Introduction to Linear Algebra
- Gilbert Strang

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Solving Linear Equations

↓

Markov Matrices, Population and Economics

↓

Vector Spaces & Subspaces

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$\boxed{2.7(B)}$ Find the symmetric factorization $S = LDL^T$

for $S = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$

Ans:
$$S = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & -14 & -22 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} = U$$

$l_{21} = 4, \; l_{31} = 5$ $\qquad$ $l_{32} = 1$

$$S = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|S| = |D| = 1(-14)(-8) = \underline{112}$$

$\boxed{2.7(C)}$ For a rectangular 'A'

Block matrix from least squares : $S = \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} = S^T$ has size $m+n$.

Apply block elimination to find block factorization, $S = LDL^T$. Then test invertability.

$$S = \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} I & A \\ 0 & -A^TA \end{bmatrix}$$

Block factorization,

$$S = LDL^T = \begin{bmatrix} I & 0 \\ A^T & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -A^TA \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}$$

$L$ & $L^T$ are invertible.

$$|L| = |L^T| = 1 \neq 0$$

$$|S| = 1 \cdot \begin{vmatrix} I & 0 \\ 0 & -A^T A \end{vmatrix} \cdot 1 = |I| \cdot |-A^T A|$$

$$= |A^T||A| = |A^T||A| = |A^2| = |A|^2$$

# The Transpose of a Derivative

The matrix changes to a derivative,

$$A = \frac{d}{dt}$$

To find the transpose of this unusual 'A', we need to define the inner product b/w two functions, $x(t)$ and $y(t)$.

Inner product of functions : $x^T y = (x, y) = \int_{-\infty}^{\infty} x(t)\, y(t)\, dt$

* The inner product changes from the sum of $x_k y_k$ to the integral of $x(t)\, y(t)$.

Linear Algebra for functions

* The word adjoint is more correct than transpose when we are working with derivatives.

$$(Ax, y) = \int_{-\infty}^{+\infty} \frac{dx}{dt} y(t)\, dt = -\int_{-\infty}^{+\infty} x(t) \frac{dy}{dt}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \left(-\frac{dy}{dt}\right) dt = (x, A^T y)$$

if $\left. x(t)\, y(t) \right]_{-\infty}^{\infty} = 0$

The transpose of a derivative is minus the derivative

$$A = \frac{d}{dt} \Longrightarrow A^T = \frac{-d}{dt}$$

The derivative is antisymmetric.

This antisymmetry of the derivative applies also to centered difference matrices :

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = -C$$

* In differential equations,
the $2^{nd}$ derivative (acceleration) is symmetric.

The $1^{st}$ derivative (damping proportional to velocity) is antisymmetric.

$$(Ax, y) = \int_{-\infty}^{+\infty} \frac{d^2x}{dt^2} \cdot y(t)\, dt = \underbrace{\cancel{y(t)\frac{dx}{dt}\Big]_{-\infty}^{+\infty}}}_{=0} - \int_{-\infty}^{+\infty} \frac{dy}{dt}\frac{dx}{dt}\, dt$$

$$= \underbrace{\cancel{-\frac{dy}{dt}x(t)\Big]_{-\infty}^{+\infty}}}_{0=} + \int_{-\infty}^{+\infty} x(t)\left(\frac{d^2y}{dt^2}\right) dt$$

$$= (x, Ay)$$

$$\boxed{A = \frac{d^2\cdot}{dt^2} \Longrightarrow A^T = \frac{d^2}{dt^2}}$$

boundary conditions : both $f(x)$ & $g(x)$ are zero at the boundaries.

$\boxed{P.S (2.1)}$

5. eq$^n$ ① + eq$^n$ ② = eq$^n$ ③

$$x+y+z=2$$
$$x+2y+z=3$$
$$2x+3y+2z=5$$

The 1st 2 planes meet along a line, because if $x, y, z$ satisfy the 1st 2 equations then they also satisfy the 3rd equation.

The eq$^n$s have infinitely many solutions (the whole line L). Find 3 solutions on L:

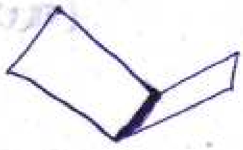
Ans: Plane passing thro' plane ① & plane ②:

$$\left.\begin{array}{l}\vec{r}\cdot(1,1,1)=2\\ \vec{r}\cdot(1,2,1)=3\end{array}\right\} \quad \vec{r}\cdot\{\alpha(1,1,1)+\beta(1,2,1)\}=2\alpha+3\beta.$$

$x=t$
$y=1,$
$z=1-t$

$$\longrightarrow \vec{r}\cdot\{(1,1,1)+(1,2,1)\}=\vec{r}\cdot(2,3,2)=2+3=5$$

passing thro' intersecting line



The line L of solutions contains $u=(1,1,0)$, $v=(2,1,-1)$ and all combinations $c\vec{u}+d\vec{v}$ with $c+d=1$

6. Move the 3rd plane in [5] to a llel plane $2x+3y+2z=9$.. Now the 3 eq's have no solutions — why not ? ~~The~~

Ans: $eq^n ① + eq^n ② - eq^n ③ = 0 \neq -4$

No solution

7. In problem [5], the columns are $(1,1,2)$, $(1,2,3)$, $(1,1,2)$. This is a singular case because 3rd column is ________ Find 2 combinations of the columns that give $b = (2,3,5)$. This is only possible for $b = (4,6,c)$ if $c =$ ________

Ans: column ③ $=$ column ①

$(x,y,z) = (1,1,0)$

$\vec{b} = (4,6,c) \longrightarrow c = 10$

9. Compute each $Ax$ by dot products of the rows with the column vector

(a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Ans: $\begin{bmatrix} 18 \\ 5 \\ 0 \end{bmatrix}$

13. (a) A matrix with 'm' rows & 'n' columns multiplies a vector with ______ components to produce a vector with ______ components.

(Ans) $A_{m \times n} x_{n \times 1} = b_{m \times 1}$

Ans:

(b) The planes from the 'm' equations $Ax = b$ are in ___n___ dimensional space.

The combination of the columns of A is in ___m___ dimensional space

Ans:

14. Write $2x+3y+z+5t=8$ as a matrix $A$?
The solutions $v=(x,y,z,t)$ fill a plane or hyperplane in 4D space?

Ans. $\begin{bmatrix} 2, 3, 1, 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 8$

$A = \begin{bmatrix} 2, 3, 1, 5 \end{bmatrix}$

The solutions $(x,y,z,t)$ fill a 3D plane or hyperplane in 4D space.
The plane is 3D with no 4D volume.

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17. Find the matrix $P$ that multiplies $(x,y,z)$ to give $(y,z,x)$. Find the matrix $Q$ that multiplies $(y,z,x)$ to bring back $(x,y,z)$.

Ans: $P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix}$

$P$ is orthogonal $\Rightarrow P^{-1} = P^T$

$Q = P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

20. What 2×2 matrix $P_1$ projects vector $(x,y)$ onto the x-axis to produce $(x,0)$? What matrix $P_2$ projects onto the y-axis to produce $(0,y)$?

~~$P_1\begin{bmatrix}5\\7\end{bmatrix}=?$~~ & ~~$P_2P_1\begin{bmatrix}5\\7\end{bmatrix}=?$~~

Ans. $P_1\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$

$P_2\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$

26. Draw the row & column pictures for the eq$^n$.

$x-2y=0$ , $x+y=6$

Ans. $x-2y=0$

$x+y=6$

two lines meeting at $(4,2)$

Row Picture :

$A\vec{x}=\begin{bmatrix}\vec{r}_1\\\vec{r}_2\end{bmatrix}\cdot\vec{x}=\begin{bmatrix}\vec{r}_1\cdot\vec{x}\\\vec{r}_2\cdot x\end{bmatrix}=\vec{b}=\begin{bmatrix}0\\6\end{bmatrix}$

Column Picture :

$4\begin{bmatrix}1\\1\end{bmatrix}+2\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\6\end{bmatrix}$ combination of 2 column vectors produce $(4,2)$

27. For 2 linear equations 3 unknowns $x, y, z$ the row picture will show (2 or 3) lines/planes in 2 or 3 -D space. The column picture is in (2 or 3) - D space. The solution normally lie on a ______.

Ans: row picture: show 2 planes in 3D space.

column picture: is in 2D space

The solutions normally fill a line in 3D space.

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28. For 4 linear equations in 2 unknowns $x$ & $y$, the row picture shows four ______. The column picture is in ______-D space. The eqⁿs have no solution unless the vector on the right side is a combination of ______

Ans: row picture: shows 4 lines in 2D plane.

column picture: in 4D space

No solution unless the RHS is a linear combination of the 2 column vectors.

(29) Start with the vector $u_0 = (1,0)$. Multiply again and again by the same "Markov matrix" $A = [0.8, 0.3 ; 0.2, 0.7]$. The next 3 vectors are $u_1, u_2, u_3$:

What property do you notice for all 4 vectors $u_0, \ldots, u_3$?

Ans:

(Stochastic matrices)

* A $n \times n$ matrix is called a Markov matrix if all entries are non-negative and the sum of each column vector equal to 1 ie, $\boxed{\begin{matrix} a_{ij} \geq 0 \text{ for } 1 \leq i, j \leq n \\ \sum_{j=1}^{n} a_{ij} = 1 \text{ for } 1 \leq i \leq n \end{matrix}}$

- If $\vec{v}$ is a stochastic vector & A is a stochastic matrix, then $A\vec{v}$ is a stochastic vector.

