Introduction to Linear Algebra
- Gilbert Strang

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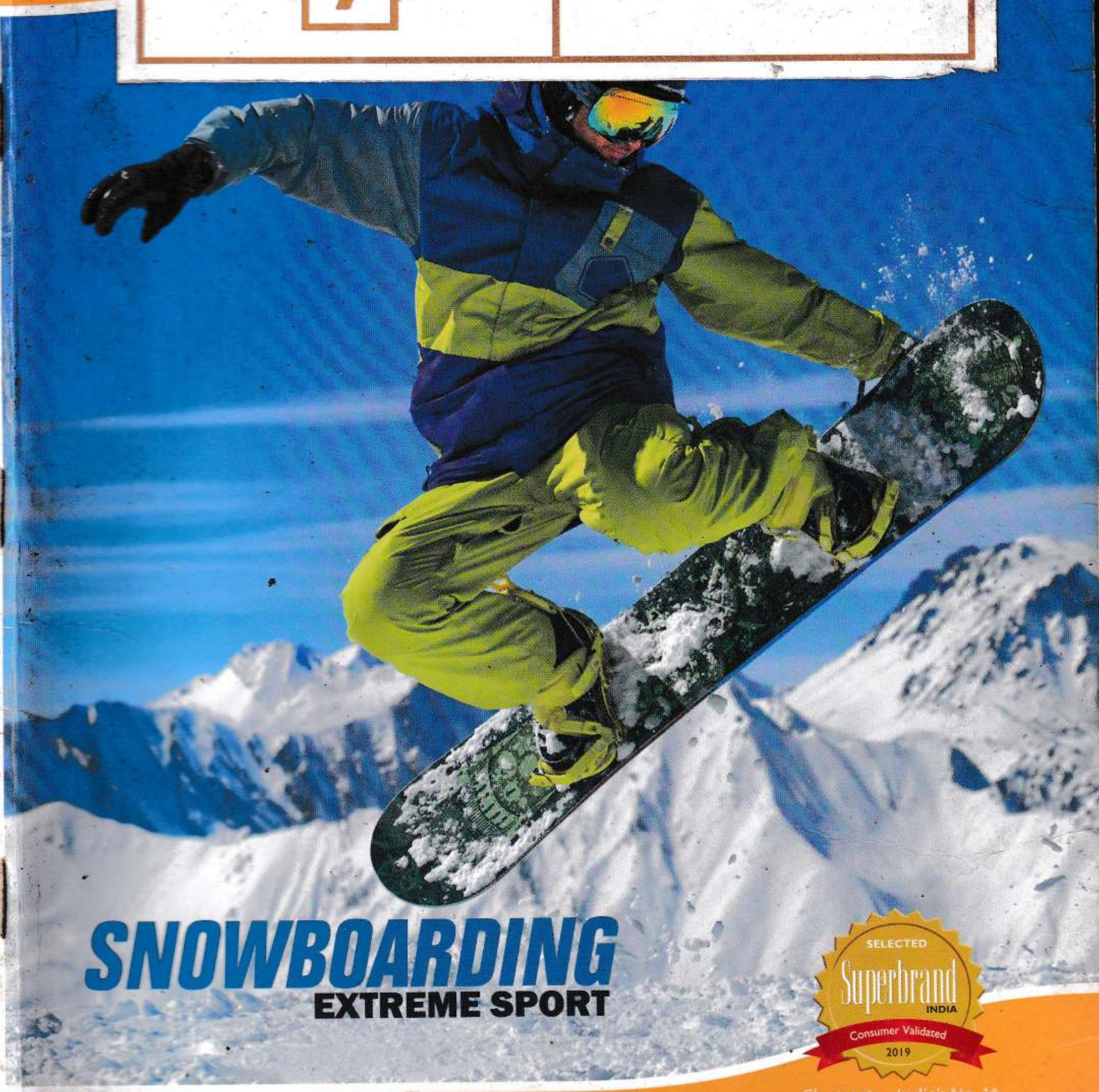
Orthogonality
↓
Eigenvalues & Eigenvectors



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		INTRODUCTION TO LINEAR ALGEBRA - Gilbert Strang, MIT (5^{th} Edition)		

24. Find a basis for the subspace S in $\mathbb{R}^4$

(a) spanned by all solutions of

$x_1 + x_2 + x_3 - x_4 = 0$

Ans: $\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

$$S = N(A) : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Find a basis for the orthogonal complement $S^\perp$

Ans: $C(A^T) \perp N(A)$

Basis for $S^\perp$ is $(1, 1, 1, -1)$

(c) Find b_1 in S and b_2 in $S^\perp$ so that $b_1 + b_2 = b = (1, 1, 1, 1)$

Ans: $$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_r + x_n = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & & 2 & -1 & 0 \\ 0 & 0 & & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$b_1 = x_n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} + 0 + 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} =$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{3} & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{2} \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array}\right]$$

$$b_1 = x_n = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$$

$$b_2 = x_r = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

25. If $ad-bc>0$, the entries in $A=QR$ are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} a & -c \\ c & a \end{bmatrix}}{\sqrt{a^2+c^2}} \frac{\begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & ad-bc \end{bmatrix}}{\sqrt{a^2+c^2}}$$

Write $A=QR$ when $a,b,c,d = 2,1,1,1$ and also $1,1,1,1$. Which entry of R becomes zero when the columns are dependent & Gram-Schmidt breaks down?

Ans: Non-singular example.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$$

Singular example:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

The Gram-Schmidt process breaks down when $ad-bc=0$.

26.

