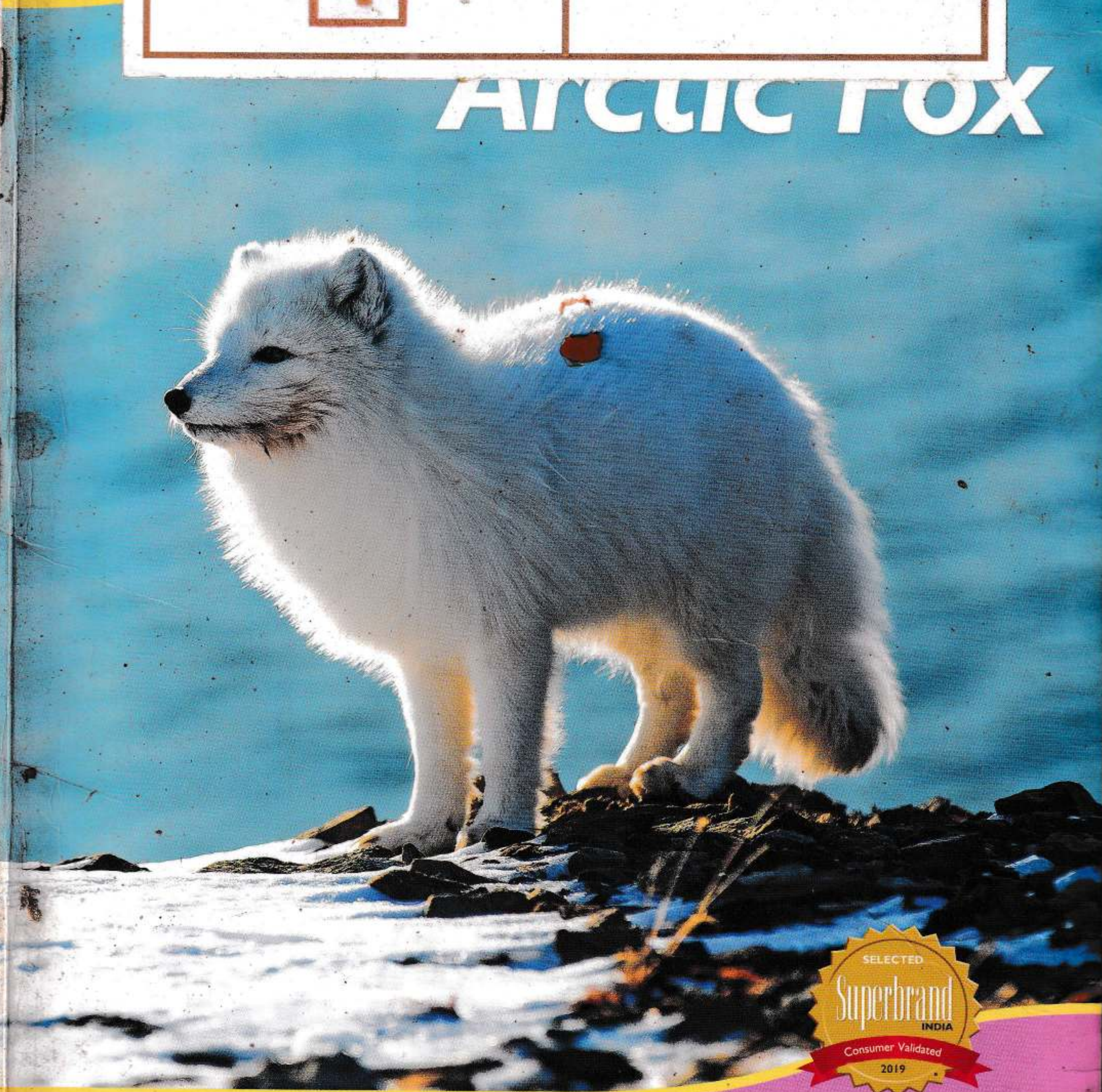
Introduction to Linear Algebra
- Gilbert Strang

4

Vector Spaces & Subspaces



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		INTRODUCTION TO LINEAR ALGEBRA - Gilbert Strang, MIT (5th edition).		

* A has the same nullspace as R. Same dimension $(n-r)$ and same basis.

$$\boxed{N(A) = N(R)} \boxed{= n-r}$$

Reasoning: Elimination steps don't change the solutions. The special solutions are a basis for this nullspace.

There are $(n-r)$ free variables, so the dimension of the nullspace is $(n-r)$.

Counting theorem :

$$\dim\left[C(A)\right] + \dim\left[N(A)\right] = r + (n-r) = n = \dim\left(\mathbb{R}^n\right)$$

* The left null space of A (the null space of A^T) has dimension $(m-r)$.

$$\boxed{\begin{array}{l} N(A^T) \neq N(R^T) \\ \dim[N(A^T)] = \dim[N(R^T)] \end{array}}$$

Row operations preserve the solutions of $Ax=0$,
but do not preserve the solutions of $x^T A = 0^T$

Reasoning: A^T is just as good a matrix as 'A'.

$\dim[C(A^T)] = r$

$A^T_{n \times m} \Longrightarrow$ the whole space is now $\mathbb{R}^m$

Counting rule for A^T:

$$\dim[C(A^T)] + \dim[N(A^T)] = r + (m-r) = m = \dim(\mathbb{R}^m)$$

Fundamental Theorem of Linear Algebra — Part 1

The column space & row space both have dimension r.

The null spaces have dimensions $(n-r)$ & $(m-r)$.

Ex:1 $A = [1, 2, 3]$

free columns

Ans: $m = 1,\ n = 3,\ r = 1$

$C(A^T)$ is a line in $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$

$N(A) = \cancel{A}\ x + 2y + 3z = 0$

$\dim[C(A^T)] + \dim[N(A)] = 1 + 2 = 3 = \dim[\mathbb{R}^3]$.

$C(A)$ is all of $\mathbb{R}^1$.

$A^T = \begin{bmatrix}1\\2\\3\end{bmatrix}$. $\qquad \underline{A^T y = 0} \Longleftarrow y^T A = 0^T$

$N(A^T) = Z$

$\dim[C(A)] + \dim(N(A^T)) = 1 + 0 = 1 = \dim[\mathbb{R}^1]$

Ex: 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has $m=2$, $n=3$, $r=1$

$\hookrightarrow R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

Ans: $C(A^T)$ is a line in $\mathbb{R}^3$

i.e., a line thro' $(1,2,3)$

$N(A)$ = plane: $x+2y+3z=0$

$\dim[C(A^T)] + \dim[N(A)] = 1 + 2 = 3 = \dim[\mathbb{R}^3]$.

$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ = line thro' $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$

$\boxed{A^T y = 0} \Longleftrightarrow \boxed{y^T A = 0^T}$

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $y_1 + 2y_2 = 0$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$N(A^T) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$ = line thro' $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

in $\mathbb{R}^2$.

(2) $(1,2)\cdot(-2,1) = -2+2=0$

$\Longrightarrow C(A)$ & $N(A^T)$ are $\perp$ lines in $\mathbb{R}^2$.

Ex: 3

Ex 3

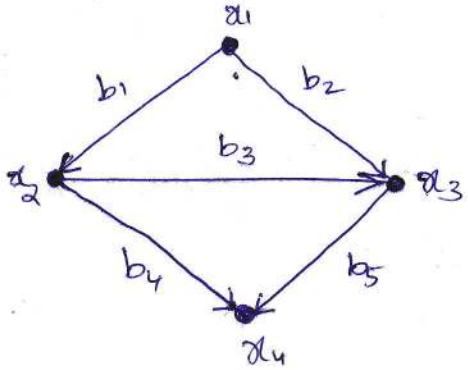
$$
\begin{aligned}
-x_1 + x_2 \qquad\qquad &= b_1 \\
-x_1 \qquad + x_3 \qquad &= b_2 \\
-x_2 + x_3 \qquad &= b_3 \\
-x_2 \qquad + x_4 &= b_4 \\
-x_3 + x_4 &= b_5
\end{aligned}
$$

$$
Ax = b \implies
\begin{array}{c c}
\text{edges} & \\
\begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{matrix} &
\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\
& \begin{matrix} x_1 & \;\; x_2 & \;\; x_3 & \;\; x_4 \end{matrix}
\end{array}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}
$$

A : incidence matrix

- matrix that shows the relationship b/w 2 classes of objects.

