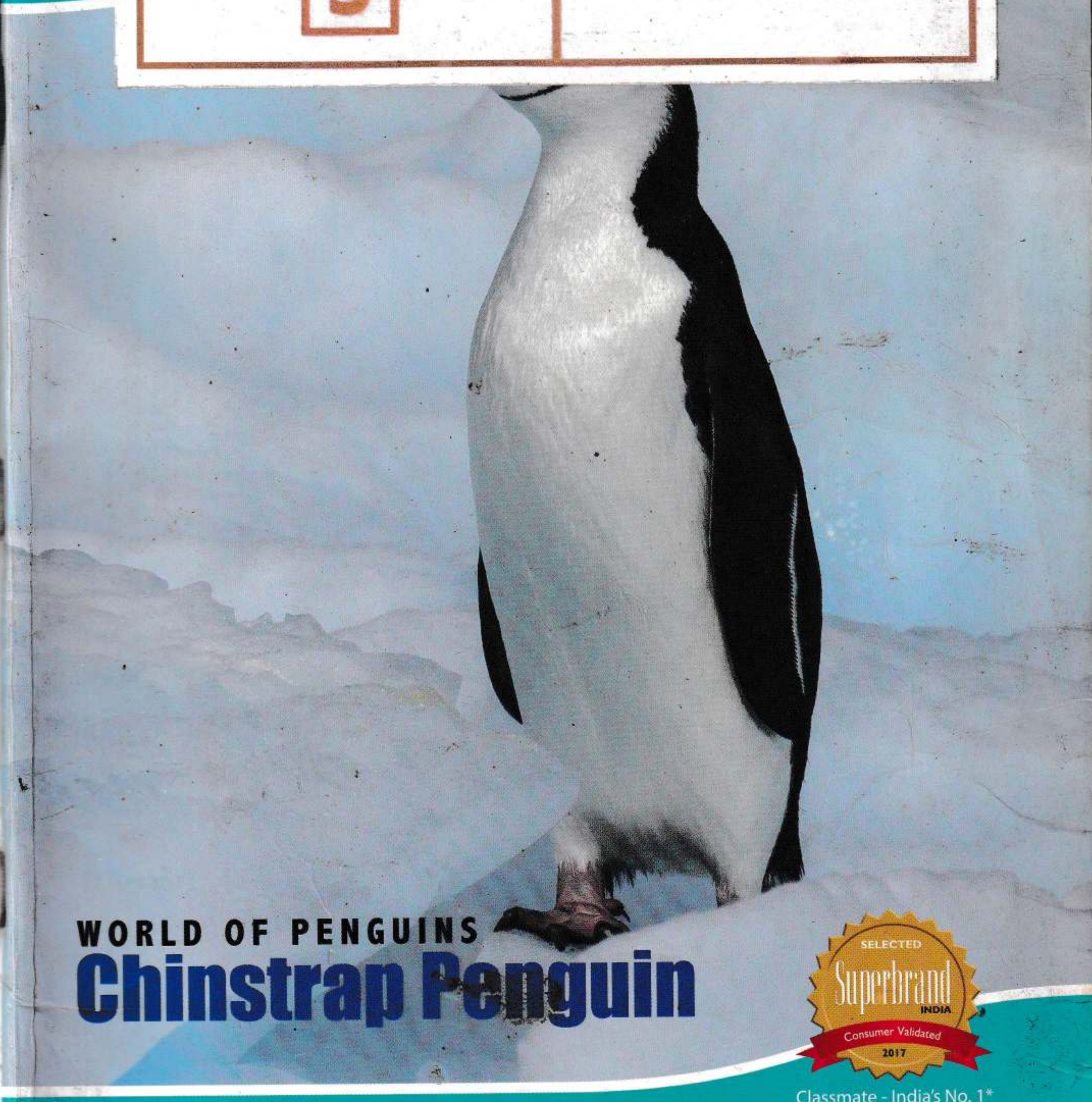
Introduction to Linear Algebra
- Gilbert Strang

3

Vector Spaces & Subspaces



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INDEX

3

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S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		INTRODUCTION TO		
		LINEAR ALGEBRA		
		- Gilbert Strang, MIT		
		(5th Edition)		

Null space of A : solving $Ax=0$ and $Rx=0$

– subspace containing all solutions to $Ax=0$

$Ax=0$, with A is
- Invertible : $x=0$ is the only solution
- Non-invertible : there must be non-zero solutions, since $\vec{0}$ is in $C(A)$

* The nullspace $N(A)$ consists of all solutions to $Ax=0$. These are vectors in $\mathbb{R}^n$.

ie.,

Nullspace is a subspace of $\mathbb{R}^n$

Columnspace is a subspace of $\mathbb{R}^m$

Proof:

If x, y are in the $N(A)$

$\therefore Ax = 0$ and $Ay = 0$

$$A(x+y) = Ax + Ay = 0 + 0 = 0$$

$$A(cx) = c\,Ax = 0$$

$\therefore x+y,\ cx$ are also in $N(A)$.

$\Longrightarrow N(A)$ is a subspace

Ex:1 Null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Ans: $x_1 + 2x_2 = 0$
$3x_1 + 6x_2 = 0$
$\longrightarrow$ $x_1 + 2x_2 = 0$
$0 = 0$

Row picture : The line $x_1 + 2x_2 = 0$ is the nullspace $N(A)$.
It contains all solutions (x_1, x_2).

(OR)

Take one special solution on the line.
All points on the line are multiples of this solution.

Take, $x_2 = 1 \longrightarrow x_1 = -2$

The special solution, $s = (-2, 1)$.

Null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

* The null space of A, $N(A)$ consists of all combinations of the special solutions to $Ax = 0$.

Ex: 2 $x + 2y + 3z = 0$ comes from $A_{1\times3} = [1, 2, 3]$

$Ax = 0$ produces a plane. All vectors on the plane are $\perp$ to $(1, 2, 3)$.

Ans: The plane is the nullspace of A.

There are 2 free variables y & z.

Set to 0 and 1.

$$Ax = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$ has 2 special solutions,

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

* The solutions to ~~...~~ $x + 2y + 3z = 6$ also lie on a plane, but that plane is not a subspace.

The vector $x = 0$ is only a solution if $b = 0$.

* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent

Check OM(23) Rank of a matrix

$\Longrightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent

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Pivot columns & Free columns

The 1^{st} column of $[1\ 2\ 3]$ contains the only pivot, so the 1^{st} component of x is not free.

⇒ The free components correspond to columns with no pivots.

