ It is impossible to get the position error and momentum error both very small.

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