30. The 1st 4 wavelets are in the columns in this wavelet matrix W :

$$W = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

What is special about the columns ?
Find the inverse wavelet transform W^{-1} ?

Ans : The columns of the wavelet matrix W are orthonormal.

$$\therefore \quad W^{-1} = W^{T}$$

32. If u is a unit vector, then $Q = I - 2uu^T$ is a reflection matrix. (Householder reflection matrix). Find Q_1 from $u = (0,1)$ and Q_2 from $u = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. Draw the reflections when Q_1 and Q_2 multiply the vectors $(1,2)$ and $(1,1,1)$

Ans:
$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ reflects across x-axis

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \times \frac{1}{2}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$ reflects across $y+z=0$ plane

33. Find all matrices that are both orthogonal & lower triangular

Ans:
$$Q = \begin{bmatrix} q_{11} & 0 & 0 & \cdots & 0 \\ q_{21} & q_{22} & 0 & \cdots & 0 \\ q_{31} & q_{32} & q_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ q_{m1} & q_{m2} & q_{m3} & \cdots & q_{mn} \end{bmatrix}$$

$$Q^T Q = I$$

$$\begin{bmatrix} q_{11} & q_{21} & q_{31} \\ 0 & q_{22} & q_{32} \\ 0 & 0 & q_{33} \end{bmatrix} \begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} q_{11}^2 + q_{21}^2 + q_{31}^2 & \overset{0}{q_{21}} q_{22} + q_{31} q_{32} & \overset{0}{q_{31}} q_{33} \\ q_{22} \overset{0}{q_{21}} + q_{32} q_{31} & q_{22}^2 + q_{32}^2 & \overset{0}{q_{32}} q_{33} \\ q_{33} \underset{=0}{q_{31}} & q_{33} \underset{=0}{q_{32}} & q_{33}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$q_{11}^2 = q_{22}^2 = q_{33}^2 = 1$$

$\Longrightarrow$ <u>±1 on the main diagonal</u> & zeros elsewhere.

EIGEN VALUES & EIGEN VECTORS

LINEAR ALGEBRA

Part 1 : $Ax = b$

Balance, equilibrium, steady state.

Part 2 : is about change — time enters the picture

cont. time in a differential equation $\frac{du}{dt} = Au$ (or)

time steps in a difference equation $u_{k+1} = Au_k$.

Those equations are not solved by elimination.

Suppose, the solution vector $u(t)$ stays in the direction of a fixed vector x. Then we need to only find the # (changing with time) that multiplies x. (A) # is easier than a vector.

⟹ We want "eigenvectors" x that don't change direction when you multiply by A.

Suppose,
you need the 100th power A^{100}.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix}$$

A^{100}'s columns are very close to the eigenvector $(0.6, 0.4)$.

$$A^{100} \approx \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

A^{100} was found by using the eigenvalues of A, not by multiplying 100 matrices.

$\lambda = 1$ & $1/2$

$$A \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = 1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \Longrightarrow A^{100} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

* Almost all vectors change direction when they are multiplied by A. Certain exceptional vectors x are in the same direction as Ax. Those are the eigenvectors.

The basic equation is $Ax = \lambda x$. The # λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged, when it is multiplied by A. The eigenvalue λ could be zero. Then $Ax = 0x$ means that this eigenvector $x \in N(A)$.

If $A = I$, every vector has $Ax = Ix = x$.

All vectors are eigenvectors of I. All eigen values, $\lambda = 1$.

$\lambda = 0$ $\Longrightarrow$ $\|A\| = 0$, A is singular	
$A\vec{x} = 0\vec{x}$	$x \in N(A)$

Ex:1 $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

Ans. $\det \begin{bmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)(\lambda-\frac{1}{2})$

For $\lambda = 1, \lambda = \frac{1}{2}$,

$\det(A - \lambda I) = 0$

$\Rightarrow$ The eigenvectors $x_1, x_2 \in N(A-I)$ & $N(A - \frac{1}{2}I)$

$$A x_1 = A \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = x_1$$

$$A x_2 = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2} x_2$$

* All other vectors are combinations of the 2 eigenvectors

$$a_1 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = x_1 + 0.2\, x_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}$$

$$A a_1 = A \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = x_1 + \frac{1}{2}(0.2)\, x_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$A^{99}\begin{bmatrix}0.8\\0.2\end{bmatrix} = x_1 + \left(\frac{1}{2}\right)^{99} 0.2\, x_2 = \begin{bmatrix}0.6\\0.4\end{bmatrix} + \begin{bmatrix}\text{very small}\\\text{vector}\end{bmatrix}$$

The eigenvector x_1 is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector x_2 is a "decaying mode" that virtually disappears (because $\lambda_2 = \frac{1}{2}$). The higher the power of A, the more closely its columns approach the steady state.

$$A = \begin{bmatrix}0.8 & 0.3\\0.2 & 0.7\end{bmatrix}$$ is a Markov matrix.

Ex: 2 The projection matrix, $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

has eigenvalue $\lambda = 1$ and $\lambda = 0$.

Ans: Eigenvectors, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Px_1 = x_1$ (steady state)

$Px_2 = 0$ (nullspace)

1. Markov matrix : each column of P adds to 1.
So, $\lambda = 1$ is an eigenvalue

2. P is singular : $\lambda = 0$ is an eigenvalue

3. P is symmetric : eigenvectors $(1,1)$ and $(1,-1)$ are $\perp$.

~~The body eigenvalues~~

P : projection matrix	$\lambda_1 = 0$	, $Px_1 = 0x_1$ fills up the nullspace
	$\lambda_2 = 1$	, $Px_2 = x_2$ fills up the column space

Nullspace is projected to 0. The column space projects onto itself. The projection keeps the column space and destroys the nullspace.

$$V = \begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix} \Rightarrow PV = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}$$

Permutations have all $|\lambda| = 1$.