A graph with 5 edges & 4 nodes $\longrightarrow$



Ans: $\underline{\underline{N(A)}}$

Set $b = 0$: $x_2 = x_1$, $x_3 = x_1$, $x_3 = x_2$, $x_4 = x_3$

$$\Rightarrow x_1 = x_2 = x_3 = x_4.$$

$$N(A) = \text{span}\left[(c, c, c, c)\right].$$

$= \text{line in } \mathbb{R}^4.$

Special solution, $= (1,1,1,1)$ is a basis for $N(A)$

$\dim[N(A)] = 1 = n - r = 4 - r \Rightarrow \boxed{r = 3}$

$\underline{\underline{C(A)}}$

$r = 3$ independent columns.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -\text{[illegible]} & \text{[illegible]} & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -\text{[illegible]} & \text{[illegible]} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \text{[illegible]} & 0 \\ 0 & 0 & -\text{[illegible]} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

$C(A) \neq C(R)$

$$C(A) = \text{span}\left(\begin{bmatrix}-1\\-1\\0\\0\\0\end{bmatrix}\begin{bmatrix}1\\0\\-1\\-1\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\\0\\-1\end{bmatrix}\right)$$

$C(A^T)$

$\dim\left[C(A^T)\right] = 3.$

1st 3 rows of A are not independent.

$C(A^T) = C(R)$

rows 1,2,4 are independent. Those 3 rows are a basis for the row space.

Graph - see

edges b_1, b_2, b_3 form a loop in the picture

— Dependent rows 1,2,3

edges b_1, b_3, b_4 form a tree in the picture

Trees have no loops — independent rows 1,2,4

$N(A^T)$

- Solve $A^T y = 0$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & +1 & +1 & +1 & 0 \\ 0 & 0 & 0 & +1 & +1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

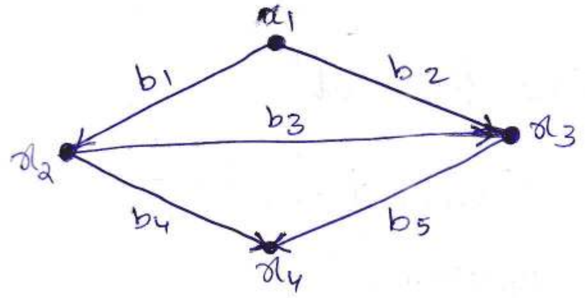
$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free colums

free colms

$$\left.\begin{aligned} y_1 - y_3 + y_5 &= 0 \\ y_2 + y_3 - y_5 &= 0 \\ y_4 + y_5 &= 0 \end{aligned}\right\} \begin{aligned} y_1 &= y_3 - y_5 \\ y_2 &= -y_3 + y_5 \\ y_4 &= -y_5 \end{aligned}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = y_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y_5 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$



$Ax = b$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

- The equations $Ax = b$ give "voltages" x_1, x_2, x_3, x_4 at the 4 ~~edges~~ nodes.

$$A^T y = 0 : \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ y_1 - y_3 - y_4 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $A^T y = 0$ give "currents" y_1, y_2, y_3, y_4, y_5 on the 5 edges.

These 2 equations are Kirchhoff's Voltage Law and Kirchhoff's Current Law.

"Flow into a node equals flow out"

→ Kirchhoff's Current Law is the balance equation.

This must be the most important equation in applied mathematics. All models in science & engineering & economics involve a balance — of force (or) heat flow (or) momentum (or) money

The balance equation + Hooke's law or Ohm's Law (or) some law connecting "potentials" to "flows", give a clear framework for applied mathematics.

Rank 2 matrices = Rank 1 + Rank 1

ILA(3) Rank: 1

Rank 2

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 7 \\ 4 & 2 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = CR$$

Basis for $C(A^T)$: $v_1^T = [1 \ 0 \ 3]$, $v_2^T = [0 \ 1 \ 4]$

Basis for $C(A)$: $u_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \text{zero row} \end{bmatrix} = u_1 v_1^T + u_2 v_2^T$$

$$\boxed{\text{rank } 2 = \text{rank } 1 + \text{rank } 1}$$

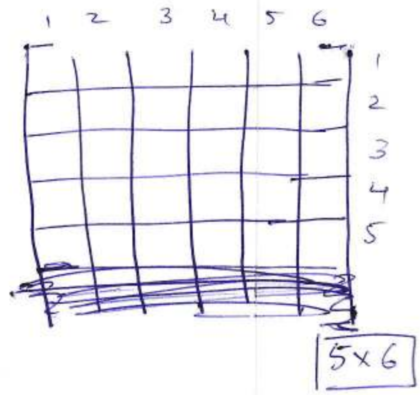
* Every $m \times n$ matrix of rank r reduces to $(m \times r)$ times $(r \times n)$

$$A = \begin{pmatrix} \text{pivot columns} \\ \text{of } A \end{pmatrix} \begin{pmatrix} 1^{st} \ r \text{ rows of } R \end{pmatrix}$$

$$\boxed{A = CR}$$

Think: Every column of A is a linear combination of columns of C & Every row of A is a linear combination of the rows of R.

$\boxed{3.5\,(A)}$ Put four 1's into a 5×6 matrix of zeros, keeping the dimension of its row space as small as possible. Describe all the ways to make the dimension of its column space as small as possible.
Describe all ways to make the dimension of its nullspace as small as possible.
How to make the sum of the dimensions of all 4 subspaces small?



Ans.
$C(A), C(A^T)$

four 1's go into the same row
(or) into the same column.

They can also go into 2 rows & 2 colums

$$a_{ii} = a_{ij} = a_{ji} = a_{jj} = 1$$

$N(A)$ $\dim[N(A)]_{min}$ when $n-r$ is small.

$$r_{max} \Longrightarrow r = 4.$$

1's must go into 4 different rows & 4 different columns.

Ans. Sum of the dim. of all 4 subspaces $= n+m = \underline{11}$.
no matter how 1's are placed.

You can't do anything about the sum.

$\boxed{3.1}$ Check for vector spaces: vector addition $x+y$ & ~~g scaly~~ scalar multiplication obey the rules

1. $(x_1, x_2) + (y_1 + y_2)$ is defined to be (x_1+y_2, x_2+y_1) with the usual multiplication $cx = (cx_1, cx_2)$,

Ans: 8 rules.

① $x+y = y+x$

② $x+(y+z) = (x+y)+z$

③ There is a unique "zero vector" such that $x+0=x$ for all x

④ For each x, there is a unique vector $-x$ such that $x+(-x)=0$.

⑤ $1x = x$

⑥ $(c_1c_2)x = c_1(c_2x)$

⑦ $c(x+y) = cx+cy$

⑧ $(c_1+c_2)x = c_1x+c_2x$

Ans: $x+y \neq y+x$

$x+(y+z) \neq (x+y)+z$

$(c_1+c_2)x \neq c_1x + c_2x$

2. Suppose the multiplication cx is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2) with the usual addition in R^2, are the 8 conditions satisfied?

Ans: $1x \neq x$

3. (a) Which rules are brocken if we keep only the +ve numbers $x>0$ in R^1? Every c must be allowed. The half-line is not a subspace

Ans: for $c \leq 0$, $cx \ngtr 0$., not closed under multiplication.

(b) The true numbers with $x+y$ and cx redefined to equal the usual xy and x^c do satisfy the 8 rules. Test rule 7 when $c=3$, $x=2$, $y=1$. Then $x+y=2$ & $cx=8$. Which # acts as the "zero vector"?

Ans: $c(x+y) = (xy)^c = x^c y^c = cx + cy$.

$c=3,\ x=2,\ y=1$

$3(2+1)=8$

zero vector is 1.

4. $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$. What matrices are in the smallest subspace containing A?

Ans: ~~All~~ all matrices cA.

5. (b) If a subspace of M does contain A and B, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, must it contain I?

Ans: Yup, $I = A - B$.

(c) Describe a subspace of M that contains no nonzero diagonal matrices.

Ans: $S = \text{span}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right) =$ matrices whose main diagonal is all zero.

6. The functions $f(x) = x^2$ & $g(x) = 5x$ are vectors in $\mathbb{F}$. This is the vector space of all real functions. (The functions are defined for $-\infty < x < \infty$). The combination $3f(x) - 4g(x)$ is the function $h(x) =$ ______

Ans: $h(x) = 3f(x) - 4g(x) = 3x^2 - 20x$

in the function space.

7. Which rule is brocken by multiplying $f(x)$ by c gives the function $f(cx)$?

Keep the usual addition $f(x) + g(x)$

Ans: $(c_1 + c_2) f(x) = f((c_1 + c_2)x)$

$\neq$

$c_1 f(x) + c_2 f(x) = f(c_1 x) + f(c_2 x)$

8. If the sum of the "vectors" $f(x)$ and $g(x)$ is defined to be the function $f(g(x))$, then the "zero vector" is $g(x) = x$. Keep the usual scalar multiplication $c f(x)$ and find 2 rules that are brocken.

Ans: $f(x) + g(x) = f(g(x)) \neq g(f(x)) = g(x) + f(x)$

Rule 4 is also brocken since there is no inverse function.

$f(x) + f^{-1}(x) = f(f^{-1}(x)) = x$

$f(x) + -f(x) = f(-f(x)) \neq 0$

Subspace requirements

9. One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbb{R}^2$ for when $x+y$ stays in the set but $\frac{1}{2}x$ may be outside

Ans: Vectors with integer components

(b) A set of vectors in $\mathbb{R}^2$ (other than 2 quarter planes) for which every cx stays in the set but $x+y$ may be outside

Ans: Remove x-axis but leave the origin, from the xy-plane.

10. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces.

(a) Plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

(b) Plane of vectors with $b_1 = 1$

(c) Vectors with $b_1 b_2 b_3 = 0$

(d) All linear combinations of $v = (1,4,0)$, $w = (2,2,2)$

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$

(f) All vectors with $b_1 \le b_2 \le b_3$.

Ans:

11. Describe the smallest subspace of the matrix space M that contains.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Ans: $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ Ans: $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans: All diagonal matrices.

15.
(c) If S & T are subspaces of $\mathbb{R}^5$, prove that their intersection $S \cap T$ is a subspace of $\mathbb{R}^5$. Here $S \cap T$ consists of the vectors that lie in both subspaces.
Check that $x+y$ and cx are in $S \cap T$ if x & y are in both spaces.

Ans:
$$x, y \in S \cap T \implies x, y \in S \;\&\; x, y \in T$$
$$\implies x+y \in S \;\&\; x+y \in T$$
$$\implies x+y \in S \cap T$$

$$x \in S \cap T \implies x \in S \;\&\; x \in T$$
$$\implies cx \in S \;\&\; cx \in T \implies cx \in S \cap T$$

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(a) The symmetric matrices in M $(A^T = A)$ form a subspace
$$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2$$

Ans (b) The skew-symmetric matrices in M form a subspace.

(c) The unsymmetric matrices in M (with $A^T \neq A$) do not form a subspace.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$ is symmetric.

22. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\& \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Ans: (a) $\rightarrow$ $\cancel{b_3} = x_3 = b_3, \; x_2 = b_2 - b_3,$

$|A| = 1 \neq 0 \implies x = A^{-1}b$

solution for all b.

(b) $|A| = 0$: $C(A)$ is xy-plane $\implies b_3 = 0$
(or in terms of rank)

(c) $|A| = 0$: $b_3 = b_2$

solvable only if

24. The columns of AB are combinations of the columns of A.
i.e., the column space of AB is contained in (possibly equal to) the column space of A.
Give an example where $C(A) = C(AB)$

Ans: $\underset{m\times n}{A}\,\underset{n\times r}{B} = \underset{m\times r}{P}$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}_{1\times n} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}_{n\times r} = \begin{bmatrix} \vec{P}_1 & \vec{P}_2 & \cdots & \vec{P}_r \end{bmatrix}_{1\times r}$$

$$(AB)_i = P_i = \vec{a}_1 b_{11} + \vec{a}_2 b_{21} + \cdots + a_n \vec{b}_{n1}$$

for all $i \in \{1, 2, \ldots n\}$.

$\rightarrow$ $\boxed{C(AB) \in C(A)}$

Example: $B = 0, \; A \neq 0$

$AB = 0$,

27. True / False

(d) The column space of $A - I$ equals the column space of A.

Ans: False $\quad C(A-I) \neq C(A)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A - I = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

28. Construct a 3×3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$.

Ans: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ do not have $(1,1,1)$ in $C(A)$.

30. Suppose S & T are a subspaces of a vector space V,

$$\boxed{\text{The sum } S+T \text{ contains all sums } s+t \text{ of a vector } s \text{ in } S \text{ and a vector } t \text{ in } T.}$$

(a)

Show that $S+T$ satisfies the requirements for a vector space.

Ans: If $u, v \in S+T$

$u = s_1 + t_1$ & $v = s_2 + t_2$

$u+v = (s_1+s_2) + (t_1+t_2) \in S+T$

& $cu = cs_1 + ct_1 \in S+T$

$S+T$ is a subspace

(b). If S & T are lines in $\mathbb{R}^n$, what is the difference b/w S+T and S∪T ?

That union contains all vectors from S or T or both.

Explain: The span of S∪T is S+T

Ans: S & T are different lines

S∪T is just the 2 lines (not a subspace)

S+T is the whole plane that they span.

31. If $S = C(A)$, $T = C(B)$, then $S+T$ is the column space of what matrix M?
The columns of A & B & M are all in $\mathbb{R}^m$.

Ans: $M = \begin{bmatrix} A & B \end{bmatrix}$

$\boxed{3.2}$

6 Put as many 1's as possible in a 4×7 echelon matrix U whose pivot columns are

(a) 2, 4, 5

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) 1, 3, 6, 7

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) 4, 6

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. Put as many 1's as possible in a 4×8 reduced echelon ~~ma~~ matrix R so that the free columns are

(a) 2, 4, 5, 6 $\Longrightarrow$ 1, 3, 7, 8

Ans: $$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & & 0 & 1 \end{bmatrix}$$

(b) $1, 3, 6, 7, 8 \Longrightarrow 2, 4, 5$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

8. Suppose column 4 of a 3×5 matrix is all zero. Then x_4 is certainly ________ variable. The special solution for this variable is the vector $x=$ ____________

Ans: x_4 is a free variable

$N(A) = N(R)$

$$R = \begin{bmatrix} 1 & \cdot & & \boxed{0} & \\ 0 & 1 & & \boxed{0} & \\ 0 & & & \boxed{0} & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

x_i, x_4 are free variables

pivot variables $x_j = 0$, ie 3 pivots.

$x_4 = 1$, $x_i = 0$

$s_1 = (0, 0, 0, 1, 0)$

9. Suppose the 1^{st} & last columns of a 3×5 matrix are the same (not zero). Then ______ is a free variable. Find the special solution for this variable.

Ans: x_5 is a free variable. $\leftarrow$ colum 1 = column 5

$x_1 + x_5 = 0 \implies x_5 = 1, \; x_1 = -1$

$s = (-1, 0, 0, 0, 1)$

10. Suppose an $\boxed{m\times n}$ matrix has r pivots. The # of special solutions is $\underline{\;n-r\;}$.

The nullspace contains only $x=0$ when $\underline{r = n}$

The column space is all of $\mathbb{R}^m$ when $r = \underline{\;m\;}$

14. Suppose column 1 + column 3 + column 5 = 0 in a 4×5 matrix with 4 pivots. Which column has no pivot? What is the special solution? Describe $N(A)$

Ans: column 5 is a combination of colm 1 & col. 3.

$\Longrightarrow$ col. 5 have no pivot

$x_5 = 1,$ $x_2 = 0$

$x_1 + x_3 + x_5 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - x_5 &= 0 \\ x_2 &= 0 \\ x_3 - x_5 &= 0 \\ x_4 &= 0 \end{aligned}\right\} S = (1, 0, 1, 0, 1)$$

15. Construct a matrix for which $N(A)$ = all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$

Ans: free variables = x_3, x_4.

$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$$

$2+a=0$ & $2+c=0$ } $a=-2,\ b=-3$

$3+b=0$ & $1+d=0$ } $c=-2,\ d=-1$

$$R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$\Rightarrow$ A can be any invertible 2×2 matrix times R

16. Construct A so that N(A) = all multiples of (4,3,2,1).
- Its rank is ______

Ans: $n-r=1$, $A_{m\times n}$, $n=4$

$\underline{r=3}$

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = 0$$

$4+a=0$ } $a=-4$

$3+b=0$ } $b=-3$

$2+c=0$ } $c=-2$.

$$R = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

A = any invertible 3×3 matrix times R

17. Construct a matrix whose columnspace contains (1,1,5), & (0,3,1) and whose nullspace contains (1,1,2)

Ans: 3×3.

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

18. Construct a matrix whose column space contains (1,1,0) & (0,1,1) and whose nullspace contains (1,0,1) and (0,0,1)

Ans: $\boxed{3\times 3}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r \geq 2$

$n - r = 2 = 3 - r$

$r = 1$

Not possible

19. Construct a matrix whose column space contains (1,1,1) & whose nullspace is the line of multiples of (1,1,1,1)

Ans: 3×4

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

20. Construct a 2×2 matrix whose nullspace equals its column space. This is possible.

$\Rightarrow 2 - r \le 2 \Rightarrow r \ge 0$

Ans: $\cancel{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ $n - r = r$

$2r = n = 2$

$\Rightarrow r = 1$

$N(A) = C(A)$

$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$

21. Why does no 3×3 matrix have a nullspace that equals its column space?

Ans: $\cancel{3 - r}$ $n - r = r \Rightarrow 2r = n = 3$

$\boxed{r = \frac{3}{2}}$, Not possible.

