Introduction to Linear Algebra
- Gilbert Strang

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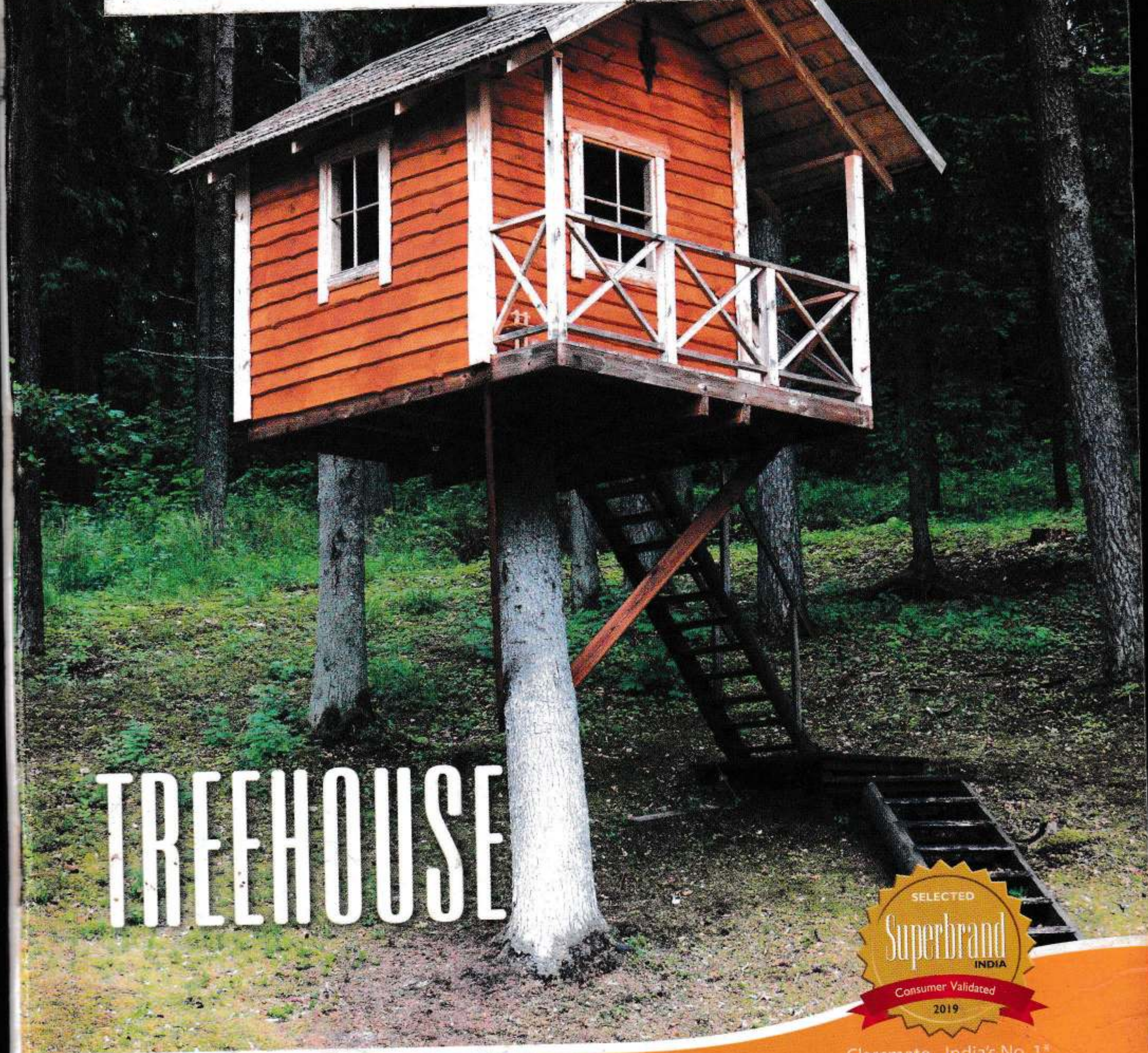
Orthogonality



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		INTRODUCTION TO LINEAR ALGEBRA – Gilbert Strang, MIT (5th Edition)		

Orthonormal Bases & Gram-Schmidt

* The vectors $q_1, \ldots, q_n$ are orthonormal ~~orthogonal~~ if

$$q_i^T q_j = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}$$

* A matrix with orthonormal columns satisfies $Q^T Q = I$:

$$Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

* When Q is square, $Q^T Q = I \Longrightarrow Q^T = Q^{-1}$

$\therefore$ Q is an orthogonal matrix

Check OM(23)

- $Q^T Q = I$ even when Q is rectangular. In this case Q^T is only an inverse from the left.
For square matrices, we also have $Q Q^T = I$, so Q^T is the two-sided inverse of Q.

- Q_1, Q_2 orthogonal $\longrightarrow$ $Q_1 Q_2$ is orthogonal

Ex:1 (Rotation) — Q rotates every vector in the plane by the angle θ:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \& \quad Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = Q^{-1}$$

$\sin^2\theta + \cos^2\theta = 1 \implies$ columns give an orthonormal basis for $\mathbb{R}^2$.

&

Ex: 2 (Permutation) —

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

Inverse = transpose : $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

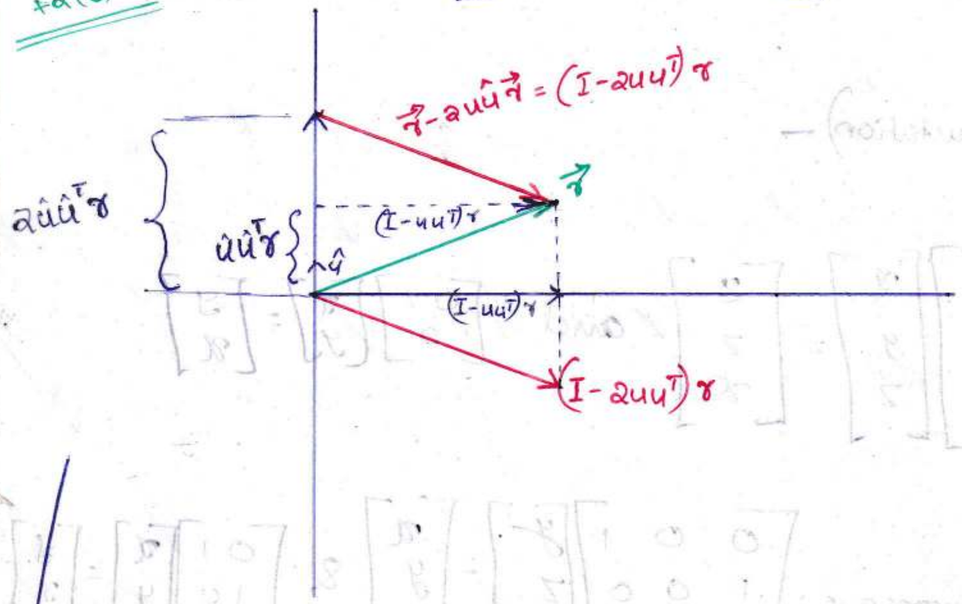
* Every permutation matrix is an orthogonal matrix.

Ex: 3 (Reflection) –

Reflection along a line ~~that~~ with slope $\tan\theta$, thro' the origin,

$$Ref(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

check $\star 2(6)$



compare Householder reflector with the projector

$$\boxed{P_{\perp u} = I - \hat{u}\hat{u}^T = I - \frac{uu^T}{u^Tu}}$$

* Householder Matrices

The Householder matrix for a reflection about the hyperplane $\perp$ to a ~~vector~~ unit vector u is:

$$\boxed{H = I - 2\hat{u}\hat{u}^T} \boxed{= I - 2\frac{uu^T}{u^Tu}}$$

where u : unit vector

- H is symmetric $(H^T = H)$ & orthogonal $(H^T = H^{-1})$.

$$H^{-1}H = H^TH = H^2 = I$$

Proof

$$H^2 = (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4u\underline{u^Tu}u^T$$

$$= I - 4uu^T + 4\underbrace{|u|^2}_{1}uu^T = I - 4uu^T + 4uu^T = I$$

- Any vector w that is $\perp$ to u is left unchanged.

$$Hw = (I - 2uu^T)w = w - 2uu^Tw = w - 2u\underbrace{(u \cdot w)}_{0} = w$$

-

$$w \parallel u \implies w = (u^T w)u = u(u^T w)$$

$$Hw = (I - 2uu^T)w = w - 2uu^T w$$

$$= w - 2u\underbrace{u^T u}u^T w = w - 2uIu^T w$$

$$= w - 2w = -w$$

- Any vector r can be written as

$$r = w + v = w + u^T r u = w + (u \cdot r)u$$

$$Hr = H(w + u^T r u) = (I - 2uu^T)(w + u^T r u)$$

$$= w + u^T r u - \cancel{2u\underbrace{u^T w}_{0}} + 2uu^T \underbrace{u^T r}u$$