Ex: 3. The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.

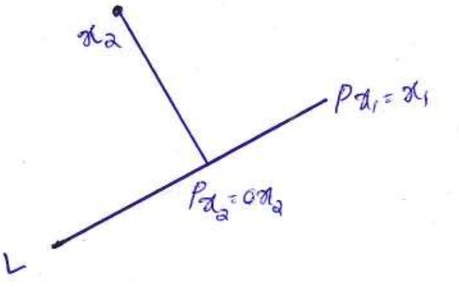
R is a reflection & at the same time a permutation.

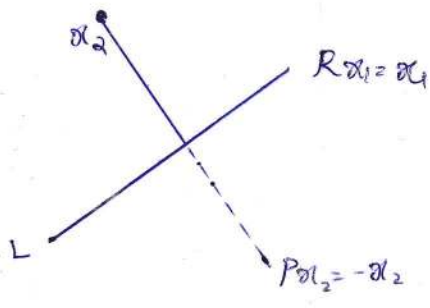
$\lambda_1 = 1$, $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The eigenvectors for R are the same as for P, because, reflection = 2(projection) − I :

$R = 2P - I \quad : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$





Projection onto L
have eigenvalues 1 and 0

Reflection across line
R have eigenvalues 1 and -1.

- When a matrix is shifted by I, each λ is shifted by 1. No changes in eigenvectors.

Eqⁿ for the Eigenvalues

$$Ax = \lambda x \implies (A - \lambda I)x = 0$$

$\implies$ The eigenvectors make up the $N(A - \lambda I)$.

If $(A - \lambda I)x = 0$ has a non-zero solution, $(A - \lambda I)$ is not invertible. $\det(A - \lambda I) = 0$.

* The # λ is an eigenvalue of A iff $A - \lambda I$ is singular. ie., $\det(A - \lambda I) = 0$

$\det(A-\lambda I)$ is a polynomial function of the variable λ of the degree n.

$\therefore$ Its term of degree 'n' is always $(-1)^n \lambda^n$.

$$\det(A-\lambda I) = (-1)^n (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots\cdots(\lambda-\lambda_n)$$

$$= (-1)^n \Big[\lambda^n - (\lambda_1+\lambda_2+\cdots+\lambda_n)\lambda^{n-1} + (\lambda_1\lambda_2+\lambda_1\lambda_3+\cdots+\lambda_{n-1}\lambda_n)\lambda^{n-2} - \cdots\cdots (-1)^n \lambda_1\lambda_2\cdots\lambda_n\Big]$$

OM(23)

$$= (-1)^n \Big[\lambda^n - \lambda^{n-1}\sum_i \lambda_i + \lambda^{n-2}\sum_{ij}\lambda_i\lambda_j - \cdots\cdots (-1)^n \prod_{i=1}^{n}\lambda_i\Big]$$

$$= (-1)^n \Big[\lambda^n - \lambda^{n-1}\cdot tr(A) + \cdots\cdots (-1)^n \cdot \det(A)\Big]$$

☐ Determinant & Trace

OM(23) Bad news : Elimination does not preserve the λ's

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \quad \lambda = 0 \ (\text{or}) \ 7$$

$$\cancel{R} = U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad \lambda = 0, \ \lambda = 1$$

Good news :

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

$$det(A) = \prod_{i=1}^{n} \lambda_i$$

* Eigenvalues of a triangular matrix lie along its diagonal.

Proof

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(A - \lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots\cdots(a_{nn}-\lambda) = 0$$

$$\Rightarrow \lambda \in \{a_{11}, a_{22}, \cdots, a_{nn}\}$$

Imaginary Eigenvalues

Ex:5 $Rot(90°) = Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalues.

$\lambda_1 = i$ & $\lambda_2 = -i$

$\lambda_1 + \lambda_2 = tr(Q) = 0$

$det(Q) = \lambda_1 \lambda_2 = 1$

After a rotation, no real vector Qx stays in the same direction as x. ($\lambda = 0$ is useless as $det(Q) \neq 0$)

$Q = Rot(90°)$

$Q^2 = Rot(180°) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

Its eigenvalues are $\lambda = -1$ and -1.

Squaring Q will square each λ, so we must have $\lambda^2 = -1$.

$\therefore$ The eigenvalues of the $90°$ rotation matrix Q are $+i$ and $-i$, because $i^2 = -1$.

$$Q = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

$$\det(Q - \lambda I) = 0 \Longrightarrow \boxed{\lambda^2 + 1 = 0}$$

$$\therefore \lambda_1 = i, \quad \lambda_2 = -i$$

Complex eigenvectors;

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- Q is orthogonal matrix $\longrightarrow |\lambda| = 1$
- Q is skew symmetric matrix $\longrightarrow \lambda$ is purely imaginary

- A symmetric matrix ($S^T = S$) can be compared to a real #.
- A skew symmetric matrix ($A^T = -A$) can be compared to an imaginary #.
- An orthogonal matrix ($Q^T Q = I$) corresponds to a complex # with $|\lambda| = 1$

☐ Eigenvalues of AB & A+B

Guess : An eigenvalue λ of A times an eigenvalue β of B usually does not give an eigenvalue of AB.