22. If $AB = 0$, then the $C(B)$ is contained in the ________ of A. Why?

Ans: $\cancel{AB = [a_1\, a_2 \cdots a_n]\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = [(\vec{ab})_1, (\vec{ab})_2, \ldots, (\vec{ab})_n] = 0}$

$AB = 0$

$A(Bx) = Ab = 0$

$N(A)$ = is all $Bx = b$ for which $ABx = Ab = 0$

$$AB = A\left[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n\right] = \left[A\vec{b}_1, A\vec{b}_2, \cdots, A\vec{b}_n\right] = 0$$

$$A\vec{b}_1 = A\vec{b}_2 = \cdots = A\vec{b}_n = 0$$

there can be other $c_i \in N(A)$ that $Ac_i = 0$ $\Rightarrow$

$$\boxed{AB = 0 \text{ iff } C(B) \text{ is a subspace of } N(A)}$$

26 If the special solutions to $Rx=0$ are in the columns of these nullspace matrices N, go backward to find the non-zero rows of the reduced matrices. R:

$$N=\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad N=\begin{bmatrix} \ \end{bmatrix} \text{ empty } 3\times 1$$

Ans:

(a) $m\times 3$

$3-r=2 \Rightarrow \boxed{r=1} \Longrightarrow \begin{bmatrix} 1 & a & b \end{bmatrix}$

$2+a=0$ & $3+b=0 \longrightarrow a=-2, b=-3$

$\therefore \begin{bmatrix} 1, -2, -3 \end{bmatrix}$

(b) $3-r=1 \longrightarrow \boxed{r=2} \Rightarrow \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \end{bmatrix}$

~~$b=0$, $c=0$~~

(c) $3-r=0 \longrightarrow \boxed{r=3} \Rightarrow R=I_{3\times 3}$

27. What are the five 2x2 reduced matrices R
(a) whose entries are all 0's and 1's ?

Ans: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

28. Why A & −A always have the same reduced echelon form R.

Ans: $N(A) = N(-A)$

$C(A^T) = C((-A)^T)$

$C(A) = C(-A)$ But that is not required for 2 matrices to share the same R.

29. If A is 4×4 & invertible, describe the nullspace of the 4×8 matrix $B=[A\ \ A]$

Ans: $$By=\begin{bmatrix}A & A\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=Ay_1+Ay_2=0$$

$$Ay_1=-Ay_2 \implies y_1=-y_2$$

$N(B)$ is all vectors $x=\begin{bmatrix}t\\ -t\end{bmatrix}$ for $t\in\mathbb{R}^4$.

30. How is $N(C)$ related to $N(A)$ and $N(B)$, if $C=\begin{bmatrix}A\\ B\end{bmatrix}$

$A_{m\times n}$, $B_{p\times n}$, C is $(m+p, n)$

Ans: $$Cx=\begin{bmatrix}A\\ B\end{bmatrix}x=\begin{bmatrix}Ax\\ Bx\end{bmatrix}=0 \quad \text{iff } \begin{matrix}Ax=0\ \&\\ Bx=0\end{matrix}$$

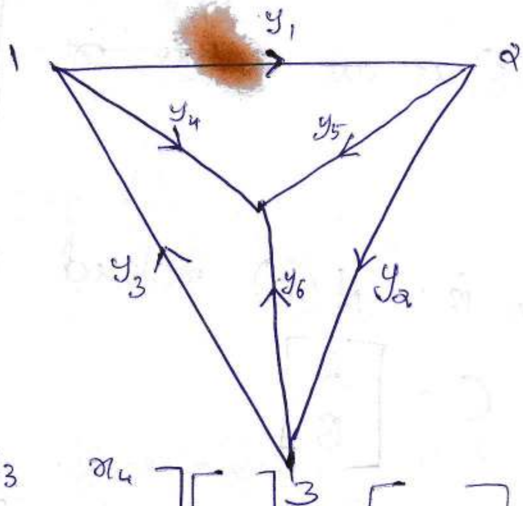
$$\underline{N(C)=N(A)\cap N(B)}$$

32. Kirchhoff's current law $A^T y = 0$ says that current in = current out at every node.

At node 1, this is $y_3 = y_1 + y_4$. Write the 4 equations for Kirchhoff's law at the 4 nodes.

Reduce A^T to R and find 3 special solutions in $N(A^T)$.

~~Ans~~



$$A = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$Ax = y$$

Ans: $A^T y = 0$

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = 0.$$

$$\boxed{\begin{aligned} y_1 - y_3 + y_4 &= 0 \\ -y_1 + y_2 + y_5 &= 0 \\ -y_2 + y_4 + y_6 &= 0 \\ -y_4 - y_5 - y_6 &= 0 \end{aligned}}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & +1 & +1 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free colums

y_3, y_5, y_6

$$y_1 - y_3 - y_5 - y_6 = 0$$
$$y_2 - y_3 - y_6 = 0$$
$$y_{\cancel{5}4} + y_5 + y_6 = 0$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = y_3 \begin{bmatrix} +1 \\ +1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y_5 \begin{bmatrix} +1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + y_6 \begin{bmatrix} +1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ flows around loop

34. Find reduced R for each of these block matrices.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} A & A \end{bmatrix}, \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

Ans: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R_A = EA$

$$B = \begin{bmatrix} A & A \end{bmatrix} \Longrightarrow R_B = \begin{bmatrix} R_A & R_A \end{bmatrix}$$

$$C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix} \Longrightarrow R_C = \begin{bmatrix} R_A & 0 \\ 0 & R_A \end{bmatrix}$$

40. If 'A' is an $m \times n$ matrix with $r = 1$, its columns are multiples of one column & its rows are multiples of one row. The column space is a <u>line</u> in $\mathbb{R}^m$.

~~The~~

The null space is a <u>hyper plane</u> in $\mathbb{R}^n$.

The null space matrix N is n by $(n-1)$

43. If A has rank r, then it has an $r \times r$ submatrix S that is invertible.

38. What are the special solutions to $Rx=0$ & $y^TR=0$ for

(a) $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 + 2x_3 + 3x_4 = 0$

$x_2 + 4x_3 + 5x_4 = 0$

Ans: $\cancel{R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

$R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$

(b) $R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow x_2 + 2x_3 = 0$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

40, 43

$$\boxed{\text{rank}(AB) \le \min\left(\text{rank}(A), \text{rank}(B)\right)}$$

46. Find the ranks of AB & AC.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}$$

Ans: $r_A = 1, \; r_B = 1, \; r_C = 1$

$r_{AB} \le 1, \quad r_{AC} \le 1$

$r_{AB} = 0$ (or) 1 | $r_{AC} = 0$ (or) 1

47. The rank 1 matrix uv^T times the rank 1 matrix wz^T is uz^T times the # ______ This product uv^Twz^T also has rank 1 unless ______ $= 0$.

Ans: $uv^Twz^T = u(v^Tw)z^T = u(v \cdot w)z^T$

$= (v \cdot w)\, uz^T = (uz^T)\, \underline{\underline{(v^Tw)}}$

$= kuz^T$

has rank 1 unless $k = v \cdot w = v^T w = 0$.

48. (a) $\text{rank}(AB) \le \text{rank}(A)$

$\text{rank}(AB) \le \text{rank}(B)$

Check OM(23) for proof $\Longrightarrow$ $\text{rank}(AB) \le \min\left(\text{rank}(A), \text{rank}(B)\right)$

(b) Find A_1 & A_2 so that $\text{rank}(A_1B) = 1$ and $\text{rank}(A_2B) = 0$ for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Ans: $\text{rank}(B) = 1$

$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $A_1B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$

$\text{rank}(A_1B) = 1$..

$|A_1| \ne 0 \Longrightarrow \text{rank}(A_1B) = \text{rank}(B)$

$A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $A_2B = \begin{bmatrix} 1 & -1 \\ +1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

50. Suppose A & B are $n \times n$ matrices, and $AB = I$. Prove from $\text{rank}(AB) \le \text{rank}(A)$ that the rank of A is n. So A is invertible & B must be its 2-sided inverse. $\therefore BA = I$

Ans: $A_{n\times n} B_{n\times n} = I_n$.

$$\text{rank}(AB) = \text{rank}(I_n) = n \le \text{rank}\, A \le n.$$

$$\Longrightarrow \underline{\underline{\text{rank}(A) = n}}$$

$\therefore$ A is invertible $\Rightarrow AB = I \longrightarrow BA = I$

$$ABA = A$$

$$A^{-1}(ABA) = I$$

51. If 'A' is 2×3 and B is 3×2 and $AB = I$, show from its rank that $BA \ne I$. Give an example of A & B with $AB = I$. For $m<n$, a right inverse is not a left inverse

Ans: $r_A, r_B \le 2$ & $\text{rank}(AB) = \text{rank}(I) = 2$

$\text{rank}(BA) \le 2$ $\qquad (BA)_{3\times 3} \ne I$

$\text{rank}(I_3) = 3$ $\qquad 2 \qquad (BA)$

52. If A & B have the same R

(b) $E_1 A = R$ & $E_2 B = R$. So A equals ______ matrix times B.

Ans: $E_1 A = E_2 B \implies A = E_1^{-1} E_2 B$

- A equals an invertible matrix times B, when they share the same R.

53. Express A & B as the sum of 2 rank 1 matrices :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$$

Ans (a)

$$A = LU \Longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ 0 \end{bmatrix} = u_1 v_1^T + u_2 v_2^T$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \text{[redacted]} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

$$= u_1 v_1^T + u_2 v_2^T$$

$$= \begin{bmatrix} \text{[redacted]} \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(OR)

$$B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$$

(55) What's the nullspace matrix N (containing the special solutions) for A, B, C?