The special choice (1 or 0) is only for the free variables in the special solutions.

Ex: 3 Find the null-spaces of A, B, C and the 2 special solutions to $Cx=0$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}$$

$$C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

Ans: $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & +2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$Ax=0$ has only the zero solution $x=0$.

$\Longrightarrow$ $N(A) = Z$, ie., it contains only the single point $x=0$ in R^2.

A is invertible.

Both columns of this matrix have pivots.

$$Bx = 0 \Longrightarrow Bx = \begin{bmatrix} A \\ 2A \end{bmatrix} x = \begin{bmatrix} Ax \\ 2Ax \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{array}{l} Ax = 0 \\ 2Ax = 0 \end{array} \right\} x = 0$$

$$\Longrightarrow N(B) = Z$$

* When we add extra equations (giving extra rows), the nullspace certainly can not become larger. The extra rows impose more conditions on the vectors x in the null space.

$$Cx = 0 \Longrightarrow \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$U_c = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

Pivot columns

Free columns

Taking $(x_3, x_4) = (0,1)$ and $(1,0)$, we get too special solutions in $N(C)$.

$(x_3, x_4) = (0,1) \Longrightarrow (x_1, x_2) = (0, -2)$

$(x_3, x_4) = (1,0) \Longrightarrow (x_1, x_2) = (-2, 0)$

Special solutions $Cs = 0$, $Us = 0$

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \quad S_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Pivot variables

Free variables

(OR)

$x_1 + 2x_2 + 2x_3 + 4x_4 = 0 \rightarrow x_1 + 2x_3 = 0$

$2x_2 + 4x_4 = 0 \longrightarrow x_2 + 2x_4 = 0 \Rightarrow x_2 = -2x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \vec{S}_1 x_3 + \vec{S}_2 x_4$$

□ The Reduced Row Echelon Form, R

$$U \to R$$

(i) Produce zeros above the pivots

(ii) Produce ones in the pivots.

$$\boxed{\begin{array}{l} N(A) = N(U) = N(R) \\ \text{reduced row echelon form } R = \text{rref}(A) \end{array}}$$

* The pivot columns of R contain I

Ex:-

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

pivot columns free columns

$$Rx = 0 \implies \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cancel{0} \\ \cancel{0} \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\cancel{2\left(x_3 + \frac{x_2}{2}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}(x_1 + 2x_3) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(x_2 + 2x_4) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + 2x_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= -2x_4 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$S_1 = (-2, 0, 1, 0)$ & $S_2 = (0, -2, 0, 1)$

For many matrices, the only solution to $Ax = 0$ is $x = 0$. Their null space $N(A) = Z$ contain only that zero vector: no special solutions.

$N(A) = Z$: columns of A are independent

ie :) No combination of columns gives the zero vector (except the zero combination)

Pivot variables & Free variables in the Echelon matrix R

$$A = \begin{array}{ccccc} p & p & f & p & f \\ \end{array}$$
$$A = \begin{bmatrix} | & | & | & | & | \\ | & | & | & | & | \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & a & 0 & c \\ 0 & 1 & b & 0 & d \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot columns p
2 free columns f
to be revealed by R.

\# of pivots = 3
$\Longrightarrow$ rank = 3.

Set $(x_3, x_5) = (1,0)$ & $(0,1)$.

$$S_1 = \begin{bmatrix} -a \\ -b \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -c \\ -d \\ 0 \\ -e \\ 1 \end{bmatrix}$$

R : column 3 = a (column 1) + b (column 2)
same must be true for 'A'.

$N(A) = N(R)$ = all combinations of s_1 & s_2
$= \text{span}(s_1, s_2)$

$$R = \begin{bmatrix} 1 & 0 & x & x & x & 0 & x \\ 0 & 1 & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 pivot variables : x_1, x_2, x_6

4 free variables : x_3, x_4, x_5, x_7

$\therefore$ 4 special solutions s in $N(R)$

$C(R)$, $N(R)$ = ?

Ans: The columns of R have 4 components
So, they lie in R^4.
The 4^{th} component of every column is zero.

$C(R)$ consists of all vectors of the form $(b_1, b_2, b_3, 0)$.

$N(R)$ is a subspace of $\mathbb{R}^7$.

The solutions to $Rx=0$ are all the combinations of the 4 special solutions - one for each free variable:

With $n>m$, there is atleast one free variable.

If $Ax=0$ has more unknowns than equations ($n>m$, more columns than rows).
There must be atleast one free column.

$\Longrightarrow$ $Ax=0$ has non-zero solutions

# of pivots can't exceed m $\Rightarrow$ there must be atleast $n-m$ free variables.

* The nullspace is a subspace. Its dimension is the # of free variables.

The Rank of a matrix

check SM(23) ** The numbers m & n give the size of a matrix, but not necessarily the true size of a linear system.

~~The~~

$\Longrightarrow$ The true size of A is given by its rank

* rank(A) = r = # of pivots

Ex:-

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(A) = rank(R) = 2

column 3 = 2(column 1) + 0(column 2)

column 4 = 3(column 1) + 1(column 2)

$s_1 = (-2, 0, 1, 0)$ & $S_2 = (-3, -1, 0, 1)$

* Every free column is a combination of pivot columns.

Rank One

Only one pivot

ILA(4)

Rank : 2

Ex:-

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The column space of a rank 1 matrix is one dimensional.

Every row is a multiple of the pivot row.
Every column is a multiple of the pivot column.

all columns are on the line $u = (1,2,3)$.

$$v^T = \begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$$

$$A = uv^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix}$$

$$= u \otimes v$$

$Ax = 0$

$(uv^T)x = u(v^Tx) = 0$

$\longrightarrow \quad v^Tx = 0 = v \cdot x$

$\therefore$ All vectors x in the null space must be orthogonal to v in the row space.

When $r = 1$: row space = line

null space = $\perp$ plane

* Every rank one matrix is one column times one row.

$$\boxed{A = uv^T = u \otimes v}$$

$\dim(\text{row space}) = \dim(\text{column space}) = r$

$\dim(\text{null space}) = n - r$

$= \#$ of free variables.

- $\boxed{A^2 = uv^T uv^T = u(v^T u)v^T = 0 \quad \text{if } v^T u = 0}$

- $A = uv^T$

Let $w \in \mathbb{R}^n$,

$$Aw = uv^T w = (v \cdot w)u$$

$\rightarrow$ $A = uv^T$ maps every vector in $\mathbb{R}^n$ to a scalar multiple of u.