False proof : $AB x = A \beta x = \beta A x = \beta \lambda x$

seems like $\beta\lambda$ is an eigenvalue of AB. ¿

Nope ——.

This proof is correct, when x is an eigenvector for A and B.

Similarly,

the eigenvalues of $A+B$ are generally not $\lambda + \beta$.

Suppose, x really is an eigenvector for both A and B. Then,

$$AB x = \lambda \beta x \quad \& \quad BA x = \lambda \beta x$$

When all n eigenvectors are shared, we can multiply eigenvalues.

* A and B share the same n independent eigenvectors iff $AB = BA$.

Proof

Assume that $AB = BA$ and that v is an eigenvector of A,

$$Av = \lambda v$$

$$A(Bv) = (AB)v = (BA)v = B(Av) = B(\lambda v) = \lambda(Bv)$$

$\therefore$ Bv is an eigenvector of A

$\therefore$ Bv is a scalar multiple of v

$\Rightarrow$ $Bv = \mu v$ $\Rightarrow$ v is also an eigenvector of B.

(OR)

If A & B have the same eigenvectors.

$A = P D_A P^{-1}$ and $B = P D_B P^{-1}$

$$AB = P D_A P^{-1} P D_B P^{-1} = P D_A D_B P^{-1} = P D_B D_A P^{-1}$$

$$= P D_B P^{-1} P D_A P^{-1} = BA$$

* Two diagonalizable matrices are simultaneously diagonalizable iff they commute.

OM(23)

Heisenberg's uncertainty principle

In quantum mechanics, the position matrix P and the momentum matrix Q do not commute.
In fact,

$$QP - PQ = I$$

To have $Px = 0$ at the same time as $Qx = 0$ would require $x = Ix = 0$.

$\Rightarrow$ If we knew the position exactly, we could not also know the momentum exactly.

$\boxed{6.1(A)}$ Find the eigenvectors & eigenvalues of A, A^2, A^{-1} and $A+4I$.

$\boxed{6.1(B)}$ How can you estimate the eigenvalues of any A?

Gershgorin Circle theorem

check ILA(1)

Every eigenvalue of 'A' must be "near" atleast one of the entries a_{ii} on the main diagonal.
For λ to be "near a_{ii}" means that $|a_{ii} - \lambda|$ is no more than the sum R_i of all other $|a_{ij}|$ in that row i of the matrix. Then
$R_i = \sum_{i \neq j} |a_{ij}|$ is the radius of a circle centered at a_{ii}.

* Every λ is in the circle around one or more diagonal entries a_{ii} : $|a_{ii} - \lambda| \leq R_i$

Reasoning: If λ is an eigenvalue, then $A - \lambda I$ is not invertible. Then $A - \lambda I$ can not be diagonally dominant. So atleast one diagonal entry $a_{ii} - \lambda$ is not larger than the sum R_i of all other entries $|a_{ij}|$ in row i.

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ First circle: $|\lambda - a| \le |b| = R_1$

2^{nd} circle: $|\lambda - d| \le |c| = R_2$

These are circles in the complex plane, since λ could certainly be complex.

$$A = \begin{bmatrix} d_1 & 1 & 2 \\ 2 & d_2 & 1 \\ -1 & 2 & d_3 \end{bmatrix}$$

$|\lambda - d_1| \le 1 + 2 = 3 = R_1$

$|\lambda - d_2| \le 2 + 1 = 3 = R_2$

$|\lambda - d_3| \le 1 + 2 = 3 = R_3$

All eigenvalues of this 'A' lie in a circle of radius $R = 3$ around one or more of the diagonal entries d_1, d_2, d_3.

Diagonalizing a Matrix

- When x is an eigenvector, multiplication by 'A' is just multiplication by a number 'λ'. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a diagonal matrix with no off-diagonal interconnections.

$\Rightarrow$ The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly.

Diagonalization: Suppose $A_{n\times n}$ has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the columns of an eigenvector matrix X. Then $X^{-1}AX$ is the eigenvalue matrix Λ:

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

The matrix A is diagonalized.

$$AX = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda x_1 & \lambda x_2 & \cdots & \lambda x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = X\Lambda$$

$$\boxed{AX = X\Lambda \Rightarrow X^{-1}AX = \Lambda \Longrightarrow A = X\Lambda X^{-1}}$$

The matrix X has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent.

$\Rightarrow$ Without n independent eigenvectors, we can't diagonalize.

- A and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. The eigenvectors are different.

$$\boxed{A^k = X\Lambda X^{-1} X\Lambda X^{-1} \cdots X\Lambda X^{-1} = X\Lambda^k X^{-1}}$$

- Suppose, the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $x_1, \ldots, x_n$ are independent. The eigenvector matrix X will be invertible.

om(23) proof

→ Any matrix that has no repeated eigenvalues can be diagonalized.

- We can multiply eigenvectors by any non-zero constant.

$A(cx) = \lambda(cx)$ is still true

- Some matrices have too few eigenvectors. Those matrices can not be diagonalized.

* There is no connection b/w invertibility and diagonalizability :

  - Invertibility is concerned with the eigenvalues ($\lambda = 0$ (or) $\lambda \neq 0$)
  - Diagonalizability is concerned with the eigenvectors (too few or enough for X).

* Eigenvectors $x_1, x_2, \ldots, x_j$ that correspond to distinct (all different) eigenvalues are linearly independent. An $n \times n$ matrix that has n different eigenvalues (no repeated λ's) must be diagonalizable.