Block matrices: $A = \begin{bmatrix} I & I \end{bmatrix}$, $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} I & I & I \end{bmatrix}$

Ans: A: n pivots n free.

$$N_A = \begin{bmatrix} I \\ -I \end{bmatrix}$$

$$N_B = \begin{bmatrix} I \\ -I \end{bmatrix}$$

C: $(n) \times (3n)$: n pivots & 2n free. columns

$$N_C = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}$$

56) Every $m \times n$ matrix of rank r reduces to $(m \times r)$ times $(r \times n)$.

$$A = (\text{pivot columns of } A)\left(1^{st} \text{ non-zero } \cancel{\text{pivot}} \text{ (pivot) rows of } R\right)$$

$$= (\text{col})(\text{row})$$

?

57. Suppose 'A' is an $m \times n$ matrix of rank r. Its reduced echelon form is R. Describe exactly the matrix Z that comes from transposing the reduced row echelon form of R^T.

?

Ans: $R = \text{rref}(A)$

$$Z = \left(\text{rref}\left(R^T\right)\right)^T$$

$$R = \begin{bmatrix} I & A' \\ 0 & 0 \end{bmatrix}_{m \times n} \longrightarrow R^T = \begin{bmatrix} I & 0 \\ A' & 0 \end{bmatrix}_{n \times m}$$

$$\text{rref}(R^T) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}_{n \times m}$$

$$Z = \left(\text{rref}(R^T)\right)^T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}_{m \times n}$$

∴ ~~rref of R^T is an~~ Z is an $m \times n$ matrix that has 1's on 1st r diagonal places, & all other elements are zero.

3.3

7. Show by elimination that (b_1, b_2, b_3) in $C(A)$ if ________

What combination of the rows of A gives the zero row?

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}$$

Ans:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right] - 2b_2 + 6b_1 = \underline{4b_1 - 2b_2 + b_3 = 0}$$

$$\underline{4(\text{row } 1) - 2(\text{row } 2) + (\text{row } 3) = 0}$$

8. Which vectors (b_1, b_2, b_3) are in $C(A)$?

Which combinations of rows of A give 0?

Ans: (a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 6 & 3 & b_2 \\ 0 & 2 & 5 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2 - 2b_1 \\ 0 & 0 & 4 & 2b_1 - b_2 + b_3 \end{array}\right]$$

independent rows.

Only the zero combination gives 0.

10. Construct a 2×3 system $Ax=b$ with particular solution $x_p=(2,4,0)$ & homogeneous solution $x_n=$ any multiple of $(1,1,1)$

Ans: $n=3,\ n-r=3-r=1 \Rightarrow r=2$

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}2\\4\\0\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$$

11. Why can't a 1×3 system have $x_p=(2,4,0)$ & $x_n=$ any multiple of $(1,1,1)$

Ans: $\begin{bmatrix}1 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\ 0$

$\begin{bmatrix}1 & 1 & -2\end{bmatrix}\begin{bmatrix}2\\4\\0\end{bmatrix}=6$

$n-r=3-r=1$

$\Rightarrow r=3-1=2$

N.P

12. (a) If $Ax=b$ has 2 solutions x_1 & x_2, find 2 solutions to $Ax=0$

Ans: x_1, x_2 are solutions.

$\implies x_1 - x_2 = 0$ is a solution.

$Ax=0$ iff $x=0$

(b). Find another solution to $Ax=0$ & another solution to $Ax=b$.

Ans: $A(10x_1 - 10x_2) = 0$

$A(2x_1 - x_2) = b$

15. Suppose, row 3 of U has no pivot. Then that row is <u>a zero row.</u> The equation $Ux=c$ is only solvable provided <u>$c_3 = 0$</u>

19. $\boxed{\text{rank}(A^T A) = \text{rank}(A A^T) = \text{rank}(A) = \text{rank}(A^T)}$

Proof

Let $x \in N(A) \Longrightarrow Ax = 0 \longrightarrow A^T A x = 0$

$\Longrightarrow x \in N(A^T A)$

$\therefore N(A) \subseteq N(A^T A)$

If $x \in N(A^T A) \Longrightarrow A^T A x = 0$

$\Rightarrow x^T A^T A x = 0 \Longrightarrow (Ax)^T (Ax) = 0$

$(Ax) \cdot (Ax) = |Ax|^2 = 0 \longrightarrow Ax = 0 \Longrightarrow x \in N(A)$

$\therefore N(A^T A) \subseteq N(A)$

$\Rightarrow N(A^T A) = N(A)$

$\Rightarrow \dim[N(A^T A)] = \dim[N(A)]$

rank-nullity theorem $\Rightarrow n - r_1 = n - r_2 \Rightarrow r_1 = r_2$

$\Rightarrow \text{rank}(A^T A) = \text{rank}(A)$

Q1. Find the complete solution in the form $x_p + x_n$ to these full rank systems :

(a) $x+y+z=4$

Ans : $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4$

$(y,z) = (1,0) \Rightarrow x = 3$

$(y,z) = (0,1) \Rightarrow x = 3$

$(y,z) = (0,0) \Rightarrow x = 4$

$x+y+z=4$

$n - r = 3 - 1 = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Q3. Choose the # q so that (if possible) the ranks are (a) 1 (b) 2 (c) 3 :

(i) $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$

Ans: $|A| = 6[-2q+6] - 4[-3q+9] + \cancel{2[-18+18]}$

$= -12q + 36 + 12q - 36 = 0$

$\begin{vmatrix} -2 & -1 \\ 6 & q \end{vmatrix} = -2q + 6 = 2(-q+3)$

$q = 3$ gives rank [illegible], all other q gives rank 1.

(ii) $B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$

Ans: $q = 6$ gives rank 1, every other q gives rank 2.

rank 3 N.P.

Q4. Give examples of matrices A for which the # of solutions to $Ax = b$ is:

(a) 0 (or) 1 depending on b

Ans: $r = n < m$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$r = 1 < 2 = m$

(b) ∞ regardless of b

Ans: $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$

$\dim(\mathcal{N}(A)) > 0$

$n - r > 0$

$\Rightarrow r < n$ & $r = m$

$\underline{\underline{r = m < n}}$

(c). 0 (or) ∞ depending on b

Ans: $n - r > 0$ &

$n > r$ $\quad r < m$

Ex: $A =$ zero matrix / null matrix

$[0]$ has rank $= 0$, $m = n = 1$

(d) 1, regardless of b

Ans: A is square & invertible.

31. Find matrices A & B with the given property (or) explain why you can't

(a) The only solution of $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Ans: $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\quad \underline{n = \cancel{3} = r}$.

$\boxed{r_{[A]} = r_{[A|b]} = n}$

(b) The only solution of $Bx = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Ans: $\begin{bmatrix} & & \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1}$

$r_{[B|b]} = r_{[B]} = n = 3$. Not possible since $\underline{r \leq 2}$

33. The complete solution to $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find A

Ans: $x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A_{2\times 2}\, x_{2\times 1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ & $n - r = 2 - r = 1 \Rightarrow \boxed{r = 1}$

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

34. The 3×4 matrix A has the vector $s = (2, 3, 1, 0)$ as the only solution to $Ax = 0$.

(a) What is the rank of A & the complete solution to $Ax = 0$?

Ans: ~~$r_{[A]} = r_{[A|b]} = n$, $r = 4$~~ $\quad n - r = 4 - r = 1$

$$\Longrightarrow r = 3$$

$x = cx$ i.e.; line of solutions.

(b) What is the exact row reduced echelon form R of A?

Ans: all 3 rows are pivot rows.

x_3 is the free variable.

$$R = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 1 \end{bmatrix} \quad \left\{ \begin{array}{l} 2 + a = 0 \\ 3 + b = 0 \\ c = 0 \end{array} \right\}.$$

$$= \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) How do you know that $Ax=b$ can be solved for all b?

Ans: $A_{3\times 4}$ & $r_A = 3$.

$r = m < n. \Longrightarrow r_{[A|b]}$ can't exceed 3.

35. Suppose K is the 9×9 second difference matrix (2's on the diagonal, -1's on the diagonal above & also below).
Solve the eq^n. $Kx = b = (10, \ldots, 10)$.
If you graph $x_1, \ldots, x_9$ above the points $1, \ldots, 9$ on the x-axis, I think the 9 points fall on a parabola.

Ans:
$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cancel{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(c) How do you know that $Ax=b$ can be solved for all b?

Ans: $A_{3\times 4}$ & $r_A = 3$.

$r = m < n. \Longrightarrow r_{[A|b]}$ can't exceed 3.

35. Suppose K is the 9×9 second difference matrix (2's on the diagonal, -1's on the diagonal above & also below).
Solve the eq^n. $Kx = b = (10, \dots, 10)$.
If you graph $x_1, \dots, x_9$ above the points $1, \dots, 9$ on the x-axis, I think the 9 points fall on a parabola.