$\therefore$ $\text{rank}(A) = \dim(C(A)) = 1$

(OR)

Assume that, $\text{rank}(A) = 1$

then for all $w \in \mathbb{R}^n$, $Aw = ku$ for some fixed $u \in \mathbb{R}^m$,

This is true for all the basis vectors of $\mathbb{R}^n$.

$\therefore$ Every column of A is a multiple of u

$$A = (v_1 u \quad v_2 u \cdots v_n u) = u(v_1 \ v_2 \cdots v_n) = uv^T$$

$A^T = vu^T \Rightarrow v \in C(A^T)$

$\boxed{3.2(A)}$ Why do A & R have the same null space if $EA = R$ and E is invertible?

Ans: If $Ax = 0 \longrightarrow Rx = EAx = E0 = 0$

If $Rx = 0 \Longrightarrow Ax = E^{-1}Rx = E^{-1}0 = 0$

$\boxed{3.2(B)}$ Create a 3×4 matrix R whose special solutions to $Rx = 0$ are s_1 and s_2:

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

pivot columns 1 & 3

free variables x_2 & x_4

Describe all possible matrices A with this null space $N(A)$ = all combinations of s_1 & s_2.

Ans: R has pivot columns 1 and 3.
The free columns are 2 & 4, which are linear combinations of the pivot columns.

$N(A) = R(A) = \text{span}(s_1, s_2)$

Every 3×4 matrix has atleast 1 special solution.
Here we have 2. $\Longrightarrow$

$$R = \begin{bmatrix} 1 & a & 0 & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

$Rs_1 = 0$ & $Rs_2 = 0$

$-3 + a = 0 \Longrightarrow \boxed{a = 3}$

$-6 + d = 0$

$-2 + c = 0 \longrightarrow \boxed{c = 2}$ $\quad \boxed{d = 6}$

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{3 \cdot 2(c)}$ Find the row reduced form R and the rank r of A and B

What are the pivot columns of A?

What are the special solutions ?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \& \quad B = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

Ans: rank (A) = 2 except if $c = 4$

$$A \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & c-4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & c-4 \\ 0 & 0 & 0 \end{bmatrix}$$

$c \neq 4$: $R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Two pivots (rank = 2) one free variable

$c = 4$: $R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ only pivot is in column 1 (rank = 1)

2nd & 3rd variables are free.
$\Rightarrow$ 2 special solutions.

$\underline{c \neq 4}$ $x_1 + 2x_2 = 0$

$s = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\underline{c = 4}$ $x_1 + 2x_2 + x_3 = 0$

$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

rank $(B) = 1$ except if $c = 0$, when the rank $= 0$

$\underline{c \neq 0}$,

$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\underline{c = 0}$

$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

nullspace $= \mathbb{R}^2$

Elimination: The big picture.

Elimination starts with the 1st pivot. It moves a column at a time (left to right) and a row at a time (top to bottom).

As it moves, elimination answers 2 questions:

$$\boxed{A \longrightarrow \text{triangular echelon matrix } U}$$

Q1: Is this column a combination of previous columns?

No, if the column contains a pivot.

Pivot columns are "independent" of previous columns.

Ex:- If column 4 has no pivot, it is a combination of columns 1, 2, 3.

Q2: Is this row a combination of previous rows?

No, if the row contains a pivot.

Pivot rows are "independent" of previous rows.

Ex:- If row 3 ends up with no pivot, it is a zero row and it is moved to the bottom of R.

Triangular echelon matrix $\longrightarrow$ reduced echelon matrix

$U \longrightarrow R$

U tells which columns are combinations of earlier columns (pivots are missing).

Then,

→ R tells us what those combinations are.

ie.,

R tells us the special solutions to $Ax = 0$.

ie., R reveals a "basis" for three fundamental subspaces:

column space (A) : pivot columns of A is a basis

row space (A) : non-zero rows of R is a basis

nullspace (A) : special solutions to $Rx=0$ (& $Ax=0$)

* When 'A' is square & invertible,

R is I & E is A^{-1}.

The Complete Solution to $Ax=b$

$b \neq 0$,

$Ax = b \longrightarrow Rx = d$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{bmatrix} = [A|b]$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R|d]$$

One Particular Solution, $Ax_p = b$

choose the free variables to be zero,

$x_2 = x_4 = 0 \implies x_1 = 1, \ x_3 = 6$

One particular solution to $Ax = b$ (& also $Rx = d$) is $x_p = (1, 0, 6, 0)$

$$Rx_p = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

$x_{particular}$: The particular solution solves $Ax_p = b$

$x_{nullspace}$: The $(n-r)$ special solutions solve $Ax_n = 0$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \quad \Longleftrightarrow \quad Rx = d$$

$$\begin{aligned} x_1 + 3x_2 + 2x_4 &= 1 \\ x_3 + 4x_4 &= 6 \end{aligned} \quad \left\{ \begin{aligned} x_1 &= 1 - 3x_2 - 2x_4 \\ x_3 &= 6 - 4x_4 \end{aligned} \right.$$

free variables

Complete solution : one x_p , many x_n

$$x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

If 'A' is a square matrix, $m = n = r$

$$x = x_p + x_n = A^{-1}b + 0$$

Ex:1 Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Find the complete $x = x_p + x_n$

Ans: $$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & \boxed{b_1 + b_2 + b_3} \end{array}\right]$$

left nullspace

$Ax = b$ is solvable iff b is in $C(A)$.

$$\longrightarrow b_1 + b_2 + b_3 = 0.$$

For consistency, the entries of b must also add to zero.

$n = 2, \quad r = 2$

$\therefore$ # of free variables, $n - r = 0$

$\therefore$ No special solution.

The null space solution is, $x_n = 0$

The particular solution to $Ax = b$ $(Rx = d)$ is:

Only solution to $Ax = b$: $x = x_p + x_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Full Column rank $r = n$: $R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} n \times n \text{ identity matrix} \\ m-n \text{ rows of zeros} \end{bmatrix}$

- All columns of A are pivot columns
- No free variables or special solutions
- $N(A) = Z$
- If $Ax = b$ has a solution (it might not) then it has only one solution.