Proof

Suppose,

$$c_1 x_1 + c_2 x_2 = 0$$

$$A(c_1 x_1 + c_2 x_2) = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0$$

$$c_1 \lambda_2 x_1 + c_2 \lambda_2 x_2 = 0$$

$$(\lambda_1 - \lambda_2) c_1 x_1 = 0$$

$\lambda_1 \neq \lambda_2$ & $x_1 \neq 0 \implies c_1 = 0$

Similarly $c_2 = 0$.

Only the combination with $c_1 = c_2 = 0$ gives $c_1 x_1 + c_2 x_2 = 0$.

$\therefore$ Eigenvectors x_1 and x_2 must be independent.

This proof extends directly to j eigenvectors.

(or) by induction

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Ex: 2. Powers of A

The Markov matrix $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

$\lambda_1 = 1$, $\lambda_2 = 0.5$

$x_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix} = X \Lambda X^{-1}$$

$$A^k = X \Lambda^k X^{-1} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

$$k \to \infty : A^{\infty} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

- $A^k \longrightarrow$ zero matrix when all $|\lambda| < 1$.

X^{-1}

Similar Matrices : Same Eigenvalues

Suppose the eigenvalue matrix Λ is fixed. As we change the eigenvector matrix X, we get a whole family of different matrices $A = X\Lambda X^{-1}$ - all with the same eigenvalues in Λ. All those matrices A (with the same Λ) are called similar.

This idea extends to matrices that can't be diagonalized. We choose one constant matrix C (not necessarily Λ). And we look at the whole family of matrices $A = BCB^{-1}$, allowing all invertible matrices B. Those matrices A and C are called similar.

C might not be diagonal.

Columns of B might not be eigenvectors.

We only require B is invertible.

Similar matrices A & C have the same eigenvalues.

* All the matrices $A = BCB^{-1}$ are similar.
They all share the eigenvalues of C

OM(23) 2

Proof

① Suppose, $Cx = \lambda x$

$$(BCB^{-1})(Bx) = BCx = B\lambda x = \lambda(Bx)$$

$\Rightarrow$ same λ

② $P_A(\lambda) = \det(A - \lambda I) = \det(BCB^{-1} - \lambda BIB^{-1})$

$$= \det(B(C - \lambda I)B^{-1}) = \det(B)\det(C-\lambda I)\det(B^{-1})$$

$$= \det(B)\det(B^{-1})\det(C-\lambda I) = \det(BB^{-1})\det(C-\lambda I)$$

$$= \det(I)\det(C-\lambda I) = \det(C-\lambda I) = P_C(\lambda)$$

$\Rightarrow$ The characteristic polynomials of A and B are equal

$\therefore$ Equal eigenvalues.

☐ Fibonacci Numbers

The Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, ... comes from

$$F_{k+2} = F_{k+1} + F_k$$

Problem: Find the Fibonacci number F_{100}

Let, $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$\boxed{u_{k+1} = A u_k}$$

$$u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$$

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{k+1} = F_{k+1}$$

$$\Longrightarrow u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k = A u_k$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, . . .

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \dots$$

$$\dots\ u_{100} = A^{100} u_0 = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad ; \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$$

$$(A - \lambda_1 I)x = 0 \Rightarrow \begin{bmatrix} (1-\lambda_1)t_1 + t_2 \\ t_1 - \lambda_1 t_2 \end{bmatrix} = 0$$

$$\lambda_1(1-\lambda_1)t_1 + \lambda_1 t_2 = 0$$

$$t_1 - \lambda_1 t_2 = 0$$

$$t_1(-\lambda^2 + \lambda + 1) = 0 \quad \& \quad t_2 = \frac{t_1}{\lambda_1} \Bigg\} \quad t_1 = \lambda_1 t_2$$

$$x_1 = \text{span}\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad x_2 = \text{span}\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}}{\lambda_1 - \lambda_2} = \frac{x_1 - x_2}{\lambda_1 - \lambda_2}$$

$$u_{100} = A^{100} u_0 = A^{100} \frac{x_1}{\lambda_1 - \lambda_2} - A^{100} \frac{x_2}{\lambda_1 - \lambda_2}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left[A^{100} x_1 - A^{100} x_2 \right] = \frac{1}{\lambda_1 - \lambda_2} \left[(\lambda_1)^{100} x_1 - (\lambda_2)^{100} x_2 \right]$$

$$u_{100} = \frac{(\lambda_1)^{100} x_1 - (\lambda_2)^{100} x_2}{\lambda_1 - \lambda_2} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$$u_{100} = \frac{\lambda_1^{100}}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \frac{(\lambda_2)^{100}}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{(\lambda_1)^{101} - (\lambda_2)^{101}}{\lambda_1 - \lambda_2} \\ \dfrac{(\lambda_1)^{100} - (\lambda_2)^{100}}{\lambda_1 - \lambda_2} \end{bmatrix} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$\lambda_1 - \lambda_2 = \sqrt{5}$ & $(\lambda_2)^{100} \approx 0$

$\boxed{\begin{array}{l} \text{100th Fibonacci \#,} \\ F_{100} = \dfrac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2} = \text{nearest integer to } \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{100} \end{array}}$

Every F_k is a whole #.

* The ratio $\frac{F_{101}}{F_{100}}$ must be very close to the limiting ratio $\frac{1+\sqrt{5}}{2}$. — golden mean. ~ 1.61

☐ Matrix Powers, A^k

Fibonacci's example is a typical difference equation

$$\boxed{u_{k+1} = A u_k}$$

Each step multiplies by A.