Ans:
$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$|K| = 10 \Longrightarrow x = A^{-1}b$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6/5 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7/6 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8/7 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9/8 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10/9 \end{bmatrix} = U_k$$

$$|K| = |U_k| = \cancel{2} \times \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \frac{\cancel{6}}{\cancel{5}} \times \frac{\cancel{7}}{\cancel{6}} \times \frac{\cancel{8}}{\cancel{7}} \times \frac{\cancel{9}}{\cancel{8}} \times \frac{10}{\cancel{9}}$$

$= 10$

$x = K^{-1}b = (45, 80, 105, 120, 125, 120, 105, 80, 45)$

$y = ax^2 + bx + c$

$$\left.\begin{aligned} 45 &= a+b+c \\ 80 &= 4a+2b+c \\ 105 &= 9a+3b+c \end{aligned}\right\} \quad \begin{aligned} 3a+b &= 35 \\ 5a+b &= 25 \\ \hline 2a &= -10 \implies \underline{\underline{a = -5}} \end{aligned}$$

$b = 25 + 25 = 50.$

$c = 45 - 45 = 0.$

lies along the parabola

$$\underline{\underline{x_i = 50i - 5i^2}}.$$

$x = K^{-1}b = (45, 80, 105, 120, 125, 120, 105, 80, 45)$

$y = ax^2 + bx + c$

$$\left.\begin{aligned} 45 &= a+b+c \\ 80 &= 4a+2b+c \\ 105 &= 9a+3b+c \end{aligned}\right\} \quad \begin{aligned} 3a+b &= 35 \\ 5a+b &= 25 \\ \hline 2a &= -10 \implies \underline{\underline{a = -5}} \end{aligned}$$

$b = 25 + 25 = 50.$

$c = 45 - 45 = 0.$

lies along the parabola

$$\underline{\underline{x_i = 50i - 5i^2}}$$

36. If $Ax=b$ & $Cx=b$ have the same (complete) solutions for every b. Is it true that A equals C ?

Ans: $Ax=b$ & $Cx=b$

A, C have same shape

$N(A) = N(C)$

If $b =$ column 1 of A,

$x=(1,0,\ldots,0)$ solves $Ax=b$.

$\Rightarrow x=(1,0,\ldots,0)$ also solves $Cx=b$.

$\therefore$ A & C share column 1.

$\Longrightarrow \underline{\underline{A = C}}$

3.4

7. If w_1, w_2, w_3 are independent vectors, show that the differences $v_1 = w_2 - w_3$ and $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$ are dependent. Find a combination of the v's that gives zero. Which matrix A in $[v_1\ v_2\ v_3] = [w_1\ w_2\ w_3] A$ is singular.

Ans: $v_1 - v_2 + v_3 = 0 \implies v_1, v_2, v_3$ are dependent

$v_1 = w_2 - w_3$

$v_2 = w_1 - w_3 \implies [v_1\ v_2\ v_3] = [w_1\ w_2\ w_3] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

$v_3 = w_1 - w_2$

$$\boxed{V = WA}$$

$|A| = 0$

8. If w_1, w_2, w_3 are independent vectors, show that the sums $v_1 = w_2 + w_3$ & $v_2 = w_1 + w_3$ & $v_3 = w_1 + w_2$ are independent.

Ans: $c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1 (w_2 + w_3) + c_2 (w_1 + w_3) + c_3 (w_1 + w_2) = 0$

$$(c_2 + c_3) w_1 + (c_1 + c_3) w_2 + (c_1 + c_2) w_3 = 0$$

W's are independent $\longrightarrow$ $c_1 + c_2 = c_2 + c_3 = c_3 + c_1 = 0$

$$c_2 = -c_1, \quad c_3 = -c_1$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = c_1 [v_1 - v_2 - v_3]$$
$$= c_1 [-2w_1] \neq 0.$$

The only solution is $c_1 = c_2 = c_3 = 0$.

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\boxed{V = WA}$$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1+1 = 2 \neq 0.$$

9. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^3$.

(a) These 4 vectors are dependent because

Ans: $A_{3\times 4}\, x = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} x = 0.$

$r \leq m < n \Rightarrow \underline{n - r \geq n - m > 0}$

$-r \geq -m$

$n - r > 0$

$\dim(N(A)) > 0$

$\therefore$ there is atleast 1 free variable.

(b) The 2 vectors v_1 & v_2 will be dependent if ______.

Ans: $[v_1 \ v_2]$

$$2 - r > 0 \Rightarrow r < 2$$

$$r = 0 \text{ (or) } 1$$

(c) The vectors v_1 & $(0,0,0)$ are dependent because ______

Ans: $0v_1 + 10(0, 0, 0) = 0$

for a non trivial combination of v_1.

10. Find 2 independent vectors on the plane $x + 2y - 3z - t = 0$ in $\mathbb{R}^4$. Then find 3 independent vectors. Why not 4?

Ans: $x + 2y - 3z - t = 0$ is $N(A)$,

$$Ax = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

12. The vector c is in the row space of A when __________ has a solution

Ans: $A^T y = c$ has a solution

14. $\underline{v+w}$ and $\underline{v-w}$ are combinations of $\underline{v}$ and $\underline{w}$. Write v and w as combinations of $v+w$ and $v-w$. The 2 pairs of vectors __________ the same space. When are they a basis for the same space?

Ans: $v = \frac{1}{2}(v+w) + \frac{1}{2}(v-w)$

$w = \frac{1}{2}(v+w) - \frac{1}{2}(v-w)$

$$\left. \begin{pmatrix} v+w,\ v-w \in \text{span}(v, w) \\ v, w \in \text{span}(v+w,\ v-w) \end{pmatrix} \right\} \text{span}(v, w) = \text{span}(v+w,\ v-w)$$

$\Rightarrow$ 2 pairs span the same space

They are a basis when v, w are independent.

$$\begin{bmatrix} v+w \\ v-w \end{bmatrix} = \cancel{\begin{bmatrix} v \\ w \end{bmatrix}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

$$c_1(v+w) + c_2(v-w) = 0$$

$$v(c_1+c_2) + w(c_1-c_2) = 0$$

~~If v, w are indepen~~

If $v+w$ & $v-w$ are independent,

$$c_1 \mp c_2 = 0 \Longrightarrow c_1 + c_2 = c_1 - c_2 = 0$$

v & w are independent.

16. Find a basis for each of these subspaces of $\mathbb{R}^4$

(b) All vectors whose components add to zero.

Ans:

$\Updownarrow$

Set of vectors $x = (x_1, x_2, x_3, x_4)$ satisfying

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Null space of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) All vectors that are $\perp$ to $(1,1,0,0)$ and $(1,0,1,1)$

Ans: All vectors $\perp$ to $(1\,1\,0\,0)$ & $(1\,0\,1\,1)$ constitute the null space of $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

solutions to the eq^n: $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} = R$$

free colums

$x_1 + x_3 + x_4 = 0$

$x_2 - x_3 - x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

17. Find 3 different bases for the column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$. Then find 2 different bases for the row space of U

Ans: $C(U) = \mathbb{R}^2$

take any bases for $\mathbb{R}^2$.

$C(U^T)$ is a plane in $\mathbb{R}^4$.

Bases : $\left\{ (1 \; 0 \; 1 \; 0 \; 1), (0 \; 1 \; 0 \; 1 \; 0) \right\}$

$\{$row 1 , row 2$\}$

(or) $\left\{ (1 \; 1 \; 1 \; 1 \; 1), (1 \; -1 \; 1 \; -1 \; 1) \right\}$

$\{$(row 1 + row 2) , (row 1 − row 2)$\}$

20- Find a basis for the plane $x-2y+3z=0$ in R^3. Then find a basis for the intersection of that plane with the xy-plane. Then find a basis for all vectors $\perp$ to the plane

Ans: The plane $x-2y+3z=0$ is the nullspace of the matrix $A=\begin{bmatrix}1 & -2 & 3\end{bmatrix}$

$$\begin{bmatrix}1 & \underbrace{-2 \quad 3}_{\text{free cols.}}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad x-2y+3z=0$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=y\begin{bmatrix}2\\1\\0\end{bmatrix}+z\begin{bmatrix}-3\\0\\1\end{bmatrix}$$

Basis for the plane $=\{(2,1,0),(-3,0,1)\}$.

Intersection with xy-plane, i.e., $z=0$.

Basis $=\{(2,1,0)\}$.