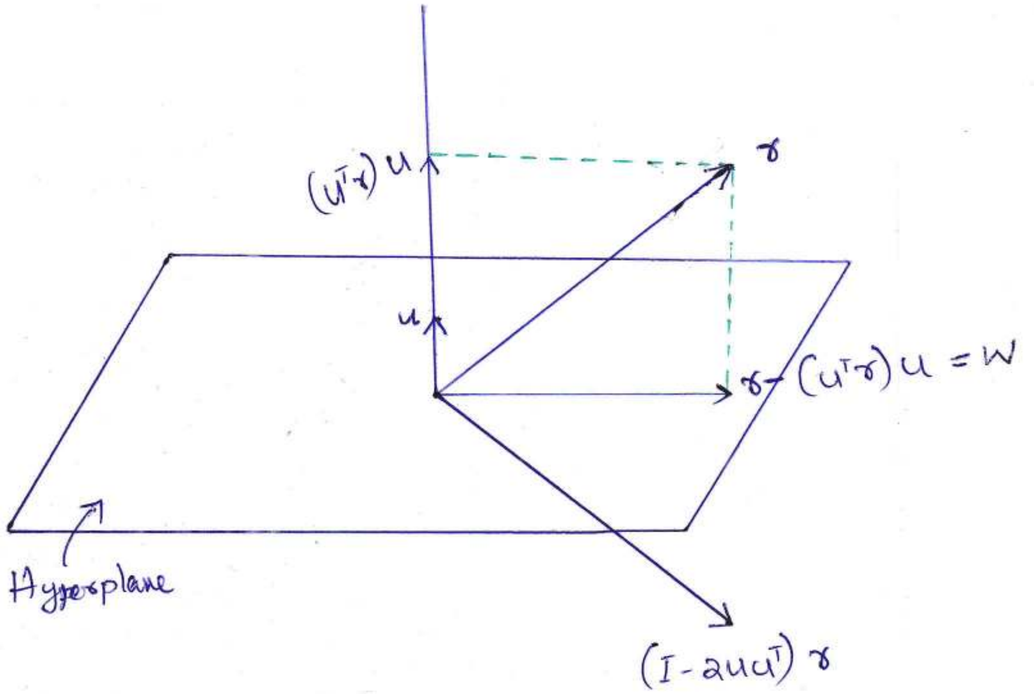
$$= w + u^T r u - 2u\underbrace{u^T u}(u^T r)$$

$$= w + u^T r u - 2u(u^T r) = w + u^T r u - 2u^T r u$$

$\cancel{H(r) = (w) - u^T r u}$

$\cancel{Hr = H(w + u^T r}$

$$Hr = H\left(w + u^T r u\right) = w - u^T r u$$



⇒ The "hyperplane" $\perp$ to u acts as a "mirror".
(A vector that is an element in that space (along the mirror) is not reflected. However,
if a vector has a component that is orthogonal to the mirror, that component is reversed in direction.

(1) reflection preserves the length of a vector.

* If Q has orthonormal columns $(Q^TQ = I)$, it leaves lengths unchanged.

$$|Qx| = |x| \text{ for every vector } x$$

Q also preserves the dot products.

$$(Qx)^T(Qy) = x^TQ^TQy = x^Ty$$

Proof. $|Qx|^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = |x|^2$

Projections using orthonormal bases : Q replaces A.

Suppose, the basis vectors are orthonormal.

$a_i \longrightarrow q_i$

$A^T A \longrightarrow Q^T Q = I$

$$\boxed{\begin{aligned} \hat{x} &= Q^T b \\ p &= Q\hat{x} = QQ^T b = Pb \\ P &= QQ^T \end{aligned}}$$

* The least square solution of $Qx = b$ is $\hat{x} = Q^T b$. The projection matrix is, $P = QQ^T$.

- There are no matrices to invert.

~~We~~

$$\hat{x} = Q^T b = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} b = \begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_n^T b \end{bmatrix} =$$

The best $\hat{x} = Q^T b$ just has dot products of $q_1, q_2, \ldots, q_n$ with b. We have <u>1-D</u> <u>projections</u> !

The "coupling matrix" or "correlation matrix" $A^T A$ is now $Q^T Q = I$

There is no coupling

When A is Q, with orthonormal columns,

$p = Q\hat{x} = QQ^T b$.

Projection onto q's :

$$p = QQ^Tb = Q\hat{x} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_n^T b \end{bmatrix}$$

$$= q_1(q_1^T b) + q_2(q_2^T b) + \cdots + q_n(q_n^T b)$$

$$= (P_1 + P_2 + \cdots + P_n)\, b = Pb$$

When Q is square $m = n$, the subspace is the whole space. Then,

$Q^T = Q^{-1}$ and $\hat{x} = Q^T b$ is the same as $x = Q^{-1} b$.

The solution is exact!

The projection of b onto the whole space is b itself.

In this case, $p = b$ and $P = QQ^T = I$.

If $q_1, \dots, q_n$ is an orthonormal basis for the whole space, then Q is square. Every $b = QQ^T b$ is the sum of its components along the q's:

$$\boxed{b = q_1(q_1^T b) + q_2(q_2^T b) + \cdots + q_n(q_n^T b) = P_1 + P_2 + \cdots + P_n}$$

$$\boxed{P_1 + P_2 + \cdots + P_n = q_1 q_1^T + q_2 q_2^T + \cdots + q_n q_n^T = I}$$

- Transforms $QQ^T = I$ is the foundation of Fourier Series and all the great transforms of applied mathematics. They break vectors b or functions $f(x)$ into $\perp$ pieces. Then by adding the pieces in the above eqn, the inverse transform puts b and $f(x)$ back together.

Ex: 4. The columns of this orthogonal Q are orthonormal vectors q_1, q_2, q_3:

$$m = n = 3 \ , \quad Q = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{has } Q^TQ = QQ^T = I$$

The separate projections of $b = (0,0,1)$ onto q_1, and q_2 and q_3 are p_1 and p_2 and p_3:

$$q_1(q_1^T b) = \frac{2}{3} q_1 \ ; \ q_2(q_2^T b) = \frac{2}{3} q_2 \ ; \ q_3(q_3^T b) = \frac{-1}{3} q_3$$

f(a) $p_1 + p_2 = q_1(q_1^T b) + q_2(q_2^T b)$ is the projection of b onto the plane of q_1 and q_2. The sum of all 3 is the projection of b onto the whole space. — which is $p_1 + p_2 + p_3 = b$ itself.

Reconstruct b

$b = p_1 + p_2 + p_3$

$$\therefore \ \frac{2}{3} q_1 + \frac{2}{3} q_2 - \frac{1}{3} q_3 = \frac{1}{9}\begin{bmatrix} -2+4-2 \\ 4-2-2 \\ 4+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b$$

□ The Gram-Schmidt Process

check +2(4)

"Orthogonal is good"

When $A^T A \longrightarrow Q^T Q = I$

$\Rightarrow$ The 1-D projections are uncoupled.

The best, $\hat{x} = Q^T b$ (just n separate dot products)

$\Leftrightarrow$ Gram-Schmidt way to create orthonormal vectors.

Input: $a, b, c,$

Begin by choosing $A = a$.

1st Gram-Schmidt step:

$$B = b - (\text{projection of } b \text{ along } A)$$

$$= b - \frac{A^T b}{A^T A} A$$

B: error vector, $e \perp$ to A.

Next Gram-Schmidt step :

$$C = c - \left[\text{projection of } c \text{ along span}(A, B) \right]$$

$$= c - \left[\frac{A^T c}{A^T A} A + \frac{B^T c}{B^T B} B \right]$$

Gram-Schmidt process ⇒ Subtract from every new vector its projections in the directions already set.

$$q_1 = \frac{A}{|A|}, \quad q_2 = \frac{B}{|B|}, \quad q_3 = \frac{C}{|C|}.$$

are orthonormal.

Ex: Suppose the independent non-orthogonal vectors a, b, c are

$$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

Ans: $A = a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$B = b - \frac{A^T b}{A^T A} A = b - \frac{2}{2} A = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = c - \frac{6}{2} A + \frac{6}{6} B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$|A| = \sqrt{2}$, $|B| = \sqrt{6}$, $|C| = \sqrt{3}$.

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The Factorization $A = QR$

We started with a matrix $A = \begin{bmatrix} a & b & c \end{bmatrix}$ & ended up with a matrix $Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$

The vectors a, b, c are combinations of the q's & (vice versa) $\Longrightarrow$ there must be a 3^{rd} matrix connecting A to Q

- The vectors a and A and q_1 are all along a single line.

- The vectors a, b and A, B and q_1, q_2 are all ~~along~~ in the same plane.

- The vectors a, b, c and A, B, C and q_1, q_2, q_3 are in one subspace

$$\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

(OR)

$$\boxed{A = Q\,R}$$

* $A = QR$ is Gram-Schmidt in a nutshell

* From independent vectors $a_1, a_2, \cdots, a_n$, Gram-Schmidt constructs orthonormal vectors $q_1, q_2, \cdots, q_n$.
  The matrices with these columns satisfy $A = QR$.
  Then, $R = Q^T A$ is upper triangular because later q's are orthogonal to earlier a's.

Ex:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR$$

* diagonal elements of R are $|A| = \sqrt{2}$, $|B| = \sqrt{6}$, $|C| = \sqrt{3}$.

Any $m \times n$ matrix A with independent columns can be factored into $A = QR$. The $m \times n$ matrix Q has orthonormal columns, and the square matrix R is upper triangular with +ve diagonal. $\rightarrow R^{-1}$ exists

This is useful for least squares ! ← Cholesky factorization

$$A^T A = (QR)^T QR = R^T Q^T Q R = R^T R$$

The least square solution $A^T A \hat{x} = A^T b$ simplifies to $R^T R \hat{x} = R^T Q^T b \Longrightarrow R\hat{x} = Q^T b$: Good.

Least squares : $R^T R \hat{x} = R^T Q^T b$ (OR) $R\hat{x} = Q^T b$

$$\Rightarrow \boxed{\hat{x} = R^{-1} Q^T b}$$

Instead of solving $Ax = b$, which is impossible, we solve $R\hat{x} = Q^T b$ by back substitution – which is very fast. The real cost is the mn^2 multiplications in the Gram-Schmidt process.

Starting from $a, b, c = a_1, a_2, a_3$

$$q_1 = \frac{a_1}{|a_1|}$$

$$B = a_2 - (q_1^T a_2)\, q_1 \implies q_2 = \frac{B}{|B|}$$

$$C^* = a_3 - (q_1^T a_3)\, q_1 \quad \& \quad C = C^* - (q_2^T C^* q_2)$$

$$\implies q_3 = \frac{C}{|C|}$$

Eq?. subtracts one projection at a time in C^* and C. That change is called modified Gram-Schmidt.

~~Good Software like LAPACK, used in good systems like~~

Good software like LAPACK will not use this Gram-Schmidt code. There is a better way, "Householder reflections" act on A to produce the upper triangular R. This happens one column at a time in the same way that elimination produces the upper triangular U in LU.

Positive Definite Matrix

ILA(8)

- A matrix is positive definite if its symmetric & all its eigenvalues are +ve.

- A matrix is positive definite if its symmetric and all its pivots are positive.

  signs of the pivots are the signs of the eigenvalues.

- A matrix is positive definite if $x^T A x > 0$ for all vectors $x \neq 0$.