The Complete Solution

Full row rank : $r = m$ & $m \le n$ (short & wide)

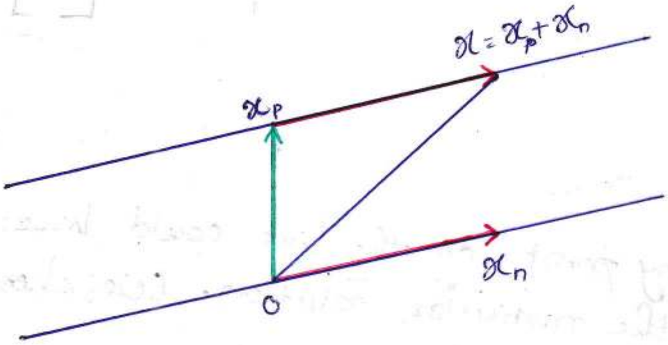
Ex: 2

$$x + y + z = 3$$
$$x + 2y - z = 4$$

$(r = m = 2)$

Ans: 2 planes in $\mathbb{R}^3$.

Planes are not ||el $\Longrightarrow$ intersect in a line.



The particular solution will be one point on the line. Adding the null space vectors x_n will move us along the line

Complete solution = one particular solution + all nullspace solutions.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] = [R|d]$$

free colums.

$x_3 = 0$: $x_p = (2, 1, 0)$

$x_3 = 1$: $s = (-3, 2, 1)$

Complete solution : $x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

* Any point on the line could have been chosen as the particular solution. We chose the point $x_3 = 0$.

Full row rank : $R = \begin{bmatrix} I & F \end{bmatrix}$

$r = m$

- All rows have pivots, & R has no zero rows.
- $Ax = b$ has a solution for every b
- $C(A) = \mathbb{R}^m$
- There are $n - r = n - m$ special solutions in $N(A)$

The 4 possibilities for linear equations depend on the rank r

Check OM(23)

$r = m = n$	Square & Invertible	$R = \begin{bmatrix} I \end{bmatrix}$	$Ax = b$ has 1 solution
$r = m < n$	Short & Wide (Full row rank)	$R = \begin{bmatrix} I & F \end{bmatrix}$	$Ax = b$ has ∞ solutions
$r = n < m$	Tall & thin (Full column rank)	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$Ax = b$ has 0 or 1 solution
$r < m,\ r < n$	Not full rank	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$	$Ax = b$ has 0 or ∞ solutions.

* $Ax = b$ has no solution $\Longrightarrow$ $r < m$

Reasoning: $Ax = b$ is inconsistent system, then $rref(A|b)$ has a row of $[0, 0, \cdots, 0 | 1]$.

$\therefore$ $r < m$

* Fredholm's Alternative

Check $\boxed{4.1}$ (6.7)

For any matrix A and column vector b, exactly one of the following must hold:

Either

① $Ax = b$ has a solution

Or

② $A^T y = 0$ has a solution (non-trivial) y with $y^T b \neq 0$.

$\Longrightarrow$ $Ax = b$ has a solution, iff for any $A^T y = 0$, $y^T b = 0$

$\boxed{3.3(A)}$ ~~This~~

$$Ax=b: \quad \begin{aligned} x_1+2x_2+3x_3+5x_4 &= b_1 \\ 2x_1+4x_2+8x_3+12x_4 &= b_2 \\ 3x_1+6x_2+7x_3+13x_4 &= b_3 \end{aligned}$$

(1) Reduce $[A|b]$ to $[U|c]$ so that $Ax=b$ becomes a triangular system $Ux=c$.

(2) Find the condition of b_1, b_2, b_3 for $Ax=b$ to have a solution

(3) Describe $C(A)$. Which plane in $\mathbb{R}^3$.

(4) Describe $N(A)$. Which special solutions in $\mathbb{R}^4$.

(5) Reduce $[U|c]$ to $[R|d]$: special solutions from R, particular solution from d.

(6) Find a particular solution to $Ax=(0,6,-6)$ and then the complete solution.

Ans:

(1)

$$\begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & -2 & -2 & b_3-3b_1 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & \boxed{-5b_1+b_2+b_3} \end{array}\right] \Rightarrow \text{left null space}$$

(2) $\boxed{-5b_1+b_2+b_3=0}$ ⟵ solvability condition

left nullspace

(3) $C(A) = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\8\\7\end{bmatrix}\right)$

$$b = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \forall \quad -5b_1+b_2+b_3=0.$$

$C(A)$ contains all vectors with $-5b_1+b_2+b_3=0$. That make $Ax=b$ solvable., b is in the column space of A.

All columns of A ~~are~~ satisfy this condition,

$\Longrightarrow \vec{x}\cdot(-5,1,1)=0$

This is the eq^n. for the plane.

(4) $Ax=0 \Longrightarrow Rx=0$

$$\begin{bmatrix}1 & 2 & 3 & 5\\0 & 0 & 2 & 2\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

free columns

$(x_2, x_4) = (1, 0)$ & $(0, 1)$

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$N(A) = \text{span}(s_1, s_2)$ in $\mathbb{R}^4$.

$x_n = c_1 s_1 + c_2 s_2$.

(5) $$[U|c] = \begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow [R|d] = \begin{bmatrix} 1 & 2 & 1 & 3 & -6 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [R|d]$$

free colums:

$$(x_2, x_4) = (0, 0) \Longrightarrow x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

(6) The complete solution to $Ax = (0, 6, -6)$ is :

$x = x_p + x_n = x_p + c_1 s_1 + c_2 s_2$

$\boxed{3.3\,(B)}$ Suppose you have this information about the solutions to $Ax = b$ for a specific b. What does this tell you about m, n, r (and A itself)? And possibly about b.

① There is exactly 1 solution

② All solutions to $Ax = b$ have the form

$$x = \begin{bmatrix} ? \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

③ There are no solutions

④ All solutions to $Ax = b$ have the form

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

⑤ There are infinitely many solutions.

Ans:

① $r = n$ (Full column rank)

$\&\ m \geq n$

② # of columns, $n = 2$, m: arbitrary

$$x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$N(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$n - r = 2 - r = 1 \implies \underline{\underline{r = 1}}.$$

columns are : $\vec{a}_1$ & ~~$k\vec{a}_1$~~

~~$x_1\vec{a}_1 + x_2 k\vec{a}_1 = 0$~~

$a_1 + a_2 = 0$

$\underline{\underline{a_2 = -a_1}}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A) \implies \cancel{\begin{bmatrix} 1+k \\ 1+k \end{bmatrix} \vec{a}_1 = 0 \implies k = -1}$$

$\therefore$ $\underline{\text{column } 2 = -(\text{column } 1)}$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a solution : $b = 2(\text{column } 1) + 1(\text{column } 2)$

(3) $b \notin C(A)$ for no solution

$b \neq 0$, else $x = 0$ would be a solution.