$\perp$ to the plane $\Longleftrightarrow$ $\perp$ to both the vectors $(2,1,0)$ & $(-3,0,1)$

Nullspace of $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

i.e.,

solutions to $\begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{bmatrix}$

$x - \frac{1}{3}z = 0$

$y + \frac{2}{3}z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} +\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

24. True/False

(a) If the columns of a matrix are dependent so are the rows

Ans: False,

Ex:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}$

25. For which numbers c & d do these matrices have rank 2?

(a) $A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$

Ans: $\begin{vmatrix} 2 & 5 & 0 \\ 0 & c & 2 \\ 0 & 0 & d \end{vmatrix} = d(2c) = 0$

$\begin{vmatrix} 2 & 5 & 5 \\ 0 & c & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2(2c) = 0 \Longrightarrow c = 0$

$\begin{vmatrix} 1 & 0 & 5 \\ 0 & 2 & 2 \\ 0 & d & 2 \end{vmatrix} = 4 - 2d = 0 \Longrightarrow \boxed{d = 2}$

(b) $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

Ans: $|B| = c^2 - d^2$

B has rank 2 except when $c = d$ (or) $c = -d$

26. Find a basis for each of these subspaces of 3×3 matrices:

(a) All diagonal matrices

Ans: dimension = 3

Basis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) All symmetric matrices

Ans: dimension = $1+2+3 = 6$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) All skew-symmetric matrices.

Ans: dimension = 3.

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

27. Construct six linearly independent 3x3 echelon matrices $U_1, \ldots, U_6$

Ans: I, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

28. Find a basis for the space of all 2×3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero

Ans:

(a) $\begin{bmatrix} a & b & c \\ -a & -b & -c \end{bmatrix}$

(OR) $\begin{bmatrix} \cancel{1} & \cancel{1} \end{bmatrix}\begin{bmatrix} \cancel{x} \\ y \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Bases : $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(b) 2×3 matrices whose rows add to zero.

$\begin{bmatrix} a & b & -a-b \\ x & y & -x-y \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \longrightarrow x+y+z=0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

28. Find a basis for the space of all 2×3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero

Ans:

(a) $\begin{bmatrix} a & b & c \\ -a & -b & -c \end{bmatrix}$

(OR) $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Bases : $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(b) 2×3 matrices whose rows add to zero.

$\begin{bmatrix} a & b & -a-b \\ x & y & -x-y \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \longrightarrow x+y+z=0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Basis: $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(c) both columns & rows add to zero

Ans:

Basis: $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

29. What subspace of 3x3 matrices is spanned by

(a) the invertible matrices

Ans: span the space of all 3x3 matrices

(b) the rank 1 matrices

Ans: span the space of all 3x3 matrices

(c) the identity matrix

Ans: span the space of all multiples cI

30. Find a basis for the space of 2x3 matrices whose null space contains $(2,1,1)$

Ans: $$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow 2a+b+c=0$$

$$\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$\Longrightarrow \text{span}\{(-1, 2, 0), (-1, 0, 2)\}$

Basis : $\begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 2 \\ -1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}$

dimension : 4

31. (a) Find all functions that satisfy $\frac{dy}{dx} = 0$

Ans: $y(x) = C$ (constant)

(b) Choose a particular function that satisfy $\frac{dy}{dx} = 3$

Ans: $y(x) = 3x$

(c) Find all functions that satisfy $\frac{dy}{dx} = 3$

Ans: $y(x) = 3x + C$

---

32. The cosine space $\mathbb{F}_3$ contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$.
Find a basis for the subspace with $y(0) = 0$

Ans: $y(0) = A + B + C = 0$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = 0 \longrightarrow$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = B \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis: $y(x) = \cos x - \cos 2x$, $y(x) = \cos x - \cos 3x$

33. Find a basis for the space of functions that satisfy

(a) $\frac{dy}{dx} - 2y = 0$

Ans: $\frac{dy}{y} = 2dx \Longrightarrow \log y = 2x + c.$

$$y = Ke^{2x}$$

Basis $y(x) = e^{2x}$.

(b) $\frac{dy}{dx} - \frac{y}{x} = 0$

Ans: $\frac{dy}{y} = \frac{dx}{x} \Longrightarrow \log y = \log x + c$

$$y = Kx$$

Basis : $y = x$.

First order linear equation $\Longrightarrow$ 1 basis function in solution space.

34. Say,
$y_1(x), y_2(x), y_3(x)$ are 3 different functions of x.
The vector space they span could have dimensions 1, 2 (or) 3. Give an example to show each possibility

Ans: $y_1(x), y_2(x), y_3(x)$ can be

dim (1) : $x, 2x, 3x$

dim (2) : $x, 2x, x^2$

dim (3) : x, x^2, x^3.

35. Find a basis for the space of polynomials $p(x)$ of degree ≤ 3. Find a basis for the subspace with $p(1) = 0$

Ans: polynomial of degree 3 : $ax^3 + bx^2 + cx + d = p(x)$

Basis : $1, x, x^2, x^3$

ie, $p(x) = 1$, $p(x) = x$, $p(x) = x^2$, $p(x) = x^3$

$x.$

$P(1) = a+b+c+d = 0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Basis for the subspace with $P(1)=0$ $\Big\}$ $-1+x,\ -1+x^2,\ -1+x^3$

(or)

$\underline{P(x) = x-1,\ P(x) = x^2-1,\ P(x) = x^3-1}$

36. Find a basis for the space S of vectors (a,b,c,d) with $a+c+d=0$ and also for the space T with $a+b=0$ and $c=2d$. What's the dimension of the intersection $S \cap T$?

Ans. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ c \\ d \end{bmatrix} = 0 \quad \Longleftarrow a+c+d=0$

$$\begin{bmatrix} a \\ c \\ d \end{bmatrix} = c\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

(or) $a+b+c+d = b \Rightarrow a+c+d=0$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Basis for } S : (0,1,0,0),\ (1,0,-1,0),\ (1,0,0,-1)}$$

T: $a+b=0$ & $c=2d$

$(a, -a, 2d, d)$

Basis for T is $(1,-1,0,0)$, $(0,0,2,1)$

$a+c+d=0$ | $a+b=0$, $b=-a$ | $c=2d$

$a+2d+d=0$

$a+3d=0$ $\Longrightarrow$ $\cancel{(a,b,c,d) = (a,-a,}$

$a=-3d, b=3d,$
$c=2d$ $\quad$ $(a,b,c,d) = (-3,3,2,1)$

Basis for $S \cap T$: $(-3,3,2,1)$

---

37. If $AS = SA$, for the shift matrix S, show that A must have this special form:

If $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow$

then $A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$.

"The subspace of matrices that commute with the shift S has dimension ________."

Ans: $AS = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} = SA$

$h = d = 0 \;|\; a = e = i \;|\; b = f \;|\; g = 0.$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace ~~of~~ matrices that have $AS = SA$ has dimension 3., because only the 3 numbers a, b, c can be choosen independently in A.

40. (e) Find a Basis for all solutions to $\frac{d^4y}{dx^4} = y(x)$

Ans: Set,

$$y = e^{ax}$$

$$y' = ae^{ax} \quad \Big| \quad y'' = a^2 e^{ax} \quad \Big| \quad y''' = a^3 e^{ax} \quad \Big| \quad y'''' = \frac{d^4y}{dx^4} = a^4 e^{ax}$$

$$\frac{d^4y}{dx^4} = y \implies a^4 e^{ax} = e^{ax} \longrightarrow a = \pm 1, \pm i$$

$$y(x) = Ae^{ix} + Be^{-ix} + Ce^{x} + De^{-x}$$

$$= (A+B)\cos x + (A - iB)\sin x + Ce^{x} + De^{-x}$$

$$= c_1 \cos x + c_2 \sin x + c_3 e^{x} + c_4 e^{-x}$$

Basis for solutions to $\frac{d^4y}{dx^4} = y(x)$:

- $y(x) = \cos x$
- $y(x) = \sin x$
- $y(x) = e^{x}$
- $y(x) = e^{-x}$

~~(b) Find a particular solution to $\frac{d^4y}{dx^4} = y(x)+1$. Find a complete solution~~

Ans:

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 (\cosh x + \sinh x) + c_4 (\cosh x - \sinh x)$$

$$= k_1 \cos x + k_2 \sin x + k_3 \cosh x + k_4 \sinh x$$

(b) Find a particular solution to $\frac{d^4y}{dx^4} = y(x) + 1$

Find a complete solution.

Ans:

A particular solution to $\frac{d^4y}{dx^4} = y(x) + 1$ is -1

Null space is $c_1 \sin x + c_2 \cos x + c_3 e^{x} + c_4 e^{-x}$.

is the solution to $\frac{d^4y}{dx^4} = y(x)$.

The complete solution is : $-1 + c_1 \sin x + c_2 \cos x + c_3 e^{x} + c_4 e^{-x}$

2 41. Write the 3×3 identity matrix as a ~~4~~ combination of the other ~~5~~ permutation matrices. Then show that those 5 matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$, & check entries to prove that c_1 to c_5 must all be zero). The 5 permutations are a basis for the subspace of 3×3 matrices with row & column sums all equal.

Ans:
5 permutation matrices are:

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$\underline{I} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} + \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} + \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} - \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} - \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}.$$

The 6 P's are linearly dependent. But those 5 P's are independent.