Proof

If x is an eigenvector of A, then $x \neq 0$ and $Ax = \lambda x$

$\rightarrow x^T A x = \lambda x^T x = \lambda |x|^2$

If $\lambda > 0$, as $|x|^2 > 0$, we must have $x^T A x > 0$

- A matrix 'A' is positive definite iff it can be written as $A = R^T R$ for some rectangular matrix R with independent columns.

Cholesky factorization

Every positive definite matrix 'A' can be factored as :

$$A = LL^T = R^TR$$

where, L is lower triangular with positive diagonal elements.

L : Cholesky factor of A

- can be interpreted as "square root" of positive definite matrix.

The $A = LDL^T$ factorization exists and is unique for positive definite matrices. D is diagonal matrix with +ve diagonal entries (~~e.g. +ve eigenvalues of the positive definite matrix~~)

$$A = LDL^T = L\sqrt{D}\sqrt{D}L^T = (L\sqrt{D})(L\sqrt{D})^T = \mathcal{L}\mathcal{L}^T$$

when 'A' is positive definite, the Cholesky factor is given by $\mathcal{L} = L\sqrt{D} = LD^{1/2}$

QR decomposition - LAPACK

A Givens rotations (Jacobi rotations)

The simple matrix that rotates the xy-plane by θ is Q_{21}:

$$Q_{21} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use Q_{21} the way we used E_{21}, to produce a zero in the $(2,1)$ position. That determines the angle θ.

Ex:- $Q_{21}A = \begin{bmatrix} 0.6 & 0.8 & 0 \\ -0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 & -153 & 114 \\ 120 & -79 & -223 \\ 200 & -40 & 395 \end{bmatrix} = \begin{bmatrix} 150 & -155 & -110 \\ 0 & 75 & -225 \\ 200 & -40 & 395 \end{bmatrix}$

$-0.8(90) + 0.6(120) = 0$

~~90 sinθ + cosθ 120~~

$90 \cdot \sin\theta + 120 \cdot \cos\theta = 0$

$$\frac{90}{\sqrt{90^2+120^2}} \sin\theta + \frac{120}{\sqrt{90^2+120^2}} \cos\theta = 0$$

$$\sin(\theta+\phi) = 0 \qquad \cos\phi = \frac{90}{\sqrt{90^2+120^2}}$$

$$\theta+\phi = 2n\pi \qquad \sin\phi = \frac{120}{\sqrt{90^2+120^2}}$$

$$\theta = -\phi + 2n\pi$$

$$\sin\theta = \frac{-120}{\sqrt{90^2+120^2}}, \quad \cos\theta = \frac{90}{\sqrt{90^2+120^2}}$$

Now the (3,1) entry.

The numbers cosθ and sinθ are determined from 150 & 200,

$$Q_{31} Q_{21} A = \begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 150 & & \\ 0 & & \\ 200 & & \end{bmatrix} = \begin{bmatrix} 250 & -125 & 250 \\ 0 & 75 & -225 \\ 0 & 100 & 325 \end{bmatrix}$$

The (3,2) entry has to go. :

$$Q_{32}Q_{31}Q_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 250 & -125 & \cdot \\ 0 & 75 & \cdot \\ 0 & 100 & \cdot \end{bmatrix} = \begin{bmatrix} 250 & -125 & 250 \\ 0 & 125 & 125 \\ 0 & 0 & 375 \end{bmatrix}$$

$$= R$$

$$Q_{32}Q_{31}Q_{21}A = R \implies A = \left(Q_{21}^{-1}Q_{31}^{-1}Q_{32}^{-1}\right)R = QR$$

* Inverse of each Q_{ij}, is

$$Q_{ij}^{-1} = Q_{ij}^{T} \quad (\text{rotations thro' } -\theta)$$

B Householder reflections

- faster than rotations because each one clears out a whole column below the diagonal.

The Householder matrix for a reflection about the hyperplane $\perp$ to a ~~[illegible]~~ vector $\vec{u}$ is:

$$\boxed{H = I - 2\hat{u}\hat{u}^T = I - 2\frac{uu^T}{u^Tu}}$$

We need to build 'H_1' so that ~~[illegible]~~
$H_1 a_1 = \alpha e_1$ for some constant α and

$$e_1 = [1 \ 0 \ 0]^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

H_1 is orthogonal : $|H_1 a_1| = |a_1|$ & $|\alpha e_1| = |\alpha||e_1| = |\alpha|$
& symmetric

$$\Rightarrow |a_1| = |\alpha| \Rightarrow \boxed{\alpha = \pm|a_1|}$$

$$H_1 a_1 = \pm|a_1| e_1 = \begin{bmatrix} |a_1| \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ (or) } \begin{bmatrix} -|a_1| \\ 0 \\ \vdots \\ 0 \end{bmatrix} = r_1$$

$$H_1 a_1 = \cancel{\pm} |a_1| e_1 = r_1$$

$$(I - 2\hat{u}_1\hat{u}_1^T) a_1 = a_1 - 2u_1u_1^T a_1 = |a_1| e_1 = r_1$$

$$u_1(2u_1^T a_1) = a_1 - |a_1| e_1 = a_1 - r_1$$

$$u_1 = \frac{a_1 - r_1}{(2u^T a_1)} \Rightarrow \text{scalar}$$

$$|u_1(2u_1^T a_1)| = |u_1| |2u_1^T a_1| = |a_1 - r_1|.$$

$$\Longrightarrow |2u_1^T a_1| = |a_1 - r_1| = |a_1 - |a_1| e_1|$$

$$\therefore \quad \hat{u}_1 = \frac{a_1 - |a_1| e_1}{|a_1 - |a_1| e_1|} = \frac{a_1 - r_1}{|a_1 - r_1|}$$

$$\Longrightarrow H_1 = I - 2\hat{u}\hat{u}^T = I - 2\frac{(a_1 - |a_1| e_1)(a_1 - |a_1| e_1)^T}{|a_1 - |a_1| e_1|^2}$$

$$= I - 2\frac{(a_1 - r_1)(a_1 - r_1)^T}{(a_1 - r_1)^2}$$

* The more numerically stable transformation is to reflect about the hyperplane with normal vector,

$$u = a_1 + \text{sign}(a_{11}) \, |a_1| \, e_1 = \begin{bmatrix} a_{11} + \text{sign}(a_{11})|a_1| \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$A^{(1)} = H_1 A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & \cdots & a_{mn}^{(1)} \end{bmatrix}$$

where, $a_{11}^{(1)} = \sqrt{a_{11}^2 + a_{21}^2 + \cdots + a_{m1}^2} = |a_1|$

If $a_{11}^{(1)} = 0$, we define $H_2 = H_2(a_2^{(1)})$
where $a_2^{(1)}$ is the entire 2nd column of $A^{(1)}$.

If $a_{11}^{(1)} \neq 0$, we do not want to touch the 1st row ~~col~~ of $A^{(1)}$ and we only want to set the last $(n-2)$ elements of the 2nd column of $A^{(1)}$ into zero.

Now, we can construct $\tilde{H}_2$ such that,

$$\tilde{H}_2 \tilde{a}_2^{(1)} = |\tilde{a}_2^{(1)}| e_1^{(1)} = \begin{bmatrix} |\tilde{a}_2^{(1)}| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\hat{u}_2 = \frac{\tilde{a}_2^{(1)} - |\tilde{a}_2^{(1)}| e_1^{(1)}}{|\tilde{a}_2^{(1)} - |\tilde{a}_2^{(1)}| e_1^{(1)}|} \quad \text{and} \quad \tilde{a}_2^{(1)} = \begin{bmatrix} a_{22}^{(1)} \\ a_{32}^{(1)} \\ \vdots \\ a_{m2}^{(1)} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix} \quad \text{where } \tilde{H}_2 = I - 2\hat{u}_2 \hat{u}_2^T$$

$$A^{(2)} = H_2 A^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix} \left[\begin{array}{c|cccc} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ \hline 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & a_{m3}^{(1)} & \cdots & a_{mn}^{(1)} \end{array}\right]$$

$$H_2 H_1 A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{m3}^{(2)} & \cdots & a_{mn}^{(2)} \end{bmatrix}$$

where, $a_{22}^{(2)} = \sqrt{\left(a_{22}^{(1)}\right)^2 + \left(a_{32}^{(1)}\right)^2 + \cdots + \left(a_{m2}^{(1)}\right)^2} = \left|\tilde{a}_2^{(1)}\right|$

Construct $\tilde{H}_3$ such that

$$\tilde{H}_3 \tilde{a}_3^{(2)} = |\tilde{a}_3^{(2)}| e_1^{(2)} = \begin{bmatrix} |\tilde{a}_3^{(2)}| \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\hat{u}_3 = \frac{\tilde{a}_3^{(2)} - |\tilde{a}_3^{(2)}| e_1^{(2)}}{|\tilde{a}_3^{(2)} - |\tilde{a}_3^{(2)}| e_1^{(2)}|} \quad \text{and} \quad a_3^{(2)} = \begin{bmatrix} a_{33}^{(2)} \\ a_{43}^{(2)} \\ \vdots \\ a_{m3}^{(2)} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} I_2 & 0 \\ 0 & \tilde{H}_3 \end{bmatrix} \quad \text{where, } \tilde{H}_3 = I - 2\hat{u}_3 \hat{u}_3^T$$

$$A^{(3)} = H_3 A^{(2)} = \begin{bmatrix} I_2 & 0 \\ 0 & \tilde{H}_3 \end{bmatrix} \left[\begin{array}{cc|cccc} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ \hline 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{m3}^{(2)} & \cdots & a_{mn}^{(2)} \end{array}\right]$$

$$= H_3 H_2 H_1 A = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & \cdots & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(3)} & a_{34}^{(3)} & \cdots & a_{3n}^{(3)} \\ 0 & 0 & 0 & a_{44}^{(3)} & \cdots & a_{4n}^{(3)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{m4}^{(3)} & \cdots & a_{mn}^{(3)} \end{bmatrix}$$

where, $a_{33}^{(3)} = \sqrt{\left(a_{33}^{(2)}\right)^2 + \left(a_{43}^{(2)}\right)^2 + \cdots + \left(a_{m3}^{(2)}\right)^2} = \left| \tilde{a}_3^{(2)} \right|$

~~Suppose, we have already computed $H_1, \cdots, H_k$~~
~~In order to construct H_{k+1}~~

and so on.