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = u_1$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = u_2$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = u_3$$

Proof

$$A\vec{v} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} = \vec{a}_1 v_1 + \vec{a}_2 v_2 + \cdots + \vec{a}_n v_n$$

$$= v_1(a_{11} + a_{21} + \cdots + a_{n1}) + \cdots + v_n(a_{11} + \cdots + a_{n1})$$

$$= v_1 + \cdots v_n = 1$$

Similarly,

* The product of 2 $n \times n$ stochastic matrices is a stochastic matrix.

* If $\lambda \in \mathbb{C}$ is an eigenvalue of a stochastic / Markov matrix, then $|\lambda| \leq 1$

Proof

Let $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ and $\vec{x} \in V_n(\mathbb{C})$ is a corresp. eigenvector & 

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for all } i = 1, \ldots, n$$

$$A\vec{x} = \lambda \vec{x}$$

Let $k$ be such that $|x_k| \geq |x_j| \; \forall \, 1 \leq j \leq n$

i.e; largest entry of $\vec{x}$ is $x_k$

Equating the $k^{th}$ components,

$$\sum_{j=1}^{n} a_{kj} x_j = \lambda x_k$$

$$|\lambda x_k| = |\lambda||x_k| = \left|\sum_{j=1}^{n} a_{kj} x_j\right| \leq \sum_{j=1}^{n} a_{kj} |x_j|$$

$$\leq \sum_{j=1}^{n} a_{kj} |x_k| = |x_k|$$

$[a_{kj} \geq 0$

$$\Rightarrow |\lambda| \leq 1$$

(OR) A is Markov.

~~$A\vec{x} = \lambda \vec{x}$~~

~~$A^2 \vec{x} = \lambda^2 \vec{x}$~~

~~$A^2$ is also Markov~~

- A Markov matrix 'A' always has an eigenvalue 1

All other eigenvalues are in absolute value smaller (or) equal to 1.

$M^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Proof

Each column of A sums to 1.

$\therefore$ Each column of $(A-I)$ sums to 0.

ie; Sum of the rows of $(A-I)$ is zero vector

$\therefore$ Rows of $(A-I)$ are linearly dependent

$\det(A-I) = 0 \rightarrow \lambda = 1$ is an eigenvalue

- The eigenvector $x_1$ corresp. to $\lambda_1 = 1$ is the steady sta

check ILA-②

## 10.3 Markov Matrices, Population, and Economics

Positive matrices : every $a_{ij} > 0$

$\Rightarrow$ The largest eigenvalue is real & +ve and so is its eigenvector.

In economics, ecology, population dynamics and random walks, that fact leads a long way:

Markov : $\lambda_{max} = 1$

Population : $\lambda_{max} > 1$

Consumption : $\lambda_{max} < 1$.

## Perron-Frobenius theorem

For $A > 0$, ie; $a_{ij} > 0 \ \forall ij$

All numbers in $Ax = \lambda_{max} x$ are strictly +ve

Proof

We'll consider all numbers $t$ such that $Ax \geq tx$ for some nonnegative vector $x$ (other than $x = 0$). [we are considering all possible nonzero $x$]

Stuck 4/2/21

We are allowing inequality in $Ax \geq tx$ inorder to have many small +ve candidates $t$.

If $V \geq W$ is not an equality,

ie; atleast one of the elements of the vector $V-W$ is greater than zero

ie; atleast one row $(V_j - W_j) > 0$

$i^{th}$ row of the vector $A(V-W)$ is: $\sum_{j=1}^{\infty} a_{ij}(V_j - W_j) > 0$

since $a_{ij} > 0$ for the positive square matrix $A$.

$\Longrightarrow A(V-W) > 0$

i.e. every entry of $Av$ is strictly greater than $Aw$.

$\therefore \Rightarrow Av > Aw$

---

Using this reasoning,

If $Ax \geq t_{max} x$ is not an equality,

Since 'A' is positive,

$$A(Ax) > t_{max}(Ax)$$

$\therefore$ the +ve vector $y = Ax$ satisfies

$$Ay > t_{max} y$$

$\therefore$ $t_{max}$ can be increased so that atleast one row of $Ay$ and $t_{max} y$ are equal.

$\therefore$ $t_{max}$ is not the maximum.

$\Longrightarrow$ $Ax = t_{max} x$ , we have an eigenvalue.

Suppose. $Az = \lambda z$ and $\lambda, z$ may involve ~~non~~ negative or complex numbers, so we take absolute values:

$$|\lambda||z| = |Az| \leq A|z|$$

$|z|$ is a non-negative vector, so this $|\lambda|$ is one of the possible candidates $t$.

$\therefore$ $|\lambda|$ can not exceed $t_{max}$, which must be $\lambda_{max}$.

## Absolute value of vectors & matrices

Given $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$

$$|x| = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix} \quad \text{and} \quad |A| = \begin{bmatrix} |a_{11}| & |a_{12}| & \cdots & |a_{1n}| \\ |a_{21}| & |a_{22}| & \cdots & |a_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |a_{m1}| & |a_{m2}| & \cdots & |a_{mn}| \end{bmatrix}$$

Let $A \in \mathbb{R}^{m \times k}$ and $B = \mathbb{R}^{k \times n}$,

$$\boxed{|AB| \leq |A||B|}$$

**Proof**

Let $C = AB$,

then the $(i,j)$ entry in $|C|$ is given by

$$|\gamma_{ij}| = \left| \sum_{p=1}^{k} \alpha_{ip} \beta_{pj} \right| \leq \sum_{p=1}^{k} |\alpha_{ip} \beta_{pj}|$$

$$= \sum_{p=1}^{k} |\alpha_{ip}||\beta_{pj}|$$

which is equal to the $(i,j)$ entry of $|A||B|$

## Markov Matrices

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad u_1 = Au_0$$

$$u_2 = Au_1 = A^2 u_0$$

After k steps we have $A^k u_0$. The vectors $u_1, u_2, \ldots$ will approach a "steady state" $u_\infty = (0.6, 0.4)$. This final outcome does not depend on the starting vector $u_0$. For every $u_0 = (a, 1-a)$ we converge to the same $u_\infty (0.6, 0.4)$

Why ?

The steady state equation $Au_\infty = u_\infty$ makes $u_\infty$ an eigenvector with eigenvalue 1:

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = u_\infty \longleftarrow \text{steady state.}$$

But this does not explain why so many vectors $u_0$ lead to $u_\infty$.

Other examples might have a steady state, but its not necessarily attractive:

Not Markov:

$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has the unattractive steady state

$$B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this case, the starting vector $u_0 = (0, 1)$ will give $u_1 = (0, 2)$ and $u_2 = (0, 4)$.
The $2^{nd}$ components are doubled.

$B$ has $\lambda = 1$ but also $\lambda = 2$ as eigenvalues.
— this produces instability.

The comp. of $u$ along that unstable eigenvector is multiplied by $\lambda$, and $|\lambda| > 1$ means blowup.

Markov matrices : ① Every entry of 'A' is +ve

$a_{ij} > 0$

② Every column of A adds to 1

① → Multiplying $u_0 \geq 0$ by 'A' produces a non-negative $u_1 = Au_0 \geq 0$.

② → If the comp. of $u_0$ add to 1, so do the components of $u_1 = Au_0$.

Every vector $A^k u_0$ is nonnegative with components adding to 1. These are "probability vectors." The limit $u_\infty$ is also a probability vector.

* $u_\infty$ is an eigenvector of 'A' corresp. to $\lambda = 1$
* $\lambda_{max} = 1$ for a +ve Markov matrix
* The size $|\lambda_2|$ of the $2^{nd}$ eigenvalue controls the speed of convergence to steady state.

Ex 1. The fraction of rental cars in Denver starts at $\frac{1}{50} = 0.02$. The fraction outside Denver is 0.98. Every month, 80% of the Denver cars stay in Denver (and 20% leave). Also 5% of the outside cars come in (95% stay outside). This means that the fractions $u_0 = (0.02, 0.98)$ are multiplied by $A$:

Ans:

1st month: $A = \begin{bmatrix} 0.80 & 0.05 \\ 0.20 & 0.95 \end{bmatrix}$ leads to

$$u_1 = Au_0 = A\begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 0.065 \\ 0.935 \end{bmatrix}$$

$0.065 + 0.935 = 1$. All cars are accounted for.

Next month:

$$u_2 = Au_1 = \begin{bmatrix} 0.09875 \\ 0.90125 \end{bmatrix} = A^2 u_0$$

$A x =$

Each

by

The $1^{st}$ components have grown from 0.02 and cars are moving toward Denver.

Since every column of 'A' adds to 1, nothing is lost or gained. We are moving rental cars or populations, and no cars or people suddenly appear (or disappear).
The fractions add to 1 and the matrix A keeps them that way.

$A^k u_0$ gives the fractions in and out of Denver after k steps. The eigenvalues of A are $\lambda = 1$ and 0.75.

$A x = \lambda x$: $A\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = 1\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$ and $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.75\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$u_0 = \begin{bmatrix} 0.02 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + 0.18\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Each $x$ is multiplied by $\lambda$: $u_1 = 1\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + (0.75)(0.18)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$u_k = A^k u_0 = 1^k \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + (0.75)^k (0.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The eigenvector $x_1$ with $\lambda = 1$ is the steady state. The other eigenvector $x_2$ disappears because $|\lambda| < 1$. The more steps we take, the closer we come to $u_\infty = (0.2, 0.8)$. In the limit, $\frac{2}{10}$ of the cars are in Denver and $\frac{8}{10}$ are outside. This is the pattern for Markov chains, even starting from $u_0 = (0, 1)$.