(42) Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself span a subspace S. Find specific vectors x so that the dimension of S is

(a) zero (b) 1 (c) 3 (d) 4

Ans: Dimension of S spanned by all rearrangements of S is: zero when

(a) $x = (0, 0, 0, 0)$

one when

(b) $x = (1, 1, 1, 1)$

(c) Dimension of S is 3 when

$x = (1, 1, -1, -1)$.

$\# = \dfrac{4!/2!\times 2!}{2} = 3.$

because all rearrangements of this $x = (1, 1, -1, -1)$ are $\perp$ to $(1, 1, 1, 1)$.

Note: No x gives $\dim S = 2$.

(d) $\dim S = 4$ when the x's are not equal & don't add to zero (If it adds to zero, will become $\perp$ to $(1,1,1,1)$ then dimension become 3)

43.

Let V be a vector space, and let W_1 & W_2 be subspaces of V. Then,

$W_1 \cap W_2 = \{ w \mid w \in W_1 \text{ and } w \in W_2 \}$ and is called the intersection of W_1 and W_2

$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1 \text{ \& } w_2 \in W_2 \}$ is called the sum of W_1 & W_2

Let W_1 & W_2 are subspaces of the vector space V.

* $W_1 \cap W_2$ and $W_1 + W_2$ are subspaces of V.

* Let W_1 & W_2 are subspaces of the vector space V,
$W_1 \cap W_2$ and $W_1 + W_2$ are subspaces of V

Proof

(1) Let $u, v \in W_1 \cap W_2 \Longrightarrow u, v \in W_1$ & $u, v \in W_2$

$\Longrightarrow u+v \in W_1$ & $u+v \in W_2$

$\Longrightarrow u+v \in W_1 \cap W_2$

Let $u \in W_1 \cap W_2 \Longrightarrow u \in W_1$ & $u \in W_2$

$\Longrightarrow au \in W_1$ & $au \in W_2$

$\Longrightarrow au \in W_1 \cap W_2$.

$\therefore$ $W_1 \cap W_2$ is a subspace.

(2) Let $u, v \in W_1 + W_2$,

For $x, x' \in W_1$ and $y, y' \in W_2$, we can write,

$u = x + y$ & $v = x' + y'$

$$u+v = (x+y)+(x'+y') = (x+x')+(y+y') \in W_1+W_2$$

Let $u \in W_1+W_2$.

$$cu = c(x+y) = cx+cy \in W_1+W_2$$

$\therefore$ W_1+W_2 is a subspace of V

* Let W_1, W_2 are subspaces of a vector space V over a field $\mathbb{F}$, then

$$\boxed{\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)}$$

Proof

Let S be a basis of $W_1 \cap W_2$

For each $i = 1, 2$, extend S to a basis B_i of W_i.

Let, $S = \{u_1, u_2, \ldots, u_r\}$, $B_1 = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$

and $B_2 = \{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_t\}$

Then,

$\dim(W_1 \cap W_2) = r$, $\dim(W_1) = r+s$,

$\dim(W_2) = r+t$

Let $B = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t\}$

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j + \sum_{k=1}^{t} c_k w_k = 0$$

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j = -\sum_{k=1}^{t} c_k w_k$$

LHS $\in W_1$ & RHS $\in W_2$

So this element must be in $W_1 \cap W_2$

$\therefore$

$$-\sum_{k=1}^{t} c_k w_k = \sum_{i=1}^{r} d_i u_i$$

$$\Longrightarrow \sum_{i=1}^{r} d_i u_i + \sum_{k=1}^{t} c_k w_k = 0$$

B_2 is linearly independent $\Longrightarrow$ $\boxed{\begin{array}{l} d_i = 0, \; C_k = 0 \\ \text{for each } i \;\&\; k \end{array}}$

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j = 0$$

B_1 is linearly independent $\longrightarrow$ $a_i = 0$, $b_j = 0$ for each i & each j.

$\therefore$

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j + \sum_{k=1}^{t} c_k w_k = 0$$

$$\Longrightarrow a_i = 0, \; b_j = 0, \; c_k = 0 \quad \text{for all } i, j, k$$

$\therefore$ B is linearly independent.

Let, $w \in W_1 + W_2$

Then $w = w_1 + w_2$ for some $w_i \in W_i$ for $i = 1, 2$

Then, $w_1 = \sum_{i=1}^{r} p_i u_i + \sum_{j=1}^{s} q_j v_j$ &

$w_2 = \sum_{i=1}^{r} g_i u_i + \sum_{j=1}^{t} h_j w_j$ ~~for~~

for $p_i, q_j, g_i, h_i \in \mathbb{F}$

Now,

$$w = \sum_{i=1}^{r} (p_i + g_i) u_i + \sum_{j=1}^{s} q_j v_j + \sum_{k=1}^{t} h_k w_k$$

which is in span $\mathcal{B}$.

$\Longrightarrow \mathcal{B}$ is a basis of $W_1 + W_2$.

$$\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) =$$

$$= (r+s) + (r+t) - r = r+s+t = \dim(W_1+W_2)$$

* The sum $W_1 + W_2$ is called direct if $W_1 \cap W_2 = \{0\}$.

A vector space V is said to be the direct sum of 2 subspaces W_1 and W_2 if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

* W_1 & W_2 are independent

When V is a direct sum of W_1 & W_2, we write $\boxed{V = W_1 \oplus W_2}$.

* Suppose, W_1 and W_2 are subspaces of a vector space V so that $V = W_1 + W_2$. Then $V = W_1 \oplus W_2$ iff every vector in V can be written in a unique way as $w_1 + w_2$ where $w_i \in W_i$

Proof Let $V = W_1 + W_2$

Suppose,

that for every $v \in V$, there is only one pair (w_1, w_2) with $w_i \in W_i$ such that $v = w_1 + w_2$.

unique

If $W_1 \cap W_2$ is non-zero,

pick a non-zero vector $u \in W_1 \cap W_2$

Then, $u = u + 0$ with $u \in W_1$, $0 \in W_2$

and $u = 0 + u$ with $0 \in W_1$, $u \in W_2$

$\Longrightarrow$ Contradicts uniqueness.

Conversely,

suppose $V = W_1 \oplus W_2$,

Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$

If for $v \in V$, we have

$v = w_1 + w_2 = w_1' + w_2'$ for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$

$w_1 - w_1' = w_2' - w_2$

LHS $\in W_1$ & RHS $\in W_2$

$\Longrightarrow \quad w_1 - w_1' = w_2' - w_2 \in W_1 \cap W_2$

By assumption $\quad W_1 \cap W_2 = \{0\}$,

~~$w_1 - w_1'$ and~~

$w_1 - w_1' = 0$ and $w_2' - w_2 = 0$

$\Longrightarrow w_1 = w_1'$ and $w_2' = w_2$

~~Let~~

Examples

① $V = \mathbb{R}^2$, $W_1 = \{(x, 2x) \mid x \in \mathbb{R}\}$, $W_2 = \{(x, 3x) \mid x \in \mathbb{R}\}$

Then $V = W_1 \oplus W_2$

② $V = M_n(\mathbb{R})$, W_1 is the subspace of all the upper triangular matrices and W_2 is the subspace of all the lower triangular matrices over $\mathbb{R}$.

$V = W_1 + W_2$ is not direct, since $W_1 \cap W_2$ is the non-empty set of all diagonal matrices.

③ $V = M_n(\mathbb{R})$, W_1 is the subspace of all the symmetric $n \times n$ matrices over $\mathbb{R}$ and W_2 is the subspace of all the skew-symmetric $n \times n$ matrices over $\mathbb{R}$.

$V = W_1 \oplus W_2$

④ The space of 2×2 matrices is this direct sum

$$\left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| a, d \in \mathbb{R}\right\} \oplus \left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| b \in \mathbb{R}\right\} \oplus \left\{\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \middle| c \in \mathbb{R}\right\}$$

It is the direct sum of subspaces in many other ways as well, ie., direct sum decompositions are not unique.

Let 'A' is a subspace of the vectorspace V, then

$$A^{\perp} \oplus A = V$$

* If $W_1, \dots, W_k$ are subspaces of a vector space V, their sum is the span of their union.

$$\boxed{W_1 + W_2 + \cdots + W_k = \text{span}\{W_1 \cup W_2 \cup \cdots W_k\}}$$

Proof.

If $x \in W_1 + W_2$,

then $x = w_1 + w_2$ for $w_i \in W_i$

$\Longrightarrow x \in \text{span}(W_1 \cup W_2)$

$\therefore W_1 + W_2 \subset \text{span}(W_1 \cup W_2)$

If $x \in \text{span}(W_1 \cup W_2)$,

x can be written as a sum of elements from $W_1 \cup W_2$, say $u_1 + v_1 + u_2 + v_2 + \cdots + u_k + v_k \cdots$

where $u_i \in W_1$ and $v_k \in W_2$

$x = (u_1 + \cdots + u_i) + (v_1 + \cdots + v_k) \in W_1 + W_2$

$\therefore \text{span}(W_1 \cup W_2) \subset W_1 + W_2$

$\Longrightarrow W_1 + W_2 = \text{span}(W_1 \cup W_2)$