The solution is $\boxed{u_k = A^k u_0}$

$$A^k u_0 = (X \Lambda X^{-1})^k u_0 = (X \Lambda X^{-1})(X \Lambda X^{-1}) \cdots\cdots (X \Lambda X^{-1}) u_0$$

$$= (X \Lambda^k X^{-1}) u_0$$

$$u_{k+1} = A u_k$$

① Write u_0 as a combination $c_1x_1 + \cdots + c_nx_n$ of the eigen vectors.

$$u_0 = Xc = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Longrightarrow \boxed{c = X^{-1}u_0}$$

$$= c_1x_1 + \cdots + c_nx_n$$

② Multiply each eigenvector x_i by $(\lambda_i)^k$.

③ Add up the pieces $c_i(\lambda_i)^k x_i$ to find the solution,

$$u_k = A^k u_0 = (X \Lambda^k X^{-1}) u_0$$

Solution for $u_{k+1} = Au_k$

$$u_k = A^k u_0 = c_1 (\lambda_1)^k x_1 + \cdots\cdots + c_n (\lambda_n)^k x_n$$

$$= X \Lambda^k X^{-1} u_0 = X \Lambda^k c = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Ex:3. Start from $u_0=(1,0)$. Compute $A^k u_0$ for this faster Fibonacci

$$A=\begin{bmatrix}1&2\\1&0\end{bmatrix} \text{ has } \lambda_1=2 \;\&\; x_1=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\lambda_2=-1 \text{ and } x_2=\begin{bmatrix}1\\-1\end{bmatrix}$$

$$F_{k+2}=F_{k+1}+2F_k$$

~~They grow~~ The new #'s start with 0, 1, 1, 3. They grow faster because of $\lambda=2$

Ans: $u_{k+1}=A^{\cancel{k}}u_k \Rightarrow u_k=A^k u_0$

$$u_0=c_1x_1+c_2x_2$$

$$u_k=c_1(\lambda_1)^k x_1+c_2(\lambda_2)^k x_2$$

$$u_0=\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}+\frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix} \Longrightarrow c_1=c_2=\frac{1}{3}$$

$$u_k=A^ku_0=\frac{1}{3}2^k\begin{bmatrix}2\\1\end{bmatrix}+\frac{1}{3}(-1)^k\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}F_{k+1}\\F_k\end{bmatrix}$$

$$F_k = \frac{1}{3}\left(2^k - (-1)^k\right)$$

Ex: $\Longrightarrow$ Fourier series built from the eigenvectors e^{ikx} of $\frac{d}{dx}$.

Non diagonalizable Matrices

Suppose λ is an eigenvalue of A.

① Eigenvectors (geometric) : There are non-zero solutions to $Ax = \lambda x$

② Eigenvectors (algebraic) : The determinant of $(A-\lambda I)$ is zero.

There are 2 ways to count the multiplicity of eigenvalues: Always $\boxed{A \cdot M \geq G \cdot M \text{ for each } \lambda}$

① (Geometric Multiplicity = GM)

- count the independent eigenvectors for λ_i

$$G \cdot M = \dim\left(N(A-\lambda_i I)\right)$$

② (Algebraic Multiplicity = AM)

- counts the repetitions of λ among the eigenvalues.

Look at the n roots of $\det(A-\lambda I)=0$

If A has $\lambda = 4, 4, 4$, then that eigenvalue has AM $= 3$ and GM $= 1, 2,$ or 3.

Ex:- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$

$\Rightarrow \lambda = 0, 0$, but

1 eigen vector

AM $= 2$
GM $= 1$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ This shortage of eigenvectors when GM is below AM means that 'A' is not diagonalizable.

Ex:- $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ has $\begin{vmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = 0$

$(\lambda - 5)^2 = 0$

$\lambda = 5, 5$

AM $= 2$ & G.M $= 1$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Ex:- $\begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}$

$\Rightarrow$ These matrices are not diagonalizable.

$\boxed{6.2\,(A)}$ The Lucas numbers are like the Fibonacci numbers except they start with $L_1 = 1$ and $L_2 = 3$. Using the same rule $L_{k+2} = L_{k+1} + L_k$ the next Lucas numbers are $4, 7, 11, 18$.

Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$

Ans.

$$L_{k+2} = L_{k+1} + L_k$$

$$L_{k+1} = L_{k+1}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k = A u_k$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\begin{aligned} \lambda_1 (1-\lambda_1) t_1 + \lambda_1 t_2 &= 0 \\ t_1 - \lambda_1 t_2 &= 0 \end{aligned} \Rightarrow t_1 = \lambda_1 t_2 \Rightarrow x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$t_1 \big[\underbrace{\lambda_1 - \lambda_1^2 + 1}_{0}\big] = 0$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} L_2 \\ L_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 x_1 + c_2 x_2 = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} c_1 \lambda_1 + c_2 \lambda_2 &= 3 \\ c_1 \lambda_2 + c_2 \lambda_2 &= \lambda_2 \\ \hline c_1 (\lambda_1 - \lambda_2) &= 3 - \lambda_2 \end{aligned} \Longrightarrow c_1 = \lambda_1 \;\&\; c_2 = \lambda_2$$

$$u_1 = \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}\lambda_1^2+\lambda_2^2\\ \lambda_1+\lambda_2\end{bmatrix} = \begin{bmatrix}tr(A^2)\\ tr(A)\end{bmatrix}$$

$$u_{100} = \begin{bmatrix}L_{101}\\ L_{100}\end{bmatrix} = A^{99}u_1$$

$$= c_1(\lambda_1)^{99}\begin{bmatrix}\lambda_1\\1\end{bmatrix} + c_2(\lambda_2)^{99}\begin{bmatrix}\lambda_2\\1\end{bmatrix}$$

$$= (\lambda_1)^{100}\begin{bmatrix}\lambda_1\\1\end{bmatrix} + (\lambda_2)^{100}\begin{bmatrix}\lambda_2\\1\end{bmatrix}$$

$$L_{100} = \lambda_1^{100} + \lambda_2^{100}$$

$\boxed{6.2(B)}$ Find the inverse & the eigenvalues and the determinant of this matrix A:

Describe an eigenvector matrix X that gives $X^{-1}AX = \Lambda$.