(4) $x = x_p + x_n = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

A must have 3 columns, m arbitrary

$m \geq 2$

$N(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

$n - r = 3 - r = 1 \Longrightarrow \underline{\underline{r = 2}}$

columns: $\vec{a}_1, \vec{a}_2, \; c_1\vec{a}_1 + c_2\vec{a}_2 = a_3$

$\vec{a}_1 + 0\cancel{\vec{a}_2} + c_1\vec{a}_1 + c_2\vec{a}_2 = 0$

$a_1 + a_3 = 0 \Longrightarrow \underline{\underline{a_3 = -a_1}}$

<u>column 3 = -(column 1)</u>

Column 2 must not be a multiple of column 1.

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a solution : $b =$ column 1 + column 2.

(5) $n - r > 0$

$N(A)$ must contain non-zero vectors.

$r < n$ & $b \in C(A)$

We don't know if every b is in $C(A)$, so we don't know if $r = m$.

$\boxed{3.3(c)}$ Find the complete solution $x = x_p + x_n$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$$

Find the # y_1, y_2, y_3 so that $y_1(\text{row } 1) + y_2(\text{row } 2) + y_3(\text{row } 3) =$ zero row.

Check that $b = (4, 2, 10)$ satisfies the condition $y_1 b_1 + y_2 b_2 + y_3 b_3 = 0$.

Why is this the condition for the equations to be solvable and b to be in the column space?

Thus:

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right] \to \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 2 & 8 & -6 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}2\\0\\0\end{matrix} & \boxed{\begin{matrix}1\\2\\0\end{matrix}} & \begin{matrix}0\\4\\0\end{matrix} & \begin{matrix}4\\-3\\0\end{matrix} \end{array}\right] = [U|c]$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & -4 & 7 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$= [R|d]$$

$$\cancel{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2}$$

$$x_1 + 2x_2 - 4x_4 = 7$$
$$x_3 + 4x_4 = -3.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$$= x_p + x_n = x_p + c_1 s_1 + c_2 s_2$$

Special solutions : $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

$2(\text{row }1) + (\text{row }2) - (\text{row }3) = (0,0,0,0)$

$\Longrightarrow$ $\underline{\underline{y = (2,1,-1)}}$

$b = (4,2,10)$: $4(2) + 2 - (10) = 0$

$\Leftrightarrow$ If a combination of the rows (on the left side) gives the zero row, then the same combination must give zero on the right side.

Otherwise no solution.

Independence, Basis & Dimension

- Independence of vectors (no extra vectors)
- Spanning ~~of~~ a space (enough vectors to produce the rest)
- Basis for a space (not too many or too few)
- Dimension of a space (the # of vectors in a basis)

Linear Independence

* The columns of 'A' are linearly independent when the only solution to $Ax=0$ is $x=0$. No other combination Ax of the columns gives the zero vector.

The columns of 'A' are independent when the nullspace $N(A)$ contains only the zero vector.

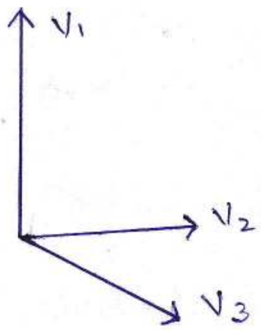
(OR)

* The sequence of vectors $v_1, \ldots, v_n$ is linearly independent if the only combination that gives the zero vector is $0v_1 + \cdots + 0v_n$.

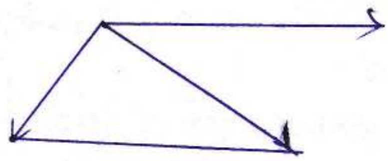
ie.,

$x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$ only happens when all x's are zero.

Ex:- 3 vectors v_1, v_2, v_3.



Not in a plane



In a plane

* The columns of 'A' are independent exactly when the rank is $r = n$. There are n pivots and no free variables. Only $x = 0$ is in the nullspace.

* Any set of vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$

Ex. in $\mathbb{R}^2$,

- $(1,0)$ and $(0,1)$ are independent
- $(1,0)$ and $(0,1.00001)$ are independent
- $(1,1)$ and $(-1,-1)$ are dependent
- $(1,1)$ and $(0,0)$ are dependent, because of the zero vector.
- In $\mathbb{R}^2$, any 3 vectors (a,b), (c,d), (e,f) are dependent.

(1,1) and (−1,−1) are on a line thro' origin.
They are dependent.

Find x_1 & x_2 so that,

$$x_1(1,1) + x_2(-1,-1) = (0,0)$$

$$x_1 = x_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } x_1 = 1,\ x_2 = 1$$

∴ Columns are dependent exactly when there is a non-zero vector in the nullspace,

---

3 vectors in $\mathbb{R}^2$ can't be independent.

← the matrix A with those 3 columns must have a free variable. and then a special solution to $Ax=0$

← If the 1st 2 vectors are independent, some combination will produce 3rd vector.

Think

Linear dependence: "One vector is a combination of the other vectors."

That sounds clear ¿

But, we said

"Some combination gives the zero vector, other than the trivial combination with every $x = 0$."

Our definition doesn't pick out one particular vector as guilty. All columns of 'A' are treated the same. We look at $Ax = 0$, and it has a non-zero solution or it hasn't.

In the end, it is better than asking if the last column (or the 1st, or a column in the middle) is a combination of the others.

Vectors that Span a Subspace

A set of vectors spans a space if their linear combinations fill the space

(OR)

The vectors $v_1, \dots, v_k$ span the space S if $S =$ all combinations of the v's.

- The columns of a matrix span its column space. They might be dependent.

Ex:-

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full 2 dimensional space $\mathbb{R}^2$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space $\mathbb{R}^2$

$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in $\mathbb{R}^2$.

So does w_1 itself.

* The rowspace of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.

$$\boxed{\text{row space}(A) = \text{column space}(A^T)}$$

- The rows of an $m \times n$ matrix have n components.
  → They are vectors in $\mathbb{R}^n$.

Ex. 5 Describe the column space & the row space of A

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}$$ &

Ans: $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 7 \end{bmatrix}$

Pivots

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$C(A)$ is the plane in $\mathbb{R}^3$ spanned by the 2 columns of A.

The rowspace of A is spanned by the 3 rows of A (which are columns of A^T). This row space is all of $\mathbb{R}^2$.

A Basis for a Vector Space

2 vectors can't span all of $\mathbb{R}^3$, even if they are independent.

4 vectors can't be independent, even if they span $\mathbb{R}^3$.

We want enough independent vectors to span the space (and not more). A "basis" is just right.