Continuing the process, we obtain

$$(H_{n-1} \cdots H_1) A = R \Longrightarrow \underline{\underline{A = (H_1 \cdots H_{n-1}) R = Q R}}.$$

where for $j = 1, 2, \ldots n-1$

$$H_j = \begin{bmatrix} I_{j-1} & 0 \\ 0 & \tilde{H}_j \end{bmatrix}$$

$\Longrightarrow$ This is how LAPACK improves on 19th century Gram-Schmidt. Q is exactly orthogonal.

Examples - QR decomposition

1. $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

Ans:

(a) Gram-Schmidt method

$q_1 = a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow \hat{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$q_2 = a_2 - \hat{q}_1(\hat{q}_1^T a_2) = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}$

$\hat{q}_2 = \frac{1}{5}\begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$q_3 = a_3 - \left[\hat{q}_1(\hat{q}_1^T a_3) + \hat{q}_2(\hat{q}_2 a_3)\right]$

$= \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$

$$\hat{q}_3 = \frac{1}{4}\begin{bmatrix} +2 \\ -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}\begin{bmatrix} q_1^T a_1 & q_1^T a_2 & q_1^T a_3 \\ & q_2^T a_2 & q_2^T a_3 \\ & & q_3^T a_3 \end{bmatrix}$$

$A = QR.$

$$\cancel{q_1^T a_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad q_1^T a_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad q_1^T a_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}}$$

$q_1^T a_1 = 2$, $q_1^T a_2 = 3$, $q_1^T a_3 = 2$

$q_2^T a_2 = 5$, $q_2^T a_3 = -2$, $q_3^T a_3 = 4$

$$A = QR \Rightarrow \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Householder method

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\hat{u}_1 = \frac{a_1 + |a_1| e_1}{|a_1 + |a_1| e_1|} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}{|u_1|}$$

$$= \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$H_1 = I - 2\hat{u}_1 \hat{u}_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 5/6 & -1/6 & -1/6 \\ -1/2 & -1/6 & 5/6 & -1/6 \\ -1/2 & -1/6 & -1/6 & 5/6 \end{bmatrix}$$

$$A^{(1)} = H_1 A = \begin{bmatrix} -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 5/6 & -1/6 & -1/6 \\ -1/2 & -1/6 & 5/6 & -1/6 \\ -1/2 & -1/6 & -1/6 & 5/6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -2 \\ 0 & 10/3 & -4 \\ 0 & 10/3 & 0 \\ 0 & -5/3 & -2 \end{bmatrix}$$

$$\tilde{u}_2 = \tilde{a}_2^{(1)} + |\tilde{a}_2^{(1)}| e^{(1)} = \begin{bmatrix} 10/3 \\ 10/3 \\ -5/3 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25/3 \\ 10/3 \\ -5/3 \end{bmatrix}$$

$$\tilde{H}_2 = I - \frac{2 u_2 u_2^T}{u_2^T u_2} = I - 2 \times \frac{3}{\cancel{250}\ 125} \begin{bmatrix} 25/3 \\ 10/3 \\ -5/3 \end{bmatrix} \begin{bmatrix} \frac{25}{3} & \frac{10}{3} & \frac{-5}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{4}{15} & \frac{-2}{15} \\ -\frac{1}{3} & -\frac{2}{15} & \frac{1}{15} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{11}{15} & \frac{2}{15} \\ \frac{1}{3} & \frac{2}{15} & \frac{14}{15} \end{bmatrix}$$

$$A^{(2)} = H_2 A^{(1)} = H_2 H_1 A = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix} \begin{bmatrix} -2 & -3 & -2 \\ 0 & 10/3 & -4 \\ 0 & 10/3 & 0 \\ 0 & -5/3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & -2 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \\ 0 & 0 & -16/5 \end{bmatrix}$$

$$u_3 = \tilde{a}_3^{(2)} + |a_3^{(2)}|\, e_1^{(2)} = \begin{bmatrix} 12/5 \\ -16/5 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 32/5 \\ -16/5 \end{bmatrix}$$

$$\tilde{H}_3 = I - 2\frac{u_3 u_3^T}{u_3^T u_3} = \cancel{I - \frac{2 \times 5}{256} \begin{bmatrix} 32/5 \\ -16/5 \end{bmatrix}}$$

$$= \cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/16 \\ -1/32 \end{bmatrix} \begin{bmatrix} 1/16 & -1/32 \end{bmatrix}}$$

$$= I - 2 \times \frac{5}{256} \begin{bmatrix} 32/5 \\ -16/5 \end{bmatrix} \begin{bmatrix} 32/5 & -16/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{5}{128} \begin{bmatrix} \frac{32 \times 32}{25} & \frac{32 \times -16}{25} \\ \frac{32 \times -16}{25} & \frac{16 \times 16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8/5 & -4/5 \\ -4/5 & 2/5 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$A^{(3)} = H_3 A_2 H_1 A = H_3 A^{(2)} = \begin{bmatrix} I_2 & 0 \\ 0 & \tilde{H}_3 \end{bmatrix} \begin{bmatrix} -2 & -3 & -2 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \\ 0 & 0 & -16/5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & -2 \\ 0 & -5 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$Q = H_1 H_2 H_3 = \begin{bmatrix} -1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 5/6 & -1/6 & -1/6 \\ -1/2 & -1/6 & 5/6 & -1/6 \\ -1/2 & -1/6 & -1/6 & 5/6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2/3 & -2/3 & 1/3 \\ 0 & -2/3 & 11/15 & 2/15 \\ 0 & 1/3 & 2/15 & 14/15 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3/5 & 4/5 \\ 0 & 0 & 4/5 & 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$A = QR \Rightarrow \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 & -3 & -2 \\ 0 & -5 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

4.4(A) Add 2 more columns with all entries 1 or −1, so the columns of this 4×4 "Hadamard matrix" are orthogonal. How do you turn H_4 into an orthogonal matrix Q?

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & x & x \\ 1 & -1 & x & x \\ 1 & 1 & x & x \\ 1 & -1 & x & x \end{bmatrix} = ? \quad \& \quad \cancel{\text{what}} \quad Q_4 = \begin{bmatrix} ? \end{bmatrix}$$

The block matrix $H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$ is the next Hadamard matrix with 1's and −1's.

$H_8^T H_8 = ?$

Ans: $H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

has orthogonal columns.

$Q = \frac{H}{2}$ has orthonormal columns.

* A 5x5 Hadamard matrix is impossible because the dot product of columns would have 5 1's and/or -1's and could not add to 0.

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$H_8^T H_8 = \begin{bmatrix} H_4^T & H_4^T \\ H_4^T & -H_4^T \end{bmatrix} \begin{bmatrix} H_4 & H_4 \\ H_4 & H_4 \end{bmatrix} = \begin{bmatrix} 2H_4^T H_4 & 0 \\ 0 & 2H_4^T H_4 \end{bmatrix}$$

$$= \begin{bmatrix} 8I & 0 \\ 0 & 8I \end{bmatrix} = 8\, I_8 \,, \quad Q_8 = \frac{H_8}{\sqrt{8}}$$

4.1

3. Construct a matrix with the required property

(a) $C(A)$ contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, nullspace contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Ans. $r=2,\ n-r=1 \Rightarrow \boxed{n=3}\ \boxed{m=3} \Rightarrow A_{3\times 3}$.

$$\begin{bmatrix}1 & 2 & -3\\ 2 & -3 & 1\\ -3 & 5 & -2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$$

(b) $C(A^T)$ contains $\begin{bmatrix}1\\2\\-3\end{bmatrix}$ and $\begin{bmatrix}2\\-3\\5\end{bmatrix}$, $N(A)$ contains $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Ans: $C(A^T) \perp N(A)$

$\begin{bmatrix}2 & -3 & 5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 4 \neq 0$. Impossible

(c) $Ax = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has solution & $A^T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Ans: $\begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A)$ & $\begin{bmatrix}1\\0\\0\end{bmatrix} \in N(A^T)$

$C(A) \perp N(A^T)$

$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = 1 \neq 0$ Impossible

(d) Every row is orthogonal to every column $(A \neq 0)$

Ans:- $AA = A^2 = 0$

Ex:- $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(e) Columns add upto a column of zeros, rows add to a row of 1's.

Ans:- $a_1 + a_2 + a_3 = 0$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in N(A)$$

& also $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in C(A^T)$

Not possible since $N(A) \perp C(A^T)$

4. If $AB = 0$ then the columns of B are in the ________ of A. The rows of A are in the ________ of B. With $AB = 0$, why can't A and B be 3×3 matrices of rank 2?

Ans: $\underset{m \times n}{A}\underset{n \times p}{B} = \underset{m \times n}{A}\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \underset{m \times p}{\begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}} = 0$

$\Longrightarrow Ab_1 = Ab_2 = \cdots = Ab_p = 0$

$\therefore$ columns of B are in the $N(A)$.

$$AB = \underset{m \times n}{\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}} \underset{n \times p}{B} = \underset{m \times p}{\begin{bmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_m B \end{bmatrix}} = 0$$

$\Longrightarrow A_1 B = 0, \; A_2 B = 0, \cdots, A_m B = 0$

$\therefore$ Rows of $A \in$ left nullspace of B.

$b_1, b_2, b_3 \in N(A) \Longrightarrow C(B) \in N(A)$

$r_1 \leq \dim(N(A))$

$2 \leq 3 - 2 = 1$

Not possible.

6. The system of equations $Ax = b$ has no solution (they lead to $0 = 1$)

$x + 2y + 2z = 5$

$2x + 2y + 3z = 5$

$3x + 4y + 5z = 9$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? Its dot product $y^T b = 1$, so no solution x

Ans. $y_1 = 1, y_2 = 1, y_3 = -1$

Linear combinations of rows $= 0$

$y = (1, 1, -1) \in N(A^T)$

$C(A) \perp N(A^T)$

$y^T b = 1 \neq 0$ No solution.