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$$

Ans: $$A = -\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -B + 5I$$

$\dim(N(B)) = 3$

$\lambda = 0, 0, 0$

$tr(B) = 4 \Rightarrow \lambda = 4$

$\lambda : 0, 0, 0, 4$

$\lambda = 5, 5, 5, 5$

$\lambda : 5, 5, 5, 1$

$\lambda_1 = 1, \quad x_1 = c(1,1,1,1)$

$A^T = A$ (A is symmetric) : $\perp$ eigenvectors.

The nicest eigenvector matrix X is the symmetric orthogonal Hadamard matrix H.

$$X = H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = H^T = H^{-1}$$

Eigenvalues of A^{-1} : $1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$

The eigenvectors are not changed.

$$\therefore \quad A^{-1} = X \Lambda^{-1} X^{-1}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

System of Differential Equations

$$\frac{du}{dt} = \lambda u \xrightarrow{\text{solution}} u(t) = u(0)e^{\lambda t}$$

↳ The solutions that start from the # $u(0)$ at time $t=0$.

If we have an $n \times n$ system of ODE's. The unknown is a vector $\vec{u}$. It starts from the initial vector $\vec{u}(0)$, which is given.

We expect n exponents $e^{\lambda t}$ in $u(t)$ from n λ's.

System of n equations

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

starting from the vector

$$u(0) = \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} \quad \text{at } t=0$$

We'll need n constants to match the n-components of $u(0)$.

1st job — find n "pure exponential solutions" $u = e^{\lambda t} x$ by using $Ax = \lambda x$.

'A' is a constant matrix.

In other linear ~~systems~~ equations 'A' changes as 't' changes.

In non-linear systems, 'A' changes as 'u' changes.

Solution of $\frac{d\vec{u}}{dt} = A\vec{u}$

Our pure exponential solution will be $e^{\lambda t}$ times a fixed vector x.

Guess – λ is an eigenvalue of A, and x is the eigenvector

Prove – Substitute $\vec{u}(t) = e^{\lambda t}\vec{x}$ into the equation

$$\frac{du}{dt} = Au \implies \lambda e^{\lambda t}\vec{x} = A e^{\lambda t}\vec{x}$$

$$\implies \lambda\vec{x} = A\vec{x}$$

Choose $u = e^{\lambda t}x$ when $Ax = \lambda x$ $\Bigg\}$ $\frac{du}{dt} = \lambda e^{\lambda t}$ agrees with $Au = Ae^{\lambda t}x$

- All components of this special solution $u = e^{\lambda t}x$ share the same $e^{\lambda t}$.
- The solution grows when $\lambda > 0$, it decays when $\lambda < 0$.
- If $\lambda \in \mathbb{C}$,

  $Re(\lambda)$ decides growth or decay

  $Im(\lambda)$ gives oscillation $e^{i\omega t}$ like a sine wave.

Ex:1 Solve $\frac{du}{dt} = Au = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u$ starting from

$u(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Ans: This is a vector equation for u.

It contains 2 scalar equations for the components y and z. They are "coupled together" because the matrix 'A' is not diagonal.

$$\frac{du}{dt} = Au \Longrightarrow \frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

$$\Longrightarrow \frac{dy}{dt} = z \quad \text{and} \quad \frac{dz}{dt} = y$$

* The idea of eigenvectors is to combine those eq^ns in a way that gets back to 1 by 1 problems.

$$\frac{dy}{dt} + \frac{dz}{dt} = \frac{d}{dt}(y+z) = z+y$$

$$(y+z) = c_1 e^{t}$$

$$\frac{dy}{dt} - \frac{dz}{dt} = \frac{d}{dt}(y-z) = -(y-z)$$

$$y - z = c_2 e^{-t}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The pure exponential solutions u_1 and u_2

$$u_1(t) = e^{\lambda_1 t} x_1 = e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Big| \quad u_2(t) = e^{\lambda_2 t} x_2 = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Complete solution : $u(t) = Ce^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} Ce^{t} + De^{-t} \\ Ce^{t} - De^{-t} \end{bmatrix}$$

$u(0) = (u_1(0), u_2(0))$ decides C and D.

$$u(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = C\begin{bmatrix} 1 \\ 1 \end{bmatrix} + D\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \underline{C = 3,\ D = 1}$$

$$\frac{du}{dt} = Au$$

① Write $u(0)$ as a combination $c_1x_1 + \cdots + c_nx_n$ of the eigenvectors of A.

$$u(0) = Xc = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

② Multiply each eigenvector x_i by its growth factor $e^{\lambda_i t}$.

③ The solution is the same combination of those pure solutions $e^{\lambda_i x}$:

$$\frac{du}{dt} = Au \quad : \quad u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$$

- If 2 λ's are equal, with only one eigenvector, another solution is needed. (It'll be $te^{\lambda t}x$)

?

A Defective Eigenvalue

$$\frac{d\vec{x}}{dt} = Ax$$

A has repeated eigenvalues.