* ~~The vectors $v_1, \ldots, v_k$ are a basis for S if they are linearly independent and they span S.~~

* The vectors $v_1, \ldots, v_k$ are a basis for a vector space S if they are linearly independent and they span the space S.

Every vector v in the space is a combination of the basis vectors, because they span the space. More than that, the combination that produces v is unique, because the basis vectors are independent.

* There is one & only one way to write v as a combination of the basis vectors.

Proof

Suppose,

$$v = a_1 v_1 + \dots + a_n v_n \quad \text{and also}$$

$$v = b_1 v_1 + \dots + b_n v_n$$

$$0 = (a_1 - b_1) v_1 + \dots + (a_n - b_n) v_n$$

Independence of basis vectors: $a_i - b_i = 0$ for all $i = 1, \dots n$

$\therefore a_i = b_i$

There are not 2 ways to produce v.

* The vectors $V_1, \ldots, V_n$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an $n \times n$ invertible matrix. Thus $\mathbb{R}^n$ has infinitely many different bases.

* The pivot columns of 'A' are a basis for its column space. The pivot rows of 'A' are a basis for its row space. So are the rows of its echelon form R.

* The columns of the $n \times n$ identity matrix give the "standard basis" for $\mathbb{R}^n$.

$\boxed{\text{Ex: 8}}$ $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

pivot column

pivot

free column

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a basis for $C(A)$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a basis for $C(R)$.

But, it doesn't belong to $C(A)$.

$\Longrightarrow C(A) \neq C(R)$

~~But~~ their bases are different.

But, their dimensions are the same.

$\Longrightarrow$ Rowspace (A) = Rowspace (R)

Basis for the row space $= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Ex:9 Find the bases for the column & row spaces of this rank 2 matrix

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Ans: $C(R) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

= xy-plane inside the full 3D xyz space

It is a subspace of $\mathbb{R}^3$, which is not $\mathbb{R}^2$.

$$= \text{span}\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right)$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$\text{Rowspace}(R) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right)$$

= subspace of $\mathbb{R}^4$.

3^{rd} ~~row~~ (zero vector) is in the row space too. But it is not in a basis for the row space. The basis vectors must be independent.

Dimension of a Vector Space

There are many choices for the basis vectors, But the # of basis vectors doesn't change.

* If $v_1, \dots, v_m$ and $w_1, \dots, w_n$ are both bases for the same vector space, then $m = n$

Proof

Suppose, there are more w's than v's.

$$n > m$$

v's are a basis $\longrightarrow$ w_1 must be a combination of the v's.

If $w_1 = a_{11}v_1 + a_{21}v_2 + \dots + a_{m1}v_m$, which is the 1st column of a matrix multiplication VA:

Each w is a combination of the v's: $W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

$$W = VA$$

$A_{m \times n}$ is a short wide matrix, since $n > m$

$n > m \geq r \implies n - r \geq n - m > 0 \implies n - r = \dim[N(A)] > 0$

$-m \leq -r$ $\therefore$ $Ax = 0$ has a non-zero solution

Stack 9/4/20

$\implies VAx = Wx = 0$ has a non-zero solution

$\therefore$

W's are not independent

W's could not form a basis.

$\therefore$ Our assumption $n > m$ is not possible.

If $m > n$,

we exchange the V's and W's and repeat the same steps.

The only way to avoid a contradiction is to have $m = n$.

The # of basis vectors depends on the space, not on a particular basis. The # is the same for every basis, & it counts the "degree of freedom" in the space.

* The dimension of a space is the # of vectors in every basis.

Ex:-

The line thro' $V = (1, 5, 2)$ has dimension 1

It is a subspace with this one vector V in its basis.

$\perp$ to that line is the plane $x + 5y + 2z = 0$.

This plane has dimension 2.

The plane is the nullspace of the matrix $A = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix}$, which has 2 free variables.

Our basis vectors $\begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are the "special solutions" to $Ax = 0$.

Bases for Matrix Spaces & Function Spaces

In differential calculus,

$\boxed{\frac{d^2y}{dx^2} = y}$ has a space of solutions.

One Basis is $y = e^{x}$ and $y = e^{-x}$

Counting the basis functions gives the dimension 2 for the space of all functions. (the dimension is 2 because of the 2nd derivative).

Set $y = e^{ax}$

$$\frac{d^2y}{dx^2} = a^2 e^{ax} = a^2 y = e^{ax} \Longrightarrow a^2 = 1 \Rightarrow a = \pm 1$$

$$y = Ae^{x} + Be^{-x}$$

Matrix Spaces

The vector space M contains all 2x2 matrices.
Its dimension is 4.

$$\text{Basis for } M: A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Those matrices are linearly independent.
Combinations of those 4 matrices can produce any matrix ~~A~~ in M, so they span the space.

$$\boxed{c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = A}$$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ iff $c_1 = c_2 = c_3 = c_4 = 0$

$\rightarrow$ A_1, A_2, A_3, A_4 are independent.

- The 3 matrices A_1, A_2, A_4 are a basis for a subspace – the upper triangular matrices. Its dimension is 3.

- A_1 & A_4 are a basis for the diagonal matrices.

- $A_1, A_2 + A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis for the symmetric matrices.

- The dimension of the whole $n \times n$ matrix space is n^2.

- The dimension of the subspace of upper triangular matrices is $\frac{n(n+1)}{2}$

- The dimension of the subspace of diagonal matrices is 'n'.

- The dimension of the subspace of symmetric matrices is $\frac{n(n+1)}{2}$

Note: The upper (or lower) part of ~~the~~ a symmetric matrix completely determines the other half.

Function spaces

- $y'' = 0$ is solved by any linear function
$$y = cx + d.$$

$\cancel{y}$

- $y'' = -y$ is solved by any combination
$$y = ae^{ix} + be^{-ix}$$
$$= c\cos x + d\sin x$$

- $y'' = y$ is solved by any combination
$$y = ce^{x} + de^{-x}.$$

The solution space for $\boxed{y'' = -y}$ has 2 basis functions : $\sin x$ and $\cos x$

The solution space for $\boxed{y'' = 0}$ has the basis x and 1. It is the "nullspace" of the 2^{nd} derivative. $\frac{d^2}{dx^2} y = 0$

The dimension is 2 in each case. (these are 2^{nd} order equations)

- The solutions of $y'' = 2$ don't form a subspace.

A particular solution is $y(x) = x^2$. The complete solution is $y(x) = x^2 + cx + d$. All those functions satisfy $y'' = 2$

Notice : $y(x) =$ particular solution $+$ Any function $cx+d$ in the nullspace.