Fredholm's Alternative

For any matrix A and vector b, exactly one of the following must hold:

Either

① $Ax = b$ has a solution

Or

② $A^T y = 0$ has a solution (non-trivial) y with $y^T b \neq 0$

ie;

$\rightarrow$ $Ax = b$ has a solution iff for any $A^T y = 0$, $y^T b = 0$

8. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_r = ?$

Ans: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x+y=0 \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$N(A) = c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, line thro' $(-1,1)$

$R(A^T) \perp N(A) \Rightarrow C(A^T)$ line $\perp$ to $(-1,1)$

$C(A^T) \Rightarrow c'\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$c(-1) + c'(1) = 1$

$c(1) + c'(1) = 0 \Rightarrow c = -c'$

$-c-c=1 \Rightarrow \underline{\underline{c = -1/2}}$, $c' = 1/2$.

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = x_r + x_n$

9. $\boxed{\text{If } A^TAx = 0 \text{, then } Ax = 0.}$

Proof: $Ax \in N(A^T)$ and $Ax \in C(A)$

$$N(A^T) \perp C(A) \implies Ax \perp Ax$$

$$\implies Ax = 0$$

10. Suppose A is a symmetric matrix ($A^T = A$).

(b) If $Ax = 0$ and $Az = 5z$, which subspaces contain these "eigenvectors" x and z?

* Symmetric matrices have perpendicular eigenvectors, $x^T z = 0$.

check $\pm 2(4)$

Ans: $Ax = 0 \implies x \in N(A)$

$Az = 5z \implies z \in C(A^T) = C(A)$ $[A^T = A]$

$N(A) \perp C(A^T) \quad \therefore \underline{x^T z = 0}$.

12. Find the pieces x_r and x_n and

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Ans:

$x - y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \therefore N(A) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$C(A^T) \perp N(A) \quad \therefore C(A^T) = c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\left.\begin{array}{l} c_1 + c_2 = 2 \\ c_1 - c_2 = 0 \end{array}\right\} \underline{c_1 = 1,\ c_2 = 1}$

$x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= x_r + x_n$

$N(A^T)$ is yz-plane

14. Find a vector in the column space of both matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

This will be a vector Ax and also $B\hat{x}$.

Think 3×4 with the matrix $\begin{bmatrix} A & B \end{bmatrix}$

Ans: I think $C([A\ B]) = C(A) + C(B)$

17

le

Ans

18.

(

Ans

17 If S is spanned by $(1,1,1)$ & $(1,1,-1)$, what's a basis for $S^\perp$?

Ans: S : subspace of $\mathbb{R}^3$

$C(A^T)$ is $\text{span}\left((1,1,1),(1,1,-1)\right)$

$$A\cdot x=\begin{bmatrix}1&1&1\\1&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\rightarrow\left[\begin{array}{ccc|c}1&1&1&0\\0&0&-2&0\end{array}\right]$$

$x+y+z=0$

$-2z=0 \Rightarrow z=0.$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=y\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$N(A)=S^\perp=\text{span}(1,-1,0)$

18. Suppose S only contains 2 vectors $(1,5,1)$ and $(2,2,2)$ (Not a subspace). Then $S^\perp$ is the nullspace of the matrix $A=$ ______.

Ans: $A=\begin{bmatrix}1&5&1\\2&2&2\end{bmatrix}$

$\Rightarrow$ $S^\perp$ is a subspace even if S is not.

21. S is spanned by the vectors $(1,2,2,3)$ & $(1,3,3,2)$. Find 2 vectors that span $S^{\perp}$.
This is same as solving $Ax=0$ for which A?

Ans:

$$Ax=0 \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & \underbrace{1 \quad -1}_{\text{free colmns}} & & 0 \end{array}\right]$$

$$\left.\begin{array}{l} a+5d=0 \\ b+c-d=0 \end{array}\right\}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

22. If P is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1+x_2+x_3+x_4=0$, write a basis for $P^\perp$. Construct a matrix that has P as its nullspace

Ans

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has P as its nullspace.

& $P^\perp$ its row space

Basis for $P^\perp = (1,1,1,1)$

23. If a subspace S is contained in a subspace V, prove that $S^\perp$ contains $V^\perp$.

Ans: $x \in V^\perp$ ~~xxx~~,

V contains all vectors in S $\Longrightarrow$ x is $\perp$ every vector in S

$S \subseteq V$

$\Longrightarrow$ Every $x \in V^\perp$ is also in $S^\perp$.

$S \subset V \rightarrow V^\perp \subset S^\perp$

24. Suppose, an $n \times n$ matrix is invertible: $AA^{-1} = I$. Then the 1st column of A^{-1} is orthogonal to the space spanned by which rows of A?

Ans:

$$AA^{-1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n \end{bmatrix} = \begin{bmatrix} a_1 \cdot \tilde{a}_1 & a_1 \cdot \tilde{a}_2 & \cdots & a_1 \cdot \tilde{a}_n \\ a_2 \cdot \tilde{a}_1 & a_2 \cdot \tilde{a}_2 & \cdots & a_2 \cdot \tilde{a}_n \\ \vdots & \vdots & & \vdots \\ a_n \cdot \tilde{a}_1 & a_n \cdot \tilde{a}_2 & \cdots & a_n \cdot \tilde{a}_n \end{bmatrix}$$

$$= I$$

1st column of A^{-1} is orthogonal to rows $2, 3, \ldots, n$. & to the space spanned by those rows.

30. A is 3×4 and B is 4×5 and $AB = 0$. So $N(A)$ contains $C(B)$. Prove from the dimensions of $N(A)$ and $C(B)$ that $\text{rank}(A) + \text{rank}(B) \le 4$

Ans:

$$AB = A \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} B = 0$$

$$Ab_1 = Ab_2 = Ab_3 = Ab_4 = Ab_5 = 0$$

$$\longrightarrow C(B) \subset N(A)$$

$$\therefore \cancel{r_A \le 5 - r_B}$$

$\cancel{\dim(C(B)) \leq \dim(N(A))}$

$\dim\left(C(B)\right) \leq \dim\left(N(A)\right)$

$$r_B \leq 4 - r_A \implies \underline{\underline{r_A + r_B \leq 4}}$$

32. Suppose, we are given 4 non-zero vectors r, n, c, l in $\mathbb{R}^2$

ⓐ What are the conditions for these to be bases for the 4 fundamental subspaces $C(A^T), N(A), C(A), N(A^T)$ of a 2×2 matrix ?

ⓑ What is one possible matrix A ?

Ans: For ~~A, A~~ r and n to be bases for $N(A)$ and $C(A^T)$,
$$r \cdot n = r^T n = 0 \impliedby N(A) \perp C(A^T)$$

For c and l to form bases for $C(A)$ and $N(A^T)$,
$$c \cdot l = c^T \cdot l = 0 \impliedby C(A) \perp N(A^T)$$

Ex:- ~~cr^T~~ cr^T

33 We have given 8 vectors $r_1, r_2, n_1, n_2, c_1, c_2, l_1, l_2$ in $\mathbb{R}^4$

(a) What are the conditions for those pairs to be bases for the 4 fundamental subspaces of a 4×4 matrix

(b) Possible matrix A. ?

Ans: The sets $\{r_1, r_2\}, \{n_1, n_2\}, \{c_1, c_2\}, \{l_1, l_2\}$ each contain linearly independent vectors.

$r_i^T n_j = 0 \longrightarrow$ for $i,j = 1,2.$

$c_i^T l_j = 0$

Ex:- $A = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{bmatrix} r_1 & r_2 \end{bmatrix}$

4.2

Q. Draw the projection of b onto a and also compute it from $p = \hat{x} a$

(a) $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ans: $P = \frac{a a^T}{a^T a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$p = \hat{x} a = Pb = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$

(b) $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ans: $P = \frac{a a^T}{a^T a} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$p = \hat{x} a = Pb = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4. Construct the projection matrices P_1 and P_2 onto the lines thro' the a's in problem 2. Is it true that $(P_1+P_2)^2 = P_1+P_2$? This would be true if $P_1P_2=0$

Ans:

$$P_1+P_2 = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$(P_1+P_2)^2 = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 10 & -4 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix} \neq P_1+P_2$$

P_1+P_2 is Not a projection matrix

5. Compute the projection matrices $\frac{aa^T}{a^Ta}$ onto the lines thro' $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$. Multiply those projection matrices & explain why their product $P_1 P_2$ is what it is

Ans: $\cancel{P_1 = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{9}}$

$$P_1 = \frac{1}{9}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$$

$$P_2 = \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$P_1 P_2 = \frac{1}{81}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} = O_{3\times 3}$$

$P_1 P_2 = 0$ because $a_1 \perp a_2$

6. Project $b=(1,0,0)$ onto the lines thro' a_1 and a_2 in problem 5 and also onto $a_3=(2,-1,2)$. Add up these projections $p_1+p_2+p_3$

Ans: $P_3 = \frac{1}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$

$p_1 = P_1 b = \frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$; $p_2 = P_2 b = \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$

$p_3 = P_3 b = \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}$

$p_1+p_2+p_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b$

Q7. Find P_3 & verify $P_1 + P_2 + P_3 = I$.

Ans: $P_1 + P_2 + P_3 = I$.

→ We can add projections onto orthogonal vectors to get the projection matrix onto the larger space.

11. Project b onto the C(A) by solving $A^TA\hat{x}=A^Tb$ and $p=A\hat{x}$: & find $e=b-p$

Q (a) $A=\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}$ and $b=\begin{bmatrix}2\\3\\4\end{bmatrix}$

Ans: $A^TA=\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}$

$$\left[\begin{array}{cc|cc}1&1&1&0\\1&2&0&1\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}1&1&1&0\\0&1&-1&1\end{array}\right]\rightarrow\left[\begin{array}{cc|cc}1&0&2&-1\\0&1&-1&1\end{array}\right]$$

$$\hat{x}=(A^TA)^{-1}A^Tb=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix}$$

$$=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}$$

$$p=A\hat{x}=\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}=\begin{bmatrix}2\\3\\0\end{bmatrix}$$

$$e=b-p=\begin{bmatrix}2\\3\\4\end{bmatrix}-\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}0\\0\\4\end{bmatrix},\quad A^Te=0$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ & $b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

Ans: $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 2 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \Rightarrow (A^T A)^{-1} = \begin{bmatrix} 3/2 & -1 \\ -1 & 1 \end{bmatrix}$

$\hat{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} 3/2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

$= \frac{1}{\cancel{2}} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \cancel{8}4 \\ \cancel{14}7 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = b.$

$b = p + e = p \Rightarrow e = 0$ since $b \in C(A)$

13. A is the 4×4 identity matrix with its last column removed. A is 4×3. Project $b=(1,2,3,4)$ onto the $C(A)$.

What is P ?