If 'A' is a +ve Markov matrix (entries $a_{ij} > 0$, each column adds to 1), then $\lambda_1 = 1$ is larger than any other eigenvalue. The eigenvector $x_1$ is the steady state:

$$u_k = x_1 + c_2 (\lambda_2)^k x_2 + \cdots + c_n (\lambda_n)^k x_n$$ always approaches $u_\infty = x_1$

Another eigenvalue has $|\lambda| = 1$ ?

Ex:2. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no steady state because $\lambda_2 = -1$.

Ans: The powers $A^k$ alternate b/w $A$ and $I$.
The 2nd eigenvector $x_2 = (-1, 1)$ will be multiplied by $\lambda_2 = -1$ at every step - and does not become smaller : No steady state.

Ex: 3. Start with 3 groups. At each time step, half of group 1 goes to group 2, & the other half goes to group 3. The other groups also split in half and move. Take one step from the starting populations $p_1, p_2, p_3$ :

Ans: New populations: $u_1 = Au_0 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_2 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_2 \end{bmatrix}$

'A' is a Markov matrix. Nobody is born or lost.

$$u_2 = A^2 u_0 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Eigenvalues of $A$ are: $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1/2$

For $\lambda_1 = 1$, the eigenvector $x_1 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ will be the steady state.

When 3 equal populations split in half and move, the populations are again equal.

## □ PageRank algorithm / Google algorithm

– an algorithm based on Classical Random walks.

Page rank is 1st proposed by Page in 1999. The purpose is to rank the web page in the World Wide Web (WWW).
ie., to define what is the importance of a webpage.

The network of webpage is considered as a graph where webpages are considered as nodes. If there is a webpage containing a hyperlink which points to another webpage, then there should be a directed edge b/w these 2 nodes. The direction of the edge is as same as the web ~~direction~~ redirection.

The most simple PageRank can be described by the following mathematical equation:

$$R(u) = c \sum_{v \in B_u} \frac{R(v)}{N_v}$$

where,

$R(u)$ : rank of the web $u$.

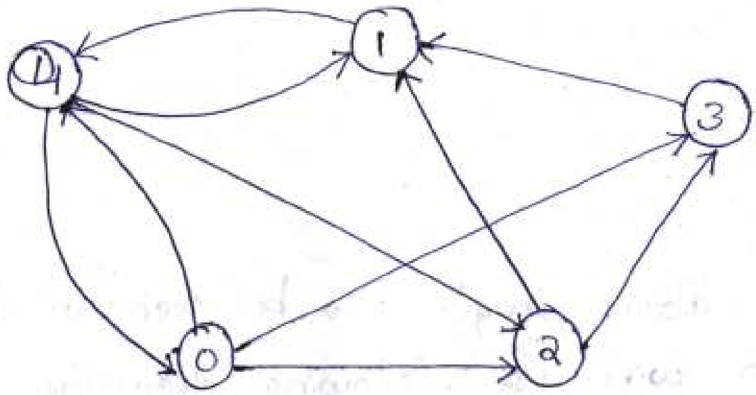
$B_u$ : set of pages pointing to page $u$.

$c$ : normalization parameter.

$F_v$ : set of pages that $v$ points to

$N_v$ : # of pages in $F_v$

## Example




$$R(0) = \frac{R(4)}{3}$$

$$R(1) = \frac{R(2)}{2} + R(3) + \frac{R(4)}{3}$$

$$R(2) = \frac{R(0)}{3} + \frac{R(4)}{3}$$

$$R(3) = \frac{R(0)}{3} + \frac{R(2)}{2}$$

$$R(4) = \frac{R(0)}{3} + R(1)$$

The above graph can be seen as a Markov chain with the following transition matrix.

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 & 0 \end{bmatrix}$$

For the initial distribution, lets consider that it is equal to

$$R_0 = \left(\frac{1}{n}, \frac{1}{n}, \ldots\ldots, \frac{1}{n}, \frac{1}{n}\right)$$

where $n$ is the total # of nodes.

ie.,

the random walker will choose randomly the initial node from where it can reach all other nodes.

At every step, the random walker will jump to another node according to the transition matrix, the probability distribution is then computed for every step. This distribution tells us where the random walker is likely to be after a certain # of steps.

The probability distribution is computed using the following equation:

$$R_{t+1} = P R_t$$

where,

R is the vector of page rank.

P is the transition probability matrix.

In ~~this~~ example,

after an infinitely long walk, the probability distribution will converge to a stationary distribution R.

⇒

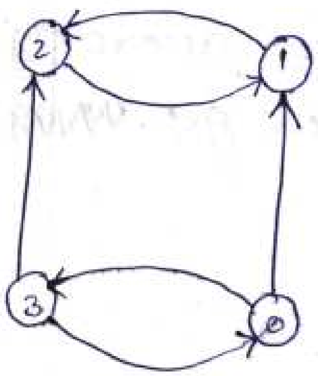
the probability distribution at time t defines the probability that the walker will be in a node after t steps.

The higher the probability, the more important is the node.

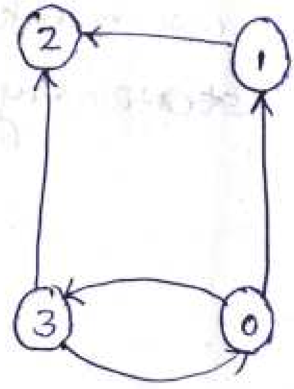
We can rank our webpages according to the stationary distribution we get. ~~using~~

## Teleportation & damping factor

In the webgraph, for example, we can find a webpage $i$ which refers only to webpage $j$ and $j$ refers only to $i$. This is what we call spider trap problem. We can also find a webpage which has no outlinks. It is commonly named Dead end.




~~Dead~~
Spider trap




Dead end

In the case of a spider trap, when the random walker reaches the node 1, he can only jump to node 2 and from node 2 he can only reach node 1, and so on. The importance of all other nodes will be taken by nodes 1 and 2.
In our example, the probability distribution will converge to $R = (0, \frac{1}{2}, \frac{1}{2}, 0)$.
This is not the desired result.

In the case of Dead ends, when the walker arrives at node 2, it can't reach any other node because it has no outlink. The algorithm can not converge.

To get over these problems, we introduce the notion of teleportation.

Teleportation consists of connecting each node of the graph to all other nodes. The graph will then be complete.

The idea is with a certain probability $\beta$, the random walker will jump to another node according to the transition matrix $P$. and with a probability $\frac{1-\beta}{n}$, it will jump randomly to any node in the graph.

The new transition matrix $P'$:

$$P' = \beta P + (1-\beta) v e^T$$

where $v = [1, \ldots, 1]^T$

$$e^T = [1/n, \ldots, 1/n]$$

$\beta$ : damping factor.

In practice it is advised to set $\beta = 0.85$.

In our example, the new transition matrix:

$$P' = \begin{bmatrix} \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1}{3}\beta + \frac{1-\beta}{5} \\ \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1}{2}\beta + \frac{1-\beta}{5} & \beta + \frac{1-\beta}{5} & \frac{1}{3}\beta + \frac{1-\beta}{5} \\ \frac{1}{3}\beta + \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1}{2}\beta + \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1}{3}\beta + \frac{1-\beta}{5} \\ \frac{1}{3}\beta + \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1}{2}\beta + \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1-\beta}{5} \\ \frac{1}{3}\beta + \frac{1-\beta}{5} & \beta + \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1-\beta}{5} & \frac{1-\beta}{5} \end{bmatrix}$$

# Markov chain

A Markov chain is a memoryless, homogeneous, stochastic process with a finite # of states.

A process is a system that changes after each time step t, and a stochastic process is a process in which the changes are random.

A process is memoryless if the probability of an $i \longrightarrow j$ transition does not depend on the history of the process.

A process is homogeneous if it does not depend on the time t.

Let $\{X_0, X_1, \dots\}$ be a sequence of discrete random variables. Then $\{X_0, X_1, \dots\}$ is a Markov chain if it satisfies the Markov property:

$$P(X_{t+1} = s \mid X_t = s_t, \dots, X_0 = s_0) = P(X_{t+1} = s \mid X_t = s_t)$$

for all $t = 1, 2, 3, \dots$ & for all states $s_0, s_1, \dots, s_t, s$

# Markov chain

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for all $t = 1, 2, 3, \dots$ & for all states $s_0, s_1, \dots, s_t, s$

The matrix describing the Markov chain is called the transition matrix.

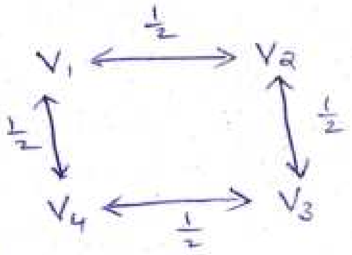
Let $\{X_0, X_1, \cdots\}$ be a Markov chain with state space $S$,

the transition probabilities of the Markov chain are

$$P_{ij} = P\left(X_{t+1} = j \mid X_t = i\right) \quad \text{for } i, j \in S, \quad t = 0, 1, 2, \cdots$$

Ex:- Consider a person on a square ~~where the~~

Suppose ~~this~~ person starts at $V_1$, ie., $q(0) = (1,0,0,0)$ and flips a coin to decide b/w going one way or the other way.