A linear differential equation is like a linear matrix equation $Ax = b$. But we solve it by calculus instead of linear algebra.

- The space Z contains only the zero vector. The dimension of Z is zero.

→ The empty set (containing no vectors) is a basis for Z.

→ We can never allow the zero vector into a basis, because then linear independence is lost.

$\boxed{3.4(A)}$ $v_1 = (1,2,0)$, $v_2 = (2,3,0)$

(c) What space V do they span?

(e) Which matrices A have V as their column space?

(f) Which matrices have V as null space?

(g) Describe all vectors v_3 that complete a basis v_1, v_2, v_3 for R^3

Ans:

(a) V contains all vectors $(x,y,0)$. xy-plane in $\mathbb{R}^3$.

(e) V is the column space of any ~~matrix~~ $3\times n$ matrix 'A' of rank 2, if every column is a linear combination of v_1 & v_2.

(f)
$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} A_1 V = 0 \\ A_2 V = 0 \\ \vdots \\ A_n V = 0. \end{matrix}$$

$$V = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right)$$

$$\rightarrow A_i = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

V is the null space of any $m \times 3$ matrix of rank 1, if every row is a multiple of $(0,0,1)$.

(g) V_3 if $c_1V_1 + c_2V_2 + c_3V_3 = 0$
only ~~if~~ $c_1, c_2, c_3 = 0$.

V_3 is not a multiple of V_1 & V_2.

Any $V_3 = (x, y, z)$ will complete a basis for $\mathbb{R}^3$, ~~if~~ $z \neq 0$.

$\boxed{3.4(B)}$ Start with the 3 independent vectors W_1, W_2, W_3. Take combinations of those vectors to produce V_1, V_2, V_3. Write the combinations in the matrix form as $\boxed{V = WB}$:

$$V_1 = W_1 + W_2$$
$$V_2 = W_1 + 2W_2 + W_3$$
$$V_3 = \quad W_2 + cW_3$$

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}}_{\text{change of basis matrix, } B}$$

What's the test on B to see if $V = WB$ has independent columns?
If $c \neq 1$ show that v_1, v_2, v_3 are linearly independent? If $c = 1$, show that the v's are linearly dependent.

Ans: For the columns of V to be independent, $N(V)$ must contain only the zero vector.

i.e., $(0,0,0)$ is the only combination of the columns that gives $Vx = 0$

If $c = 1$, $v_1 + v_3 = v_2 \Longrightarrow v_1 - v_2 + v_3 = 0$

$\therefore$ V's are not independent.

(OR)

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{\text{free}}{} \Rightarrow N(B) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$$

$x_1 + x_2 = 0$ & $x_2 + x_3 = 0$

$x_1 = -x_2 = x_3$ $\quad x_2 = -x_3$.

$\neq Z$

$Bx = 0 \longrightarrow Vx = WBx = 0$

$c \neq 1$, B is invertible.

$$B x = 0 \quad \text{iff} \quad x = (0,0,0)$$

$$W B x = 0 \quad \text{iff} \quad x = (0,0,0)$$

$\therefore$ ~~Ws~~ V's are independent.

The general rule is "independent V's from independent W's when B is invertible".
And if these vectors are in $\mathbb{R}^3$, they are not only independent - they are a basis for $\mathbb{R}^3$.

* Basis of V's from basis of W's when the change of basis matrix B is invertible.

$\boxed{3.4(C)}$ Suppose $v_1, \dots, v_n$ is a basis for $\mathbb{R}^n$ and the $n \times n$ matrix 'A' is invertible. Show that $Av_1, \dots, Av_n$ is also a basis for $\mathbb{R}^n$.

Ans :

Matrix language :

Let $V = [v_1 \cdots v_n]$

$$AV = A[v_1 \cdots v_n] = [Av_1 \cdots Av_n] = W.$$

A is invertible $\longrightarrow$ AV is invertible

$Av_1 \dots Av_n$ give a basis.

Vector language :

Suppose $c_1 Av_1 + \dots + c_n Av_n = 0$

$$A(c_1 v_1 + \dots + c_n v_n) = Av = 0$$

A is invertible $\longrightarrow$ $c_1 v_1 + \dots + c_n v_n = v = 0$

Linear independence of v's $\longrightarrow$ all $c_i = 0$.

$\therefore$ ~~the~~ Av's are independent.

Dimensions of the 4 subspaces

Rank $\longleftrightarrow$ Dimension

Rank: # of the pivots

Dimension: # of vectors in a basis

The rank of A reveals the dimension of all 4 fundamental subspaces.

4 fundamental subspaces

- The row space is $C(A^T)$, a subspace of $\mathbb{R}^n$
- The column space is $C(A)$, a subspace of $\mathbb{R}^m$
- The null space is $N(A)$, a subspace of $\mathbb{R}^n$
- The left nullspace is $N(A^T)$, a subspace of $\mathbb{R}^m$

* For the left nullspace $N(A^T)$, we solve

$$\boxed{A^T y = 0 \implies y^T A = 0^T}$$.

This is the null space of A^T.

4 subspaces for R / rref(A)

* The dimension of the row space is the rank r.
The nonzero rows of R form a basis.

Reasoning: Look only at the pivot columns, we see the r by r identity matrix.

There is no way to combine its rows to give the zero row (except by the combination with all coeff. zero). So the r pivot rows are a basis for the row space.

* The dimension of the column space is the rank r. The pivot columns form a basis.

Reasoning: The pivot columns start with the $r \times r$ identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coeff. zero). And they span the column space. Every other column is a combination of the pivot columns.

$\therefore$ The pivot columns are a basis for $C(R)$.

* The nullspace has dimension $(n-r)$.
The special solutions form a basis.

Reasoning: There is a special solution for each free variable. With 'n' variables and 'r' pivots that leaves $(n-r)$ free variables and special solutions. The special solutions are independent.

$\therefore$ $N(R)$ has dimension $(n-r)$

* The nullspace of R^T (left nullspace of R) has dimension $m-r$

Reasoning : $R^T y = 0 \iff y^T R = 0^T$

The eq^n. $R^T y = 0$ looks for combinations of the columns of R^T (the rows of R) that produce zero.

$$\boxed{y^T R = 0^T \qquad \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} \begin{bmatrix} \vec{R_1} \\ \vec{R_2} \\ \vdots \\ \vec{R_m} \end{bmatrix} = y_1 \vec{R_1} + y_2 \vec{R_2} + \cdots + y_m R_m = 0}$$

R ends with $(m-r)$ zero rows. Every combination of these $(m-r)$ rows gives zero. These are the only combinations of the rows of R that gives zero, because the pivot rows are linearly independent.