Ans:

$$A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix},$$

$$b=(P_1+P_2+P_3)b=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$p=Pb=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}1\\2\\3\\0\end{bmatrix}$$

last

14. $b = 2 \times 1^{st}$ column of A.
What is the projection of b onto the $C(A)$?
Is $P = I$ for sure in this case?

Compute p and P when $b = (0,2,4)$ and the columns of A are $(0,1,2)$ & $(1,2,0)$.

Ans. $p_1 = P_1 b = b$, $a_1 = \frac{b}{2} = (0,1,2)$

$$P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{1}{5}\begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}0 & 1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}0&0&0\\0&1&2\\0&2&4\end{bmatrix} \neq I$$

$$A = \begin{bmatrix}0&1\\1&2\\2&0\end{bmatrix}, \quad A^T A = \begin{bmatrix}0&1&2\\1&2&0\end{bmatrix}\begin{bmatrix}0&1\\1&2\\2&0\end{bmatrix} = \begin{bmatrix}5&2\\2&5\end{bmatrix}$$

$$\left[\begin{array}{cc|cc}5&2&1&0\\2&5&0&1\end{array}\right] \to \left[\begin{array}{cc|cc}1&-8&1&-2\\0&21&-2&1\end{array}\right] \to \left[\begin{array}{cc|cc}1&-8&1&-2\\0&1&-2/21&1/21\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1&0&5/21&-2/21\\0&1&-2/21&5/21\end{array}\right]$$

$1-\frac{16}{21}$

$-2+\frac{8}{21}$

$$\hat{x} = (A^TA)^{-1}A^T b = \frac{1}{21}\begin{bmatrix}5&-2\\-2&5\end{bmatrix}\begin{bmatrix}0&1&2\\1&2&0\end{bmatrix}\begin{bmatrix}0\\2\\4\end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix}-2&1&10\\5&8&-4\end{bmatrix}\begin{bmatrix}0\\2\\4\end{bmatrix}$$

$$p = Ax \Longrightarrow P = A(A^TA)^{-1}A^T = \begin{bmatrix}0&1\\1&2\\2&0\end{bmatrix}\frac{1}{21}\begin{bmatrix}-2&1&10\\5&8&-4\end{bmatrix}$$

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5

$$= \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$$

15. If 'A' is doubled, then $P' = 2A\left(4A^TA\right)^{-1}2A^T =$

$2A \cdot \frac{1}{4}\left(A^TA\right)^{-1} 2A^T = A\left(A^TA\right)^{-1}A^T = P$

~~The columnspace~~ $C(2A) = C(A)$

Is $\hat{x}$ the same for A and $2A$?

Ans: $\hat{x}_A = \left(A^TA\right)^{-1}A^Tb$

$\hat{x}_{2A} = \left(4A^TA\right)^{-1} 2A^Tb = \frac{1}{2}\left(A^TA\right)^{-1}A^Tb$

$\hat{x}_{2A} = \frac{1}{2}\hat{x}_A$

16. What linear combination of $(1,2,-1)$ and $(1,0,1)$ is closest to $b=(2,1,1)$?

Ans: $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$, $A^TA = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$

Algebra

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} = \hat{x}$$

$$\frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = b \Longrightarrow \boxed{b \in C(A)}$$

(OR)

Calculus

$$E = |Ax-b|^2 = (x+y-2)^2 + (2x-1)^2 + (-x+y-1)^2$$

$$\frac{\partial E}{\partial x} = 2(x+y-2) + 4(2x-1) - 2(-x+y-1) = 12x-6 = 0 \Longrightarrow \boxed{x = 1/2}$$

$$\frac{\partial E}{\partial y} = 2(x+y-2) + 2(-x+y-1) = 4y-6 = 0 \longrightarrow \boxed{y = \frac{3}{2}}$$

19, 20 Find the projection matrix onto the plane $x - y - 2z = 0$

Ans: $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & 3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1/2 & 0 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5/6 & -1/3 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right] \Rightarrow \qquad \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

(OR) $e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$$P = \frac{e e^T}{e^T e} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$$

$$P' = I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/6 & -1/6 & -2/6 \\ -1/6 & 1/6 & 2/6 \\ -2/6 & 2/6 & 4/6 \end{bmatrix}$$

$$= \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

(95) The projection matrix P onto an n-D subspace of $\mathbb{R}^m$ has rank $r = n$.

Ans.

Reasoning: $C(P)$ is the space that P projects onto

$C(A)$ contains all outputs Ax.

Outputs Px fill the subspace S.

$\therefore \text{rank}(P) = \dim(S) = n = \dim(C(P))$

26. If an $m \times m$ matrix has $A^2 = A$ & its rank is m. prove that $A = I$

Ans: $\text{rank}(A_{m \times m}) = m \Longrightarrow A^{-1}$ exists.

$$A^{-1} A A = A \Rightarrow A = I$$

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30. (a) Find the projection matrix P_c onto the $C(A)$

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

(b) Find the 3×3 projection matrix P_R onto the $C(A^T)$. Multiply $B = P_c A P_R$.

Explain.

Ans:

(a) $C(A) = \text{span}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = a$

$$P_c = \frac{aa^T}{a^Ta} = \frac{1}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

(b) $C(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right) = b$

$$P_R = \frac{bb^T}{b^Tb} = \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

$$P_c A = P_c\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} P_c a_1 & P_c a_2 & \cdots & P_c a_n \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = A$$

(or)

$$P_c A = A(A^TA)^{-1}A^T \cdot A = AI = A$$

$$AP_R = AA^T\left((A^T)^TA^T\right)^{-1}(A^T)^T = AA^T(AA^T)^{-1}A = IA = A$$

$$\Longrightarrow \boxed{P_c A P_R = A}$$

4.2 B

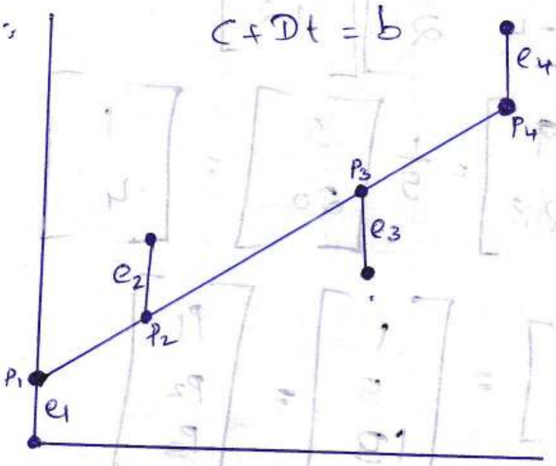
33. Kalman filter.

$\boxed{4 \cdot 3}$

1. With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$. Setup and solve the normal equations $A^T A \hat{x} = A^T b$. For the best-line, find its 4 heights P_i and 4 errors e_i. What's min. value ~~of~~ $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$

Ans:

$C + Dt = b$



$$\left.\begin{aligned} C + D(0) &= 0 \\ C + D(1) &= 8 \\ C + D(3) &= 8 \\ C + D(4) &= 20 \end{aligned}\right\} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$\underline{Ax = b}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 4 & 8 & 1 & 0 \\ 8 & 26 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 4 & 8 & 1 & 0 \\ 0 & 10 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{10} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{13}{20} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{1}{10} \end{array}\right] \qquad \frac{1}{4} + \frac{2}{5} = \frac{13}{20}$$

$$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{20}\begin{bmatrix} 13 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 13 & -4 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 28 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ +3 \end{bmatrix}$$

$$E = |e|^2 = 44 \qquad \Sigma e_i = -1+3-5+3 = 0$$

4. (By calculus) $E = \|Ax - b\|^2$ as a sum of 4 squares —

Find the derivative equations $\frac{\partial E}{\partial C} = 0$ and $\frac{\partial E}{\partial D} = 0$

Obtain the normal eqⁿ. $A^T A \hat{x} = A^T b$

Ans: $E = e_1^2 + e_2^2 + e_3^2 + e_4^2 = |Ax - b|^2$

$$= (C - 0)^2 + (C + D(1) - 8)^2 + (C + D(3) - 8)^2 + (C + D(4) - 20)^2$$

$$\frac{\partial E}{\partial C} = \cancel{2C + 2(C+D) + 2(C+3D-8) + 2(C+4D)}$$

$$= 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$$

$$\frac{\partial E}{\partial D} = 2(C + D - 8) + 2(3)(C + 3D - 8) + 2(4)(C + 4D - 20) = 0.$$

$$\frac{\partial E}{\partial C} = 0 \Rightarrow 4C + 8D = 36$$

$$\frac{\partial E}{\partial D} = 0 \Rightarrow 8C + 26D = 112$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

5. Find the height C of the best horizontal line to fit $b = (0, 8, 8, 20)$. An exact fit would solve the unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4×1 matrix A in these equations and solve $A^T A \hat{x} = A^T b$. Draw the horizontal line at height $\hat{x} = C$ and the 4 errors in e.

Ans:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$P = \frac{a a^T}{a^T a}$,

$A^T A = 4 \; ; \; A^T b = 36$

$\hat{x} = (A^T A)^{-1} A^T b = 9$

The best height for the horizontal line, $C = 9$

$p = Pb = A\hat{x} = (9, 9, 9, 9)$

$e = b - p = (-9, -1, -1, 11)$ still add to zero

$\Sigma e_i = -9 - 1 - 1 + 11 = 0$

7. Find the closest line $b = Dt$, thro origin, to the same 4 points. An exact fit would solve $D(0) = 0,\ D(1) = 8,\ D(3) = 8,\ D(4) = 20.$

Ans:
$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} D = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$A^T A = 26$, $A^T b = 112$ $\Longrightarrow$ Best, $\hat{D} = \frac{112}{26} = \frac{56}{13}.$

$\therefore$ $b = \frac{56}{13} t$

8. ~~Project (0,8,8,20)~~
$(C, D) = (9, 56/13)$ do not match $(C, D) = (1, 4)$ from previous problem because, columns of 'A' were not $\perp$, so we can't project separately to find C and D.