After $n$ steps, regardless of where our person has been, the probability of the going one of the 2 possible directions is still $\frac{1}{2}$ & this only depends on the current state.

$$P(X_n = V_j) = P(X_n = V_j \mid X_{n-1} = V_i) \quad \text{where } i \to j$$

Transition matrix
$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

# Population Growth

Divide the population into 3 age groups

- age < 20
- age 20 to 39
- age 40 to 59

At year $T$ the sizes of those groups are $n_1, n_2, n_3$.

20 years later, the sizes have changed for 3 reasons : births, deaths, getting older.

① Reproduction :

$$n_1^{new} = F_1 n_1 + F_2 n_2 + F_3 n_3$$ gives a new generation.

② Survival :

$$n_2^{new} = P_1 n_1 \quad \text{and} \quad n_3^{new} = P_2 n_2$$

gives older generations.

The fertility rates are $F_1, F_2, F_3$ ($F_2$ largest).

The Leslie matrix 'A' might look like:

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{new} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0.04 & 1.1 & 0.01 \\ 0.98 & 0 & 0 \\ 0 & 0.92 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

This is population projection in its simplest form, the same matrix 'A' at every step. In a realistic model, 'A' will change with time (from the environment or internal factors).

The matrix 'A' has $A \geq 0$ but not $A > 0$. The Perron-Frobenius theorem still applies because $A^3 > 0$. The largest eigenvalue is $\lambda_{max} \approx 1.06$.

$\lambda = 1.06, -1.01, -0.01$

$$A^2 = \begin{bmatrix} 1.08 & 0.05 & 0. \\ 0.04 & 1.08 & 0.01 \\ 0.90 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0.10 & 1.19 & 0.01 \\ 0.06 & 0.05 & 0.00 \\ 0.04 & 0.99 & 0.01 \end{bmatrix}$$

The middle group will reproduce 1.1 and also survive 0.92.

## Sensitivity of eigenvalues

Ex: 10.3 (19)

$$AX = X\Lambda \Longrightarrow A = X\Lambda X^{-1}$$

The eigenvalues and eigenvectors change by $\Delta\Lambda$ and $\Delta X$:

$$(A+\Delta A)(X+\Delta X) = (X+\Delta X)(\Lambda+\Delta\Lambda)$$

$$\cancel{AX} + A(\Delta X) + (\Delta A)(X) = \cancel{X\Lambda} + X(\Delta\Lambda) + (\Delta X)\Lambda$$

Small terms $(\Delta A)(\Delta X)$ and $(\Delta X)(\Delta\Lambda)$ are ignored.

$$A(\Delta X) + (\Delta A)X = X(\Delta\Lambda) + (\Delta X)\Lambda$$

$$\underline{X^{-1}A(\Delta X)} + X^{-1}(\Delta A)X = \Delta\Lambda + \underline{X^{-1}(\Delta X)\Lambda}$$

$$X^{-1}A = \Lambda X^{-1} \Longleftarrow A = X\Lambda X^{-1}$$

$$\underline{\Lambda X^{-1}(\Delta X)} + X^{-1}(\Delta A)X = \Delta\Lambda + \underline{X^{-1}(\Delta X)\Lambda}$$

$$\operatorname{diag}(B\Lambda) = \operatorname{diag}(\Lambda B)$$

$$\operatorname{diag}\left(\Lambda X^{-1}\Delta X\right) = \operatorname{diag}\left(X^{-1}\Delta X\Lambda\right)$$

$$\operatorname{diag}\left(\Lambda X^{-1}(\Delta X) - X^{-1}(\Delta X)\Lambda\right) = 0$$

$$\Rightarrow \boxed{\operatorname{diag}\left(X^{-1}(\Delta A)X\right) = \Delta\Lambda}$$

⇒ A matrix change $\Delta A$ produces eigenvalue changes $\Delta\Lambda$. Those changes $\Delta\lambda_1, \ldots, \Delta\lambda_n$ are on the diagonal of $X^{-1}(\Delta A)X$.

□ ~~Linear~~

## □ Consumption Matrix — Linear Algebra in Economics

The consumption matrix tells how much of each i/p goes into a unit of output. This describes the manufacturing side of the economy.

We have n industries like chemicals, food and oil. To produce a unit of chemicals may require 0.2 units of chemicals, 0.3 units of food, and 0.4 units of oil. Those #s go into row 1 of the consumption matrix A:

$$\begin{bmatrix} \text{Chemical o/p} \\ \text{food o/p} \\ \text{oil o/p} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.1 \\ 0.5 & 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} \text{Chemical i/p} \\ \text{food i/p} \\ \text{oil i/p} \end{bmatrix}$$

Row 2 shows the i/p to produce food - a heavy use of chemicals and food, not so much oil. Row 3 of A shows the i/ps consumed to refine a unit of oil.

The real consumption matrix for the US in 1958 contained 83 industries. The model in the 1990's are much larger and more precise.

Ddn

Can this economy meet demands $y_1, y_2, y_3$ for chemicals, food & oil ?

To do that, the i/p $p_1, p_2, p_3$ will have to be higher - because part of $p$ is consumed in producing $y$. The i/p is $p$ and the consumption is $Ap$, which leaves the o/p $p - Ap$.

This net production is what meets the demand $y$:

Problem: Find a vector $p$ such that

$$p - Ap = y \quad \text{(or)} \quad p = (I - A)^{-1} y$$

$I - A$ invertible ?

The vector $y$ of required o/p is non-negative and so is 'A'.

The production levels in $p = (I-A)^{-1} y$ must also be non-negative.

When is $(I-A)^{-1}$ a non-negative matrix ?

If 'A' is small compared to $I$, then $Ap$ is small compared to $p$. There is plenty of output.

If 'A' is too large then production consumes too much and the demand $y$ can not be met.

"Small" or "Large" is decided by the largest eigenvalue $\lambda_1$ of $A$ (which is +ve):

$\lambda_1 > 1$ :

$\lambda_1 = 1$ : $(I-A)^{-1}$ fails to exist

$\lambda_1 < 1$ :

$$(I-A)\sum_{k=0}^{n} A^k = \sum_{k=0}^{n} A^k - \sum_{k=0}^{n} A^{k+1}$$

$$= \sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k = I - A^{n+1}$$

If $|\lambda_i| < 1$ for each eigenvalue $\lambda_i$ of $A$

then $(I-A)$ is invertible

$$S_n = \sum_{k=0}^{n} A^k = I + A + A^2 + \cdots + A^n$$

$$= (I-A)^{-1}(I - A^{n+1})$$

$$\lim_{n\to\infty} S_n = \sum_{k=0}^{\infty} A^k = (I-A)^{-1}$$

## Neumann series

$$(I-A)^{-1} = I + A + A^2 + \cdots\cdots$$

$$= \sum_{k=0}^{\infty} A^k$$

* it converges if all eigenvalues of 'A' have, $|\lambda| < 1$

Ex: 4 $A = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.1 \\ 0.5 & 0.1 & 0.3 \end{bmatrix}$

Ans. $\lambda_{max} = 0.9$

$$(I-A)^{-1} = \frac{1}{93}\begin{bmatrix} 41 & 25 & 27 \\ 33 & 36 & 24 \\ 34 & 23 & 36 \end{bmatrix}$$

$A$ is small compared to $I$

$p - Ap = y$ (or) $p = (I-A)y$

$Ap$ is consumed in production, leaving $p - Ap$.

This is $(I-A)p = y$ and the demand is met.

Ex: 5. $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

Ans: $\lambda_{max} = 2$ and $(I-A)^{-1} = \frac{-1}{3}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

This consumption matrix $A$ is too large.

Demands can't be met, because production consumes more than it yields.

The series $I + A + A^2 + \cdots$ does not converge to $(I-A)^{-1}$ because $\lambda_{max} > 1$.

The series is growing while $(I-A)^{-1}$ is -ve.

10.3

4. For every 4x4 Markov matrix, what eigenvector of $A^T$ corresp. to the eigenvalue $\lambda = 1$ ?

Ans:
Each row of $A^T$ adds to 1.

$\rightarrow$ $A^T$ always has the eigenvector $(1, 1, \cdots, 1)$ for $\lambda = 1$.

5. Every year 2% of young people become old and 3% of old people become dead. (No births). Find the steady state for

$$\begin{bmatrix} \text{Young} \\ \text{Old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} 0.98 & 0 & 0 \\ 0.02 & 0.97 & 0 \\ 0 & 0.03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_k .$$

Ans: The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1)$ = everyone is dead.

6. For a Markov matrix, the sum of the components of $x$ equals the sum of the components of $Ax$. If $Ax = \lambda x$ with $\lambda \neq 1$ prove that the components of this ~~steady~~ non-steady eigenvector $x$ add to zero.

Ans: Adding the components of $Ax = \lambda x$

$$s = \lambda s \quad \text{given } \lambda \neq 1$$

$$\Longrightarrow \underline{\underline{s = 0}}$$

7. Find the eigenvalues and eigenvectors of $A$. Explain why $A^k$ approaches $A^\infty$.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, \quad A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

Which Markov matrices produce that steady state $(0.6, 0.4)$?