Say A is 2×2.

To form the general solution, we need 2 linearly independent solutions, but we only have one:

$$x_1(t) = e^{\lambda t} x_1$$

When we faced similar problem in the 2nd order linear case, we were able to work around it by multiplying the solution by t.
Lets try it here.

$$u_2(t) = t e^{\lambda t} x_1$$

$$u' = Au$$

$$e^{\lambda t} x_1 + \underline{\lambda t e^{\lambda t}} x_1 = \underline{A t e^{\lambda t} x_1}$$

$$x_1 = 0 \implies x_1 = 0$$

$$\lambda x_1 e^{\lambda t} = A x_1 e^{\lambda t} \implies (A - \lambda I) x_1 = 0$$

x_1 is an eigenvector of A & $x_1 = 0$.

N.P $\Longrightarrow$ We need another approach

Lets guess

$$u(t) = e^{\lambda t} x_1 + t e^{\lambda t} x_2$$

$$u' = Au$$

$$\lambda e^{\lambda t} x_1 + e^{\lambda t} x_2 + \underline{\lambda t e^{\lambda t} x_2} = A\left(e^{\lambda t} x_1 + \underline{t e^{\lambda t} x_2}\right)$$

$$\left(\lambda x_1 + x_2 + \underline{\lambda t x_2}\right) e^{\lambda t} = \left(A x_1 + \underline{t A x_2}\right) e^{\lambda t}$$

$$A x_2 = \lambda x_2 \Longrightarrow (A - \lambda I) x_2 = 0$$

$$\lambda x_1 + x_2 = A x_1 \Longrightarrow (A - \lambda I) x_1 = x_2$$

$(A - \lambda I) x_2 = 0$: x_2 is an eigenvector of A.

$(A - \lambda I) x_1 = x_2$
(or)
$(A - \lambda I)^2 x_1 = 0$

: $x(t) = e^{\lambda t} x_1 + t e^{\lambda t} x_2$ will be a solution to the differential equation $u' = Au$.

A vector x_1 satisfying $(A-\lambda I)x_1 = x_2$ is called a generalized eigenvector.

Ex: 2 Solve $\frac{du}{dt} = Au$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A:

$$\frac{du}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} u \quad \text{starting from } u(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}$$

Eigenvectors are: $x_1 = (1, 0, 0)$, $x_2 = (1, 1, 0)$, $x_3 = (1, 1, 1)$

Step 1:–; $u(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3$

Step 2: The factors $e^{\lambda t}$ give exponential solutions

$e^{t} x_1$ and $e^{2t} x_2$ and $e^{3t} x_3$

Step 3; Combination that starts from $u(0)$ is:

$u(t) = 2e^{t} x_1 + 3e^{2t} x_2 + 4e^{3t} x_3$.

□ 2^{nd}-order equations

$$my'' + by' + ky = 0$$

Substitute $y = e^{\lambda t}$,

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \implies (m\lambda^2 + b\lambda + k)\, e^{\lambda t} = 0$$

$$m\lambda^2 + b\lambda + k = 0$$

The equation for y has 2 pure solutions,

$$y_1 = e^{\lambda_1 t} \quad \text{and} \quad y_2 = e^{\lambda_2 t}$$

$c_1y_1 + c_2y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

Linear Algebra : we turn the scalar equation (with y'') into a vector equation for y and y'

$y'' + by' + ky = 0$

Let $m = 1$,

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$\left.\begin{aligned} \frac{dy}{dt} &= y' \\ \frac{dy'}{dt} &= -ky - by' \end{aligned}\right\} \quad \frac{du}{dt} = \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

Solve $\frac{du}{dt} = Au$ by eigenvalues of A.

$$\begin{vmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{vmatrix} = \det(A - \lambda I) = \lambda^2 + b\lambda + k = 0$$

$$-\lambda t_1 + t_2 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

1^{st} component of $u(t)$: $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

2^{nd} component of $u(t)$: $\underset{\uparrow \text{ velocity}}{\frac{dy}{dt}} = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}$

$A_{2\times 2} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}$: companion matrix

a companion to the 2^{nd} order equation with y''!

$\boxed{\text{Ex :3}}$ Motion around a circle with $y''+y=0$ and $y=\cos t$.

Ans: $m=1$, stiffness $k=1$, and $d=0$: no damping.

Calculus: Put $y=e^{\lambda t}$ into $y''+y=0$

$$\lambda^2+1=0 \implies \lambda_1=i,\ \lambda_2=-i$$

$$y(t)=c_1e^{it}+c_2e^{-it}=\cos t\,(c_1+c_2)+i\sin t\,(c_1-c_2)$$

$y(0)=1,\ y'(0)=0,$

$y'(t)=-\sin t\,(c_1+c_2)+i\cos t\,(c_1-c_2)$

$$\left.\begin{aligned} 1 &= (c_1+c_2) \\ 0 &= i(c_1-c_2) \\ \hline 2c_1=1 &\Rightarrow c_1=\tfrac{1}{2},\ c_2=\tfrac{1}{2} \end{aligned}\right\}\quad y(t)=\frac{e^{it}+e^{-it}}{2}=\cos t$$

Algebra

$$\frac{du}{dt}=\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}=Au$$

$\lambda_1=i,\ \lambda_2=-i$ [A is antisymmetric]

$x_1=(1,i),\ x_2=(1,-i)$

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{x_1 + x_2}{2} = \frac{1}{2}x_1 + \frac{1}{2}x_2$$

$$u(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

$u = (\cos t, -\sin t)$ goes around a circle of radius 1.