9. For the closest parabola, $b = C + Dt + Et^2$ to the same 4 points

Ans:
$$C + D(0) + E(0)^2 = 0$$
$$C + D(1) + E(1)^2 = 8$$
$$C + D(3) + E(3)^2 = 8$$
$$C + D(4) + E(4^2) = 20$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}}_{\text{Vandermonde matrix}} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \Longleftrightarrow Ax = b$$

$$A^T A x = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

10. For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same 4 points

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$\underbrace{\hphantom{xxxxxxxx}}$

Vandermonde matrix

$$\begin{bmatrix} \cancel{1} & \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{4} & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$$

11. The avg. of the 4 times is, $\hat{t} = \frac{0+1+3+4}{4} = 2$;
The avg. of the 4 b's is $\hat{b} = \frac{0+8+8+20}{4} = 9$.

ⓐ Verify that the best line goes thro' the center point $(\hat{t}, \hat{b}) = (2, 9)$

Ans: The best line, $x = 1 + 4t$ gives the centre point $\hat{b} = 9$ at center time $\hat{t} = 2$

ⓑ Explain why $C + D\hat{t} = \hat{b}$ comes from the 1st equation in $A^T A\hat{x} = A^T b$.

Ans:
$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b \Rightarrow \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$$

$$Cm + D \sum t_i = \sum b_i$$

$\div$ing by m, $C + D\hat{t} = \hat{b}$

$\therefore$ The best line goes through $\hat{b}$ at time $\hat{t}$

12 Project $b = (b_1, \ldots, b_m)$ onto the line thro' $a = (1, \ldots, 1)$: We solve m equations $ax = b$ in 1 unknown (by least squares).

(a) Solve $a^T a \hat{x} = a^T b$ to show $\hat{x}$ is the mean (average) of the b's.

Ans: $\hat{x} = \dfrac{a^T b}{a^T a} = \dfrac{\Sigma b_i}{m}$ = Mean of the b's.

(b) Find $e = b - a\hat{x}$ & the variance $|e|^2$ and the standard deviation $|e|$

Ans: $|e|^2 = (b_1 - \hat{x})^2 + (b_2 - \hat{x})^2 \ldots + (b_m - \hat{x})^2$

$= (b_1 - \text{mean})^2 + (b_2 - \text{mean})^2 + \ldots + (b_m - \text{mean})^2$

$=$ variance $= \sigma^2$

$|e| =$ s.d $= \sigma$.

(c) The horizontal line $\hat{b} = 3$ is closest to $b = (1, 2, 6)$ check $p = (3, 3, 3) \perp e$ and find the 3×3 projection matrix P.

Ans: $e = (-2, -1, 3)$

$$P = \frac{aa^T}{a^T a} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(13) 1^{st} assumption behind least squares:

$$Ax = b - (\text{noise 'e' with mean zero}).$$

Multiply the error vectors, $e = b - Ax$ by $(A^TA)^{-1}A^T$ to get $\hat{x} - x$ on the right. The estimation errors $\hat{x} - x$ also average to zero. The estimate $\hat{x}$ is unbiased.

Ans: $Ax = b - e = \cancel{b}$

$$(A^TA)^{-1}A^T(Ax - b)$$

$$(A^TA)^{-1}A^T(b - Ax) = \hat{x} - x = (A^TA)A^Te$$

$\Rightarrow$ when the components of ~~the~~ $e = Ax - b$ add to zero, so do the components of $\hat{x} - x$.

(14). 2^{nd} assumption behind least squares :

The m errors e_i are independent with variance σ^2, so the average of $(b-Ax)(b-Ax)^T$ is $\sigma^2 I$. Multiply on the left by $(A^TA)^{-1}A^T$ and on the right by $A(A^TA)^{-1}$ to show that the average matrix $(\hat{x}-x)(\hat{x}-x)^T$ is $\sigma^2(A^TA)^{-1}$.
This is the covariance matrix, W - section 10.2

Ans: $(A^TA)^{-1}A^T \cdot (b-Ax)(b-Ax)^T \cdot A(A^TA)^{-1} = (\hat{x}-x)(\hat{x}-x)^T$

When, the avg. of $(b-Ax)(b-Ax)^T$ is $\sigma^2 I$,

the avg. of $(\hat{x}-x)(\hat{x}-x)^T$ will be the output covariance matrix $(A^TA)^{-1}A^T\sigma^2 A(A^TA)^{-1}$ which simplifies to $\sigma^2(A^TA)^{-1}$.

That gives the average of the squared output errors $\hat{x}-x$.

15) A doctor takes 4 readings of your heart rate. The best solution to $x = b_1, \dots, x = b_4$ is the average $\hat{x} = b_1, \dots, b_4$. The matrix A is a column of 1's. Problem 14 gives the expected error $(\hat{x}-x)^2$ as $\sigma^2(A^TA)^{-1} =$ ______.

By averaging, the variance drops from σ^2 to $\frac{\sigma^2}{4}$.

Ans: When A has 1 column of 1s.

the expected error

17. ~~Ulrit~~ Find the least square solution $\hat{x} = (C, D)$
& draw the closest line

Ans: $\underline{\underline{C + Dt = b}}$

$b = 7$ at $t = -1$
$b = 7$ at $t = 1$
$b = 21$ at $t = 2$

$$C + D(-1) = 7$$
$$C + D(1) = 7$$
$$C + D(2) = 21$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

$$Ax = b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -4 & 1 & -1 \\ 2 & 6 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -4 & 1 & -1 \\ 0 & 14 & -2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -4 & 1 & -1 \\ 0 & 1 & \frac{-1}{7} & \frac{3}{14} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{7} & \frac{-1}{7} \\ 0 & 1 & \frac{-1}{7} & \frac{3}{14} \end{array}\right]$$

$$1 - \frac{4}{7} = \frac{3}{7}$$

$$-1 + \frac{12}{14} = -1 + \frac{6}{7}$$
$$= \frac{-1}{7}$$

$$\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = (A^TA)^{-1} A^T b = \frac{1}{\cancel{14}2} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 18 \\ 8 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

$9 + 4t = b$

(OR)

$$\hat{t} = \frac{-1+1+2}{3} = \frac{2}{3} \quad , \quad T = t - \frac{2}{3}$$

$T_1 = -1 - \frac{2}{3} = \frac{-5}{3}$, $b_1 = 7$

$T_2 = 1 - \frac{2}{3} = \frac{1}{3}$, $b_2 = 7$

$T_3 = 2 - \frac{2}{3} = \frac{4}{3}$, $b_3 = 21$

$$A = \begin{bmatrix} 1 & -5/3 \\ 1 & 1/3 \\ 1 & 4/3 \end{bmatrix} , \quad A^TA =$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -5/3 & 1/3 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & -5/3 \\ 1 & 1/3 \\ 1 & 4/3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{42}{9} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{14}{3} \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ -5/3 & 1/3 & 4/3 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} = 7 \begin{bmatrix} 1 & 1 & 1 \\ -5/3 & 1/3 & 4/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 7 \begin{bmatrix} 5 \\ 8/3 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 7/3 & 0 \\ 0 & \frac{3}{24} \end{bmatrix} \times \begin{bmatrix} 5 \\ 8/3 \end{bmatrix}$$

$$\begin{bmatrix} C' \\ D' \end{bmatrix} = \begin{bmatrix} 7/3 & 0 \\ 0 & 3/2 \end{bmatrix} \begin{bmatrix} 5 \\ 8/3 \end{bmatrix} = \begin{bmatrix} 35/3 \\ 4 \end{bmatrix}$$

$$\frac{3}{2} \times \frac{8}{3} = 4$$

$$C + DT = b \Longrightarrow C + D(t - \hat{t}) = b$$

$$\frac{35}{3} + \frac{4}{9}\left(t - \frac{2}{3}\right) = b = \frac{35}{3} + \frac{4}{9}t - \frac{8}{27}$$

$$C + DT = b \Longrightarrow \frac{35}{3} + 4T = b$$

$$\frac{35}{3} + 4\left(t - \frac{2}{3}\right) = b = 4t + \frac{35-8}{3} =$$

$$\underline{4t + 9 = b}$$

22. Find the best line to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$

Ans: $\sum t_i = 0 \implies A^T A = \text{diagonal}$,

$$A \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix} 5 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\implies$ $\underline{1 - t = b}$

23. Show that $|e|^2 = e^T b = b^T b - p^T b$.
This is the smallest total error.

Ans: $e \perp p$ in $\mathbb{R}^m$

$$|e|^2 = e^T e = e^T(b-p) = e^T b - e^T p = e^T b$$

$$= \cancel{(b-p)^T(b-p)} = \cancel{(b^T-p^T)(b-p)}$$

$$= \cancel{b^T b - b^T p - p^T b + p^T p}$$

$$= (b-p)^T b = (b^T - p^T) b = b^T b - p^T b$$

(24) The partial derivatives of $|Ax|^2$ w.r.t $x_1, \ldots, x_n$ fill the vector $2A^TAx$. The derivatives of $2b^TAx$ fill the vector $2A^Tb$. So the derivatives of $|Ax-b|^2$ are zero when.

Ans: $|Ax-b|^2 = (Ax-b)^T(Ax-b) = (x^TA^T-b^T)(Ax-b)$

$= x^TA^TAx - \underline{x^TA^Tb} - b^TAx + b^Tb$

$= |Ax|^2 - (b^TAx)^T - b^TAx + |b|^2$

Q25. What condition on $(t_1, b_1), (t_2, b_2), (t_3, b_3)$ puts those 3 pts onto a straight line?

A column space ans. is: (b_1, b_2, b_3) must be a combination of $(1,1,1)$ and (t_1, t_2, t_3).