8. The steady state eigenvector of a permutation matrix is $(1/4, 1/4, 1/4, 1/4)$. This is not approached when $u_0 = (0,0,0,1)$. What are $u_1$ and $u_2$ and $u_3$ and $u_4$? What are the 4 eigenvalues of $P$, which solve $\lambda^4 = 1$?

9.

Ans

Permutation matrix

||

Markov matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ans: $P$ = cyclic permutation.

$u_0 = (1,0,0,0)$, $u_1 = (0,0,1,0)$, $u_2 = (0,1,0,0)$

$u_3 = (1,0,0,0)$, $u_4 = u_0$

Eigenvalues are : $1, i, -1, -i$

(9.) Prove that the square of a Markov matrix is also a Markov matrix.

Ans: $M^2$ is non-negative

$$[1\ 1 \cdots 1]M = [1\ 1 \cdots 1]$$

$$[1\ 1 \cdots 1]M^2 = [1\ 1 \cdots 1]MM =$$

$$= [1\ 1 \cdots 1]M = [1\ 1 \cdots 1]$$

12. A Markov differential equation is not $\frac{du}{dt} = Au$ But,

$$\boxed{\frac{du}{dt} = (A-I)\,u}.$$

The diagonal is -ve, the rest of $A-I$ is +ve. The columns add to zero, not 1.

Find $\lambda_1, \lambda_2$ for $B = A - I = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$.

Why does $A-I$ have $\lambda_1 = 0$?

When $e^{\lambda_1 t}, e^{\lambda_2 t}$ multiply $x_1, x_2$,

What's the steady state as $t \to \infty$?

Ans:

$B$ has $\lambda = 0$ and $-0.5$

$x_1 = (0.3, 0.2)$ and $x_2 = (-1, 1)$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$= c_1 e^{0t} x_1 + c_2 e^{-0.5t} x_2$$

$$= c_1 x_1 + c_2 e^{-0.5t} x_2$$

$e^{-0.5t}$ approaches zero and the solution approaches $c_1 x_1$

15. For which of these matrices does $I+A+A^2+\cdots$ yield a non-negative matrix $(I-A)^{-1}$? Then the economy can meet any demand:

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad A=\begin{bmatrix}0&4\\0.2&0\end{bmatrix},\quad A=\begin{bmatrix}0.5&1\\0.5&0\end{bmatrix}$$

If the demands are $y=(2,6)$, what are the vectors $p=(I-A)^{-1}y$?

(b)

Ans:

(c)

Ans:

(a) $\lambda_1\lambda_2=0$ & $\lambda_1+\lambda_2=0$

$\cancel{0=2\lambda_1^2-2=2(\lambda_1^2-1)=0 \Rightarrow \lambda_1\neq 1}$

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

$$|A-\lambda I|=\begin{vmatrix}-\lambda&1\\0&-\lambda\end{vmatrix}=\lambda^2=0 \Rightarrow \underline{\lambda=0,0}$$

$\lambda_{max}<1$

$$p=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}2\\6\end{bmatrix}=\begin{bmatrix}8\\6\end{bmatrix}$$

(b) $A = \begin{bmatrix} 0 & 4 \\ 0.2 & 0 \end{bmatrix}$

Ans: $\lambda_1 = \pm 0.894427$ , $\lambda_{max} < 1$

$$P = \begin{bmatrix} 5 & 20 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 130 \\ 32 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$

$\lambda = 1, 0.5 \Longrightarrow \lambda_{max} = 1$

$I - A$ has no inverse

18. For the Leslie matrix show that $\det(A-\lambda I) = 0$ gives $F_1\lambda^2 + \lambda_2 P_1 \lambda + F_3 P_1 P_2 = \lambda^3$. The right side $\lambda^3$ is larger as $\lambda \to \infty$. The left side is larger at $\lambda = 1$ if $F_1 + F_2P_1 + F_3 P_1 P_2 > 1$. In that case the 2 sides are equal at an eigenvalue $\lambda_{max} > 1$: growth.

Ans: $$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{new} = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} F_1 - \lambda & F_2 & F_3 \\ P_1 & -\lambda & 0 \\ 0 & P_2 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + F_1\lambda^2 + F_2 P_1 \lambda + F_3 P_1 P_2$$

$|A - \lambda I| < 0$ for $\lambda \to \infty$

$|A - \lambda I| > 0$ for $\lambda = 1$ given

$F_1 + F_2 P_1 + F_3 P_1 P_2 > 1$

$\Rightarrow |A - \lambda I| = 0$ at some $\lambda \in (1, \infty)$

That eigenvalue means that the population grows.

20. Suppose $B > A > 0$, meaning that each $b_{ij} > a_{ij} > 0$. How does the Perron-Frobenius discussion show that $\lambda_{max}(B) > \lambda_{max}(A)$ ?

Ans: $Ax = \lambda_{max}(A) x > 0$

$\cancel{Bx = \lambda_{max} x}$

$Bx > Ax = \lambda_{max}(A) \, x > 0$

$Bx = \lambda_{max}(B) \, x > \lambda_{max}(A) \, x$

$$\lambda_{max}(B) > \lambda_{max}(A)$$

30. Cont. problem (29) from $u_0 = (1,0)$ to $u_7$, and also from $v_0 = (0,1)$ to $v_7$. What do you notice about $u_7$ & $v_7$?

Ans: $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_1 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$

$u_4 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}$, $u_5 = \begin{bmatrix} 0.6125 \\ 0.3875 \end{bmatrix}$

$u_6 = \begin{bmatrix} 0.60625 \\ 0.39375 \end{bmatrix}$, $u_7 = \begin{bmatrix} 0.603125 \\ 0.396875 \end{bmatrix}$

$0.603125 + 0.396875 = 1$

$u_7$ is close to, $s = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \text{Steady state, } S$$

* If $A$ is a +ve Markov matrix, $\lambda = 1$ is the only eigenvalue of modulus 1.

* If $A$ is a +ve Markov matrix, then $A^n$ has 1 as the only eigenvalue of modulus 1.

31. Invent a 3x3 "magic matrix" $M_3$ with entries $1, 2, \ldots, 9$. All rows & columns & diagonals add to 15. The $1^{st}$ row could be 8, 3, 4.

$M_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = ?$ $M_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ if a 4x4 magic matrix has entries $1, \ldots, 16$. ?

Ans:

$$M_3 = \begin{bmatrix} 8 & \cancel{1} & \cancel{6} \\ 3 & 5 & 7 \\ \cancel{4} & 9 & 2 \end{bmatrix}^{T} = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$$

$$M_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$$

$$M_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 34 \\ 34 \\ 34 \\ 34 \end{bmatrix}$$

$$1 + 2 + \cdots 16 = \frac{16 \times 17}{2} = 136 = 4(34)$$

$\boxed{2.2}$

7. For which # a does elimination breakdown

Ques: $ax + 3y = -3$
$4x + 6y = 6$

(a) Permanently

Ans: $a = 2 \Rightarrow$ 2 ||el lines in row picture

(b) temporarily

Ans: $a = 0 \rightarrow$ elimination will stop for row exchange.

8. For which 3 # k does elimination break down? Which is by a row exchange? In such case, is the # of solutions 0 (or) 1 (or) $\infty$?

$kx + 3y = 6$
$3x + ky = -6$

Ans: $k = 3$ : No solution.
$k = -3$ : infinitely many solutions.
$k = 0$ : row exchange is needed.

11. (a) system of linear eq. can't have exactly 2 solutions why?

(a) If $(x, y, z)$ & $(X, Y, Z)$ are 2 solutions. What is another solution?

(b) If 25 planes meet at 2 pts, where else do they meet?

Ans:

(a) Another $\frac{1}{2}(x+X, y+Y, z+Z)$

(b) If the planes meet at 2 pts, they meet along the whole line thro' those pts.

14. Which # d forces a row exchange, & what is the triangular system (not singular) for that d? Which 'd' makes the system singular (no 3rd pivot)?

$$2x+5y+z=0$$
$$4x+dy+z=2$$
$$y-z=3.$$

Ans.

$$2x+5y+z=0$$
$$(d-10)y-z=2$$
$$y-z=3.$$

If $d=10$, exchange rows 2 & 3.

If $d=11$, the system becomes singular

15. Which # b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a non-zero solution x, y, z

$$x+by=0$$
$$x-2y-z=0$$
$$y+z=0$$

Ans:

$$x+by=0$$
$$(-2-b)y-z=0$$
$$y+z=0.$$

If $b=-2$; we exchange rows 2 & 3.

If $b=-1$ : singular case.

$x=by$ & $z=-y$. $\Longrightarrow (t, t, -t)$

$(1, 1, -1)$ //

16. Construct a 3x3 system that needs 2 row exchanges to reach a triangular form & a solution.

(a)

(b) Construct a 3x3 system that needs a row exchange to keep moving, but breaks down later.

Ans:

(a)

$$2z = 4$$
$$x + 2y + 2z = 5$$
$$3y + 4z = 6$$

(b)

$$3y + 4z = 4$$
$$x + 2y + 2z = 5$$
$$3y + 4z = 10$$

17. If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange). If columns 1 & 2 are the same, which pivot is missing?

Ans. Equal Rows

$$\begin{aligned} 2x - y + z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + z &= 2 \end{aligned} \;\longrightarrow\; \begin{aligned} 2x - y + z &= 0 \\ 0 &= 0 \\ 3y - z &= 0 \end{aligned} \;\longrightarrow\; \begin{aligned} 2x - y + z &= 0 \\ 3y - z &= 0 \\ 0 &= 0 \end{aligned}$$

Equal columns

$$2x+2y+z=0$$
$$4x+4y+z=0$$
$$6x+6y+z=2$$

$\rightarrow$

$$2x+2y+z=0$$
$$-z=0$$
$$-2z=2$$

column 2 has no pivot

20. 3 planes can fail to have an intersection point, even if no planes are ||el. The system is singular if row 3 of A is a ________ of the 1st two rows. Find a 3rd eqn. that can't be solved together with $x+y+z=0$ & $x-2y-z=1$.