$\therefore$ 'y' in the left nullspace has $y_1 = 0, \ldots, y_r = 0$.

Ex:-

$\rightarrow$

$-2($

$-$

Ex:-

$$\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix} \Longrightarrow Ax = b.$$

$$\rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & -4b_1 + b_2 + b_4 \end{bmatrix} \Longrightarrow$$

for it to have a solution

$b_3 - 2b_1 = 0$

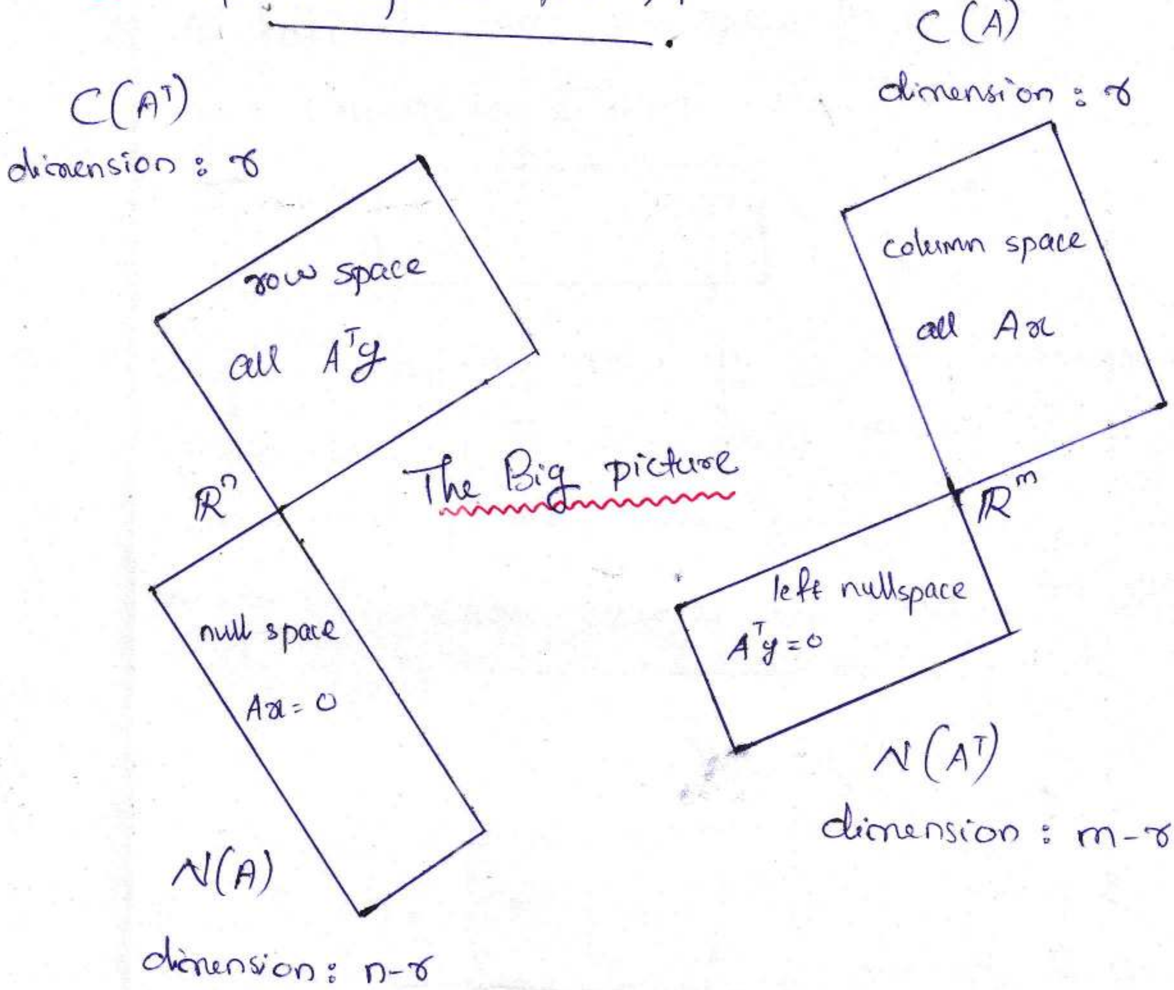
$-4b_1 + b_2 + b_4 = 0$

$-2(\text{row}1) + \text{row}3 = 0$

$-4(\text{row}1) + \text{row}2 + \text{row}4 = 0.$

$N(A^T)$ contains $(-2, 0, 1, 0), (-4, 1, 0, 1)$

4 Subspaces for A



$$\dim[C(A^T)] + \dim[N(A)] = n = \dim(\mathbb{R}^n)$$
$$\dim[C(A)] + \dim[N(A^T)] = m = \dim(\mathbb{R}^m)$$

* In $\mathbb{R}^n$, the rowspace & nullspace have dimensions r and $n-r$ (adding to n)

In $\mathbb{R}^m$, the column space and left nullspace have dimensions r and $m-r$ (total m).

* 'A' has the same row space as R.
Same dimension r and same basis.

$$\boxed{C(A^T) = C(R^T)}$$

Reasoning : Every row of A is a combination of the rows of R. Also, every row of R is a combination of the rows of A.
$\Longrightarrow$ Elimination changes the rows, but not row spaces.

Since $C(A^T) = C(R^T)$,
we can choose the 1^{st} r rows of R as a basis. Or we could choose r suitable rows of the original A. They might not always be the 1^{st} r rows of A, because those could be dependent.

The good r rows of A are the ones that end up as pivot rows in R.

* The column space of 'A' has dimension r.
The column rank equals the row rank.

Rank theorem:

# of independent columns = # of independent rows

column rank = row rank

$$\boxed{C(A) \neq C(R)}$$

$$\boxed{\dim[C(A)] = \dim[C(R)] = r}$$

The columns of R often end in zeros.
The columns of A don't end in zeros.

$$\Longrightarrow \quad C(A) \neq C(R)$$

Reasoning: The same combinations of the columns are zero (or non-zero) for A and R.

Dependent in $A \Longleftrightarrow$ Dependent in R

$Ax = 0$ exactly when $Rx = 0$.

$\Rightarrow$ The column spaces are different, but their dimensions are the same, equal to r.

$\Rightarrow$ The r pivot columns of 'A' are a basis for its column space $C(A)$.