Ans: $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $C + Dt = b$

$b \in C(A)$ & $C(A) \perp N(A^T)$

$A^T y = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\cancel{a + b + c = 0}$

$\cancel{t_1 a + t_2 b + t_3 c = 0}$

$\cancel{\begin{bmatrix} a \\ b \\ c \end{bmatrix} = t_2 \begin{bmatrix} -t_2/t_1 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -t_3/t_1 \\ 0 \\ 1 \end{bmatrix}}$

$\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Updownarrow$

$b^T y = 0$

$a b_1 + b b_2 + c b_3 = 0$

$\cancel{\begin{bmatrix} a \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -b_2/b_1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -b_3/b_1 \\ 0 \\ 1 \end{bmatrix}}$

$y = (t_2 - t_3, t_3 - t_1, t_1 - t_2) \in N(A^T) \perp (1,1,1)$ & (t_1, t_2, t_3)

$y^T b = 0 \Rightarrow b_1(t_2 - t_3) + b_2(t_3 - t_1) + b_3(t_1 - t_2) = 0.$

$t_3(b_2 - b_1) - t_2(b_3 - b_1) = t_1(b_2 - b_3)$

$\cancel{(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)}$

$t_3(b_2 - b_1) - t_2(b_2 - b_1 + \underline{b_3 - b_2}) = t_1(b_2 - b_3)$

$(t_3 - t_2)(b_2 - b_1) = (b_3 - b_2)(t_2 - t_1)$ // $\Rightarrow$ slope condition

26. Find the plane that gives the best fit to the 4 values $b=(0,1,3,4)$ at the corners $(1,0)$ and $(0,1)$ and $(-1,0)$ and $(0,-1)$ of a square. The equations $C+Dx+Ey=b$ at those 4 pts are $Ax=b$ with 3 unknowns $x=(C,D,E)$. What is A? At the center $(0,0)$ of the square, show that $C+Dx+Ey=$ avg. of the b's.

Ans.

$$\left.\begin{aligned} C+D(1)+E(0) &= 0 \\ C+D(0)+E(1) &= 1 \\ C+D(-1)+E(0) &= 3 \\ C+D(0)+E(-1) &= 4 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

$Ax=b$

$\Sigma x_i = 0$, $\Sigma y_i = 0$

$$A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad A^Tb = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$$

$$\Longrightarrow \hat{x} = \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix} : \quad \underline{2 - x - \tfrac{3}{2}y = b}$$

At $(0,0)$: the best plane $2 - x - \frac{3}{2}y$ has height, $= 2 = \dfrac{0+1+3+4}{4}$

27. Distance b/w lines.

The points $P = (x, x, x)$ and $Q = (y, 3y, -1)$ are on 2 lines in space that don't meet. Choose x and y to minimize the squared distance $|P-Q|^2$. The line connecting the closest P and Q is $\perp$ to ______

Ans:

The shortest link connecting two lines in space is $\perp$ to those lines.

28. Suppose the columns of A are not independent. How could you find a matrix B so that $P = B(B^T B)^{-1} B^T$ does give the projection onto the $C(A)$?

The usual formula will fail when $A^T A$ is not invertible

Ans:

Replace A in the formula by B that keeps only the pivot columns of A

29. Usually there will be exactly one hyperplane in $\mathbb{R}^n$ that contains the n given points $x = 0, a_1, \ldots, a_{n-1}$.

[Example for $n = 3$: there will be one plane containg $0, a_1, a_2$ unless ___].

What's the test to have exactly one plane in $\mathbb{R}^n$?

Ans: Only one plane contains $0, a_1, a_2$ unless a_1, a_2 are dependent.

4.4

4. Give examples

(a) A matrix Q that has orthonormal columns but $QQ^T \neq I$

Ans: $Q^TQ = I$

$QQ^T \neq I \Rightarrow Q$ is rectangular

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Any Q with $n < m$

(b) 2 orthogonal vectors that are not linearly independent

Ans: $(1,0), (0,0)$

$0(1,0) + 2(0,0) = 0 \Rightarrow$ dependent

$(1,0)^T(0,0) = 0 \Rightarrow$ orthogonal.

(c) An orthogonal basis for $\mathbb{R}^3$, including the vector $q_1 = \frac{(1,1,1)}{\sqrt{3}}$

Ans: $q_2 = \frac{(1,-1,0)}{\sqrt{2}}$, $q_3 = \frac{(1,1,-2)}{\sqrt{6}}$

6. If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also an orthogonal matrix

Ans: $(Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2$

$= Q_2^T I Q_2 = I$.

7. If Q has orthonormal columns, what's the least square solution $\hat{x}$ to $Qx=b$?

Ans: $Q^TQ\hat{x} = Q^Tb$

$Q^TQ = I \Longrightarrow \underline{\hat{x} = Q^Tb}$

9.
(b) Prove that always $(QQ^T)^2 = QQ^T$ by using $Q^TQ = I$. Then $P = QQ^T$ is the projection matrix onto $C(Q)$.

Ans: $(QQ^T)^2 = QQ^TQQ^T = QQ^T$

(a) Compute $P = QQ^T$ when $q_1 = (0.8, 0.6, 0)$ and $q_2 = (-0.6, 0.8, 0)$. Verify that $P^2 = P$.

Ans: $P = QQ^T = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection on the xy-plane

10. Orthogonal vectors are automatically linearly dependent

(a) Vector proof:

When q_1, q_2, q_3 are orthonormal,

OM (23)

$c_1 q_1 + c_2 q_2 + c_3 q_3 = 0$

$q_1 \cdot (c_1 q_1 + c_2 q_2 + c_3 q_3) = c_1 = 0$.

Similarly, $c_2 = c_3 = 0$.

$\therefore$ 3 q's are independent.

(b) Matrix proof :

$$Qx = 0 \Longrightarrow Q^T Q x = 0 \Longrightarrow x = 0$$

11. (a) Find orthonormal vectors q_1 and q_2 in the plane spanned by $a=(1,3,4,5,7)$ and $b=(-6,6,8,0,8)$

Ans: $A = a$,

$$B = b - \text{Proj}_A b = b - \frac{AA^T}{A^TA} b$$

$$= (-6,6,8,0,8) - \frac{1}{\cancel{200}\ \cancel{100}\ 50} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} -6 & 3 \\ 6 & 3 \\ 8 & 4 \\ 0 & 6 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \frac{1}{\cancel{1}50} \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ 3 & 9 & 12 & 15 & 21 \\ 4 & 12 & 16 & 20 & 28 \\ 5 & 15 & 20 & 25 & 35 \\ 7 & 21 & 28 & 35 & 49 \end{bmatrix} \begin{bmatrix} -3 \\ 3 \\ 4 \\ 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \frac{1}{50} \begin{bmatrix} 50 \\ 150 \\ 200 \\ 250 \\ 350 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{(1,3,4,5,7)}{10}, \quad q_2 = \frac{(-7,3,4,-5,1)}{10}$$

(b) Which vector in this plane is closest to $(1,0,0,0,0)$?

Ans: $Q = \frac{1}{10}\begin{bmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{bmatrix}$

$$Pb = QQ^Tb = \frac{1}{100}\begin{bmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ -7 & 3 & 4 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -7 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.18 \\ -0.24 \\ 0.4 \\ 0 \end{bmatrix}$$

15. Find orthonormal vectors q_1, q_2, q_3 such that q_1, q_2

(a) span $C(A)$

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 4 \end{bmatrix} \\ a_1 \quad a_2 \end{array}$$

Ans: $u_1 = a_1 = (1,2,-2) \implies q_1 = \frac{(1,2,-2)}{3}$

$u_2 = a_2 - Proj_{u_1} a_2 = a_2 - \frac{a_2^T u_1}{a_2^T a_2} a_2$

$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \\ 6 \end{bmatrix} =$

$\frac{9}{4} + \frac{9}{4} + 36 = \frac{18+144}{4} = \frac{162}{4}$

$$u_2 = a_2 - \text{Proj}_{u_1} a_2$$

$$= a_2 - \frac{u_1 u_1^T}{u_1^T u_1} a_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9} \times -9 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow q_2 = \frac{(2, 1, 2)}{3}$$

$$u_3 = a_3 - (\text{Proj } a_3 \text{ on span}(u_1, u_2))$$

$$= a_3 - \left[\text{Proj}_{u_1} a_3 + \text{Proj}_{u_2} a_3\right]$$

$$= a_3 - \frac{u_1 u_1^T}{u_1^T u_1} a_3 - \frac{u_2 \cdot u_2^T}{u_2^T u_2} a_3$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{9} \times \begin{bmatrix} 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_1 - \frac{1}{9} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_2$$

$$= \frac{1}{9} \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{9} \times \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \frac{2}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -4 \\ -2 \end{bmatrix}$$

$$q_3 = \frac{(2 \; -2 \; -1)}{3}$$

(b) Which of the 4 fundamental subspace contains q_3?

Ans: $q_3 \perp \text{span}(q_1, q_2)$

$\longrightarrow q_3 \perp \text{span}(a_1, a_2)$.

$q_3 \in C(A) \implies q_3 \in N(A^T)$

(c) Solve $Ax = (1, 2, 7)$ by least squares

Ans: $Ax = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix} = 9 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

$= \frac{1}{9} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ 27 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

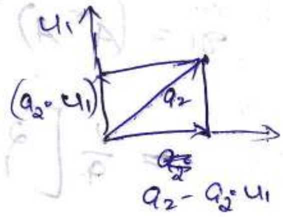
19. If $A=QR$, then $A^TA=R^TR = \_\_\_ \times \_\_\_$.
Gram-Schmidt on A corresp. to elimination on A^TA. The pivots for A^TA must be the squares of diagonal entries of R. Find Q and R by Gram-Schmidt for this A:

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix}$$

Ans: If $A=QR$, then $A^TA = R^TQ^TQR = R^TR =$ lower triangular times upper triangular.

- this Cholesky factorization of A^TA uses the same R as Gram-Schmidt.

$$u_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = a_1 \implies q_1 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$



$$u_2 = a_2 - \frac{u_1u_1^T}{u_1^Tu_1} a_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} - \frac{1}{\cancel{9}} \times \cancel{9} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$q_1^Ta_1 = 3$

$q_1^Ta_2 = 3$; $q_2^Ta_2 = 3$

$$\implies q_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = QR$$

$$A^T A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 9 & 9 \\ 0 & 9 \end{bmatrix} \rightarrow \quad = U \rightarrow \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If $S^T = S, \Rightarrow S = LDL^T$

$$(A^T A)^T = A^T A$$

$$(A^T A) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = LDL^T$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} = R^T R$$