Ans: Linear comb of 1st 2 rows.

$$x+y+z=0$$
$$x-2y-z=1$$
$$2x-y = 4 \neq 1$$

$\rightarrow$ 3 planes form a $\Delta$ tunnel. No solution.

24. For which 2 # a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$

Ans. $a=0$ & $a=2$

25. For which 3 #'s 'a' will elimination fail to give 3 pivots?

$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}$ is singular for 3 values of a?

Ans: $a=0$ (zero row)

$a=4$ (equal rows)

$a=2$ (equal columns)

26. Look for a matrix that has row sums 4 & 8, and column sums 2 & S:

Matrix $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a+b=4$, $a+c=2$
$c+d=8$, $b+d=S$.

The 4 eq's are solvable only if $S =$ ______. Then find 2 different matrices that have the correct row & column sums.
Write down the 4x4 system $Ax=b$ with $x=(a,b,c,d)$ & make A triangular by

elimination.

Ans. $a+b+c+d = 12 = 5+2 \longrightarrow \underline{5=10}$

$$\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Suppose, elimination takes A to U without row exchanges. Then row j of U is a combination of which rows of A? If $Ax=0$, is $Ux=0$? If $Ax=b$, is $Ux=b$? If 'A' starts out lower triangular, what is upper triangular U?

Ans: Row j of U is a combination of rows $1, \dots, j$ of A (when no row exchanges).

$Ax=0 \longrightarrow Ux=0$

$Ax=b \not\longrightarrow Ux=b$

→ U keeps the diagonal of 'A' when A is lower triangular.

32. Start with 100 eqⁿ's $Ax=0$ for 100 unknowns $x=(x_1, \dots, x_{100})$. Suppose elimination reduces the 100th eqⁿ. to $0=0$, so the system is "singular".

(c) Invent a 100×100 singular matrix with no zero entries

(d) For that matrix, describe in words the row picture & the column picture of $Ax=0$.

Ans: 99 random rows
100th row is the sum or any linear combination of those rows with no zeros.

Row picture: 100 hyper planes meeting along a common line thro' 0.

Column picture: has 100 vectors all in the same 99-D hyperplane

$\boxed{2.3}$

2. $E_{21}$ subtracts 5 times row 1 from row 2.

$E_{32}$ subtracts $-7$ times row 2 from row 3.

$E_{32}E_{21}b =$ ________

$E_{21}E_{32}b =$ ________ $b = (1,0,0)$

When $E_{32}$ comes $1^{st}$, row ______ feels no effect from row ______

Ans: $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +7 & 1 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_{32}E_{21}b = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ -35 & 7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -35 \end{bmatrix}$$

$$E_{21}E_{32}b = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix} b = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$$

When $E_{32}$ comes $1^{st}$, row 3 has no effect from row 1.

3. Which 3 matrices $E_{21}, E_{31}, E_{32}$ put 'A' into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \text{ and } E_{32} E_{31} E_{21} A = U$$

Multiply these E's to get one matrix M that does elimination: $MA = U$.

Ans:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = U$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

16. Write these ancient problems in a 2x2 matrix form $Ax = b$ & solve them:

(a) X is twice as old as Y and their ages add to 33

Ans:

$$\left.\begin{aligned} x &= 2y \\ x+y &= 33 \end{aligned}\right] \quad \left.\begin{aligned} x - 2y &= 0 \\ x + y &= 33 \end{aligned}\right\} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 33 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 33 \end{bmatrix} \longrightarrow \begin{aligned} y &= 11 \\ x &= 22 \end{aligned}$$

17. The parabola $y = a + bx + cx^2$ goes thro' the pts $(x, y) = (1, 4)$ & $(2, 8)$ & $(3, 14)$. Find & solve a matrix equation for the unknowns $(a, b, c)$?

Ans:

$$a + b + c = 4$$
$$a + 2b + 4c = 8 \rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}}_{\text{Vandermonde Matrix}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix}$$
$$a + 3b + 9c = 14$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$c = 1$, $b + 3c = b + 3 = 4 \Rightarrow b = 1$

$a = 4 - 2 = 2$

$(a, b, c) = (2, 1, 1)$

## Vandermonde matrix

In linear algebra,

a Vandermonde matrix is a matrix with the terms of a G.P in each row.

Ex:-

$$V_{m\times n} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \cdots & \alpha_m^{n-1} \end{bmatrix}$$

(or) $V_{i,j} = \alpha_i^{j-1} \quad \forall\ i, j$

The identical term Vandermonde matrix was used by Macon & Spitzbart.

The Vandermonde matrix used for DFT ma[trix] satisfies both definitions.

An ~~N~~ N-point DFT is expressed as $X = Wx$,
$x$: original input signal, $W$: $N \times N$ square DFT matrix,
$X$: DFT of the signal.

$W$ is a Vandermonde matrix

$$W = \left( \frac{\omega^{jk}}{\sqrt{N}} \right)_{j,k = 0,1,\dots,N-1}$$

(OR)

$$W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

where, $\omega = e^{-\frac{2\pi}{N} i}$ : $n^{th}$ root of unity

The determinant of a square Vandermonde matrix can be expressed as:

$$\boxed{\det(V) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)}$$

This is called the Vandermonde determinant (or) Vandermonde polynomial.

- $\det(V)$ is non-zero iff all $\alpha_i$ are distinct matrix

* The discriminant of a polynomial of degree '$n$' is equal to the square of the Vandermonde determinant of the roots of the polynomial times $a_n^{2n-2}$

For

$$A(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x_1 + a_0 \quad , \quad a_n \neq 0$$

in terms of the roots $r_1, r_2, \cdots, r_n$, the discriminant is equal to

$$\boxed{\mathrm{Disc}_x(A) = a_n^{2n-2} \times \prod_{i<j} (r_i - r_j)^2}$$

→ • If the polynomial has a multiple root (atleast 2 roots are equal), then its discriminant is zero.

• If all the roots are real and simple, then the discriminant is +ve.

Ex:- <u>Degree 2</u>. $A(x) = ax^2 + bx + c$

$$\Delta = a^{2(2)-2} \times (r_1 - r_2)^2$$

$$= a^2 \times \left[\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{b+\sqrt{b^2-4ac}}{2a}\right]^2$$

$$= a^2 \left[\frac{\sqrt{b^2-4ac}}{a}\right]^2 = b^2 - 4ac$$

<u>Degree 3</u> $A(x) = ax^3 + bx^2 + cx + d$

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

19. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $PQ = QP = ?$
$P^2 = ?$

Find another non-diagonal matrix whose square is $M^2 = I$ ?

Ans: $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ & $P^2 = I$

$M^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

if $\underline{a^2 + bc = 1}$

22. The entries of $A$ and $x$ are $a_{ij}$ & $x_j$.
So the 1st comp. of $Ax$ is $\sum a_{1j} x_j =$
$a_{11}x_1 + \cdots + a_{1n}x_n$. $E_{21}$ subtracts row 1 from row 2.

$C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

(a) 3rd comp. of $Ax$

Ans: $\sum a_{3j} x_j$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) the $(2,1)$ entry of $E_{21}A$

Ans: $\sum a_{2j} a_{j1} = a_{21} - a_{11}$

(c) (2,1) entry of $E_{21}(E_{21}A)$

Ans: $E_{21}(E_{21}A) = E_{21}^2 A =$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$C_{21} = \sum_{j=1}^{n} e'_{2j} a_{j1} = a_{21} - 2a_{11}$$

(d) $1^{st}$ component of $E_{21} A x$

Ans: $(EAx)_{11} = (Ax)_{11} = \underline{\sum a_{1j} x_j}$

27. Choose the # a, b, c, d in the augmented matrix so that there is

(a) No solution (b) infinitely many solutions

$$[A\ b] = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

Which of the # a, b, c, d have no effect on the solvability ?

Ans: $|A| = 0$ for both the cases.

$|A| = 4d = 0 \Longrightarrow \boxed{d = 0}$ & $\underline{\text{rank}(A) = 2}$

(b) <u>Infinitely many solutions</u>

$\text{rank}(A) = \text{rank}(A|b) < n = 3$.

$\cancel{\text{rank}(A)} = D_1 = D_2 = D_3 = 0$

$$\begin{vmatrix} 1 & 2 & a \\ 0 & 4 & b \\ 0 & 0 & c \end{vmatrix} = 4c = 0 \longrightarrow \underline{c = 0}$$

$$\begin{vmatrix} 2 & 3 & a \\ 4 & 5 & b \\ 0 & 0 & c \end{vmatrix} = 0 \Rightarrow \underline{c = 0}$$

$\boxed{d = 0 \ \& \ c = 0}$

(a) <u>No solution</u>

$\text{rank}(A) < \text{rank}(A|b)$

$2 < \text{rank}(A|b) \Longrightarrow \text{rank}(A|b) = 3$

$c \neq 0$.

$\boxed{d = 0, c \neq 0}$

30. Write $M = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ as a product of many factors $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Ans: $M \Longrightarrow \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 0 & \text{[illegible]} \\ \text{[illegible]} & \text{[illegible]} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

~~$E_{21}$ $E_{21}$~~

(a) What matrix E subtracts row 1 from row 2 to make row 2 of EM smaller?

Ans: $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = A^{-1}$

(b) What matrix F subtracts row 2 ~~of~~ EM from row ~~1~~ to reduce row 1 of FEM?

Ans- $F = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = B^{-1}$

any

EM
EM ?

ⓒ. Continue E's & F's until (many E's & F's) times M is (A or B)

~~Ans~~: ⓓ $M =$ product of A's & B's.

Ans: $M = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} \cancel{3} & \cancel{4} \\ -2 & -\cancel{3} \end{bmatrix} \xrightarrow{F} \begin{bmatrix} 5 & 7 \\ -2 & -3 \end{bmatrix}$

2.4

2. What rows/columns or matrices do you multiply to find

(a) $2^{nd}$ column of AB

Ans: $A\left[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n\right] = \left[A\vec{b}_1\ \ A\vec{b}_2 \cdots A\vec{b}_n\right]$

A (column 2 of B)

(b). $1^{st}$ row of AB ?

Ans: $\begin{bmatrix} \vec{A}_1 \\ \vec{A}_2 \\ \vdots \\ \vec{A}_n \end{bmatrix} B = \begin{bmatrix} \vec{A}_1 B \\ \vec{A}_2 B \\ \vdots \\ \vec{A}_n B \end{bmatrix} \Bigg\}$ (row 1 of A) B

(c) entry in row 3, column 5 of AB ?

Ans: (row 3 of A) • (column 5 of B)

(d) entry in row 1, column 1 of CDE

Ans: (row 1 of C) D (column 1 of E)

5. Comute $A^2, A^3$ & make a prediction for $A^5$ & $A^n$.

(a) $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

Ans: $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} \Longrightarrow A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$

Ans: $A^2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$

$A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$

8. How is each row of DA and EA related to the rows of A, when $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

How is each column of AD & AE related to the column of A?

Ans: $DA = \begin{bmatrix} 3a & 3b \\ 5c & 5d \end{bmatrix}$, $EA = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$

The rows of DA are 3(row 1 of A) and 5(row 2 of A).

Both rows of EA are row 2 of A.

9. EA : row 1 of A is added to row 2.

(EA)F : column 1 of EA is added to column 2

$$EA = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ a+c & b+d\end{bmatrix}$$

$$(EA)F = \begin{bmatrix}a & b\\ a+c & b+d\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}a & a+b\\ a+c & a+b+c+d\end{bmatrix}$$

(a) Do those steps in the opp. order.

i.e., $E(AF)$

Ans: $$E\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = E\begin{bmatrix}a & a+b\\ c & c+d\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}a & a+b\\ c & c+d\end{bmatrix}$$

$$= \begin{bmatrix}a & a+b\\ a+c & a+b+c+d\end{bmatrix}$$

$$\Longrightarrow \underline{(EA)F = E(AF)}$$

11. $(EA)F = E(AF) \Rightarrow$ If you do a ~~column~~ row operation on A, and then a column operation, the result is the same as if you did the column operation first.

12. 3×3 matrices. Choose the only B so that every matrix A satisfies

(a) $BA = 4A$

Ans: $B = 4I$

(b) $BA = 4B$

Ans: $B = 0$

(c) BA has rows 1 and 3 of A reversed & row 2 unchanged.

Ans: $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(d) All rows of BA are the same as row 1 of A

Ans: $BA = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \vec{A_1} \\ \vec{A_2} \\ \vdots \\ \vec{A_3} \end{bmatrix} = \begin{bmatrix} \vec{A_1} \\ \vec{A_1} \\ \vdots \\ \vec{A_1} \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

13. $AB = BA$ & $AC = CA$ for these 2 particular matrices $B$ & $C$:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Prove that $a = d$ and $b = c = 0$. Then $A$ is a multiple of $I$. The only matrices that commute with $B$ & $C$ and all other $2 \times 2$ matrices are $A$ = multiple of $I$.

Ans. $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $b = c = 0$

$AC = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

$CA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $a = d$

$c = 0$

$$\Longrightarrow A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$$

$\Rightarrow$ The only matrices that commute with all other $2 \times 2$ matrices are multiples of $I$.

15. True/False

(a) If $AB = B$ then $A = I$

Ans: False : true for $B = 0$

16. If 'A' is m by n, how many separate multiplication are involved when

(a) 'A' multiplies a vector $x$ with n components?

Ans: $mn$

(b). 'A' multiplies an $n \times p$ matrix B

Ans. $A_{m \times n} B_{n \times p} = C_{m \times p}$

$\#$ = mp terms of C & each term has n multiplication

$= mnp$

(c) A multiplies itself to produce $A^2$. ? $m = n$

Ans: $n^3 = m^3$

17. For $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, compute these answers and nothing more:

(a) column 2 of AB

Ans: $= A\vec{b}_2 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) row 2 of AB

Ans: $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ $\quad \vec{A}_2 B = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(c) row 2 of $AA = A^2$

Ans: $\begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$

(d) row 2 of $A^3$

Ans: $\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \end{bmatrix}$

19. What words would you use to describe each of these classes of matrix?
Which matrix belongs to all 4 classes?

(a) $a_{ij} = 0$ if $i \neq j$

Ans: $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \longrightarrow$ Diagonal matrix

(b) $a_{ij} = 0$ if $i < j$

$\begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{bmatrix} \Longrightarrow$ lower triangular

(c) $a_{ij} = a_{ji} \Longrightarrow$ Symmetric

(d) $a_{ij} = a_{1j} \Longrightarrow$ all rows equal.

Zero matrix fits all 4.

23. ⓐ Find a non-zero matrix $A$ for which $A^2 = 0$

Ans: ~~$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$~~

If $A = uv^T = u \otimes v$

$$\boxed{A^2 = uv^T uv^T = 0 \quad \text{if } v^T u = 0 \text{ (or) } v \cdot u = 0}$$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A = uv^T = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

matrix that has $A^2 \neq 0$ but $A^3 = 0$

Ans

$\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

24. By experiment with $n=2$, $n=3$ predict $A^n$ for these matrices.

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

Ans: $A_1^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$

$$A_1^3 = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix} \rightarrow A_1^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$$

$$A_2^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A_2^3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$A_2^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix} = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3^2 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ 0 & 0 \end{bmatrix}$$

$$A_3^3 = \begin{bmatrix} a^2 & ab \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3 & a^2 b \\ 0 & 0 \end{bmatrix}.$$

$$A_3^n = \begin{bmatrix} a^n & a^{n-1} b \\ 0 & 0 \end{bmatrix}$$

25. Multiply A times I using columns of A (3×3) times rows of I.

Ans: $AI = [\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n]$

$$AI = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

$$= a_1 i_1 + a_2 i_2 + a_3 i_3$$

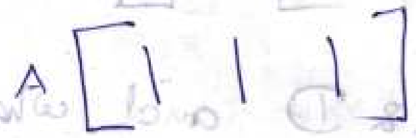
$$= \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b \\ e \\ h \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & 0 \\ d & 0 & 0 \\ g & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & e & 0 \\ 0 & h & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & f \\ 0 & 0 & i \end{bmatrix}$$

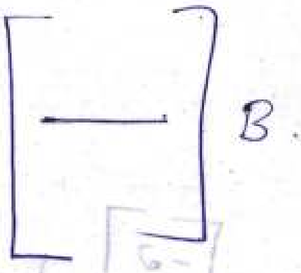
$$= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A$$

28. Draw the cells in A(2x3) and B(3x4) and AB to show how each of the 4 multiplication rules is really a block multiplication:

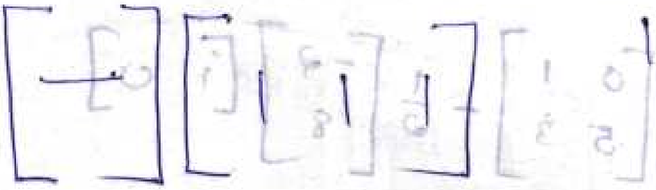
(i) A times colums of B.

Ans: 

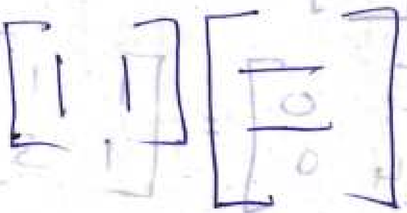
(ii) Rows of A times B.

Ans. 

(iii) Rows of A times columns of B

Ans:  Inner products.

(iv) Columns of A times rows of B

Ans:  outer products.

30. Block multiplication says that column 1 is eliminated by

$$EA = \left[\begin{array}{c|c} 1 & 0 \\ \hline -c/a & I \end{array}\right]\begin{bmatrix} a & b \\ c & D \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & D - \frac{c}{a}b \end{bmatrix}$$

What # go into c & D and what's $D - \frac{c}{a}b$ ?

$$A = \left[\begin{array}{c|cc} 2 & 1 & 0 \\ \hline -2 & 0 & 1 \\ 8 & 5 & 3 \end{array}\right]$$

Ans: $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $a = 2$, $c = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $b = [1, 0]$

$$D - \frac{cb}{a} = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 \\ 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

31. The product of $(A+iB)$ & $(x+iy)$ is

$Ax+iBx+iAy-By$.

Use blocks to separate the real part without $i$ from the imaginary part that multiplies $i$:

Ans: $$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax-By \\ Bx+Ay \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part} \end{matrix}$$

$\Longrightarrow$ Complex matrix times complex vector needs 4 times real multiplications.

32. Suppose you solve $Ax=b$ for 3 special right sides $b$

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the 3 solutions $x_1, x_2, x_3$ are the columns of a matrix $X$, what is $A$ times $X$?

Ans: $$AX = A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

33. If 3 solutions in (32) are $x_1 = (1,1,1)$, $x_2 = (0,1,1)$, $x_3 = (0,0,1)$, Solve $Ax = b$ when $b = (3,5,8)$? What is $A = ?$

Ans: $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\quad\quad\quad x_1 \quad\quad\quad\quad\quad x_2 \quad\quad\quad\quad\quad x_3$

$$b = \begin{bmatrix}3\\5\\8\end{bmatrix} = 3\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix} + 8\begin{bmatrix}0\\0\\1\end{bmatrix} = 3Ax_1 + 5Ax_2 + 8Ax_3$$

$$A(3x_1 + 5x_2 + 8x_3) = b$$

$$\Longrightarrow x = 3x_1 + 5x_2 + 8x_3 = 3\begin{bmatrix}1\\1\\1\end{bmatrix} + 5\begin{bmatrix}0\\1\\1\end{bmatrix} + 8\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$x = \begin{bmatrix}3\\8\\16\end{bmatrix}$$

$$\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}3\\5\\8\end{bmatrix} = \begin{bmatrix}3\\8\\16\end{bmatrix} \Longleftrightarrow A^{-1}b = x$$

$$A^{-1} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix} \longrightarrow E_{32}E_{31}E_{21}A^{-1} = I$$

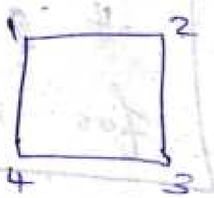
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \Longrightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

35. Suppose a "circle graph" has 4 nodes connected (in both directions) by edges around a circle. What is its adjacency matrix S ~~for~~? $S^2 = ?$
All 2-step paths predicted by $S^2$.



Ans: $S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$$S^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

36. $A_{m\times n}$, $B_{n\times p}$, $C_{p\times q}$

# of multiplications for $(AB)$ $= mnp$

$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$ : $mp$ dot products & each has $n$ Xions.

$\boxed{\text{\# of multiplication for } (AB)C = (mpq) + \text{\# for } AB = \underline{mnp + mpq}}$

$(AB)_{m\times p}\, C_{p\times q}$ $(mq)p$

$\boxed{\text{\# of multiplications for } A(BC) = mnq + npq}$

$mnq$

$A_{m\times n}(BC)_{n\times q}$

(a) If $A$ is $2\times 4$ & $B$ is $4\times 7$, $C$ is $7\times 10$ do you prefer $(AB)C$ or $A(BC)$ ?

Ans: # of multiplications for $(AB)_{2\times 7}$ $= 2\times 4\times 7$

~~#(AB~~ # for $(AB)C = (2\times 4\times 7) + (2\times 7\times 10)$

$= 2[28+70] = 2\times 98$

$= \underline{196}$

# for $(BC)_{4\times 10} = 4\times 7\times 10$

# for $A(BC) = 2\times 4\times 10 + 4\times 7\times 10$

$= 4[20+70] = 4\times 90 = 360.$

$\Longrightarrow$ We prefer $\underline{(AB)C}$

(b) With N-component vectors, would you choose $(u^T v) w^T$ (or) $u^T (v w^T)$ ?

Ans: $u^T : 1\times N$ | $v = N\times 1$ | $w^T = 1\times N$

# $(u^T v) w^T = 1\times N\times 1 + N = 2N$

# $u^T (v w^T)_{N\times N} = 1\times N\times N + N\times 1\times N = 2N^2$

$\Longrightarrow$ prefer $(u^T v) w^T$

(c) Divide by mnpq to show that

(AB)C is faster when $n^{-1}+q^{-1} < m^{-1}+p^{-1}$

Ans: # $(AB)C <$ # $A(BC)$

$mnp + mpq < mnq + npq$

÷ing by mnpq,

$$\frac{1}{q} + \frac{1}{n} < \frac{1}{p} + \frac{1}{m}$$

$$\boxed{q^{-1}+n^{-1} < p^{-1}+m^{-1}}$$

If $C_{p \times q} = v_{p \times 1} \rightarrow q = 1$

→ If matrices A and B are multiplying v for ABv,
don't multiply the matrices first.
Better to multiply Bv and then A(Bv)

$A_{n \times n} B_{n \times n} v_{n \times 1}$ : $\frac{1}{n}+1 < \frac{1}{n}+\frac{1}{n} \Rightarrow \frac{1}{1} < \frac{1}{n} \Rightarrow n < 1$
Not true

$(AB)v$ is not better.

37. To prove that $(AB)C = A(BC)$, use the column vectors $b_1, \dots, b_n$ of $B$.

1st suppose that $C$ has only one column $c$ with entries $c_1, \dots, c_n$:

$$AB = A[\vec{b}_1 \cdots \vec{b}_n] = [Ab_1, \dots, Ab_n]$$

$$(AB)c = c_1 A b_1 + \cdots + c_n A b_n$$

$$Bc = c_1 b_1 + \cdots + c_n b_n$$

$$A(Bc) = A(c_1 b_1 + \cdots + c_n b_n) = (AB)c$$

By linearity

The same is true for all other columns of $C$.

$$\therefore \quad (AB)C = A(BC)$$

2.5

2 For these permutation matrices find $P^{-1}$

(a) $P^{-1} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}^{-1} = P^{T} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} = P$

also it is ~~...~~ orthogonal matrix $\longleftrightarrow$ symmetry transformations $\Longleftrightarrow$ Norm preserved.

(b) $P^{-1} = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} = P^{T} = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

5. Find an upper triangular $U$ (not diagonal) with $U^2 = I$ which gives $U = U^{-1}$

Ans:

$$U = \begin{bmatrix}a&b\\0&c\end{bmatrix} \Longrightarrow U^{-1} = \frac{\begin{bmatrix}c&0\\-b&a\end{bmatrix}^{T}}{ac} = \frac{\begin{bmatrix}c&-b\\0&a\end{bmatrix}}{ac}$$

$$\begin{bmatrix}a&b\\0&c\end{bmatrix} = \begin{bmatrix}\frac{1}{a}&\frac{-b}{ac}\\0&\frac{1}{c}\end{bmatrix} \Longrightarrow a^2=1,\ c^2=1$$

$b=0$ (or) $ac=-1$.

$$U = \begin{bmatrix}1&b\\0&-1\end{bmatrix}$$ for any $b$

$$(\text{or}) = \begin{bmatrix}-1&b\\0&1\end{bmatrix}$$

6. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find 2 different matrices such

(b) that $AB = AC$

Ans: $|A| = 0$

$$\boxed{A(B-C) = 0}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x+z & y+w \\ x+z & y+w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$z = -x$ & $w = -y$

$$\boxed{B - C = \begin{bmatrix} x & y \\ -x & -y \end{bmatrix}}$$

7. If 'A' has row 1 + row 2 = row 3,

(a) Why $Ax = (0,0,1)$ can't have a solution.

(b) which $(b_1, b_2, b_3)$ might allow a solution to $Ax = b$ ?

(c) What happens to ~~elimination~~ eqⁿ (3), in elimination

Ans:
(a) $Ax = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ A is not invertible

row 1 + row 2 − row 3 $\Rightarrow$ $0 = 1$

No solution

(b) $b_1 + b_2 - b_3 = 0 \longrightarrow b_1 + b_2 = b_3$.

(c) row 3 becomes a row of zeros.
$\longrightarrow$ No $3^{rd}$ pivot.

8. If 'A' has column 1 + column 2 = column 3,
$A_{3\times 3}$.

(a) Find a non-zero solution $x$ to $Ax = 0$.

(b) Elimination keeps column 1 + column 2 = column 3
Why no $3^{rd}$ pivot?

Ans:
(a)
$|A| = 0 \Longrightarrow A^{-1}$ does not exist.
$\longrightarrow$ Only non-zero solution to $Ax = 0$.

$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x\vec{a}_1 + y\vec{a}_2 + z(\vec{a}_1 + \vec{a}_2) = 0$

$a_1 + a_2 = a_3$

$\vec{a}_1(x+z) + \vec{a}_2(y+z) = 0$
$x = -z,\ y = -z,\ z = z$

$\Bigg\}\ (-1,-1,1)$
$(1,1,-1)$ etc.

(b). $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

~~A~~ $\vec{a}_1 + \vec{a}_2 = \vec{a}_3$

$E(\vec{a}_1 + \vec{a}_2) = E\vec{a}_3$

$E\vec{a}_1 + E\vec{a}_2 = E\vec{a}_3 \Rightarrow \vec{A}_1 + \vec{A}_2 = \vec{A}_3$

$\Rightarrow$ Elimination keeps column(1) + column(2) = column (3)

After elimination : $1^{st}$ & $2^{nd}$ entries in the $3^{rd}$ row will be zero.

$\Rightarrow$ $3^{rd}$ entry must be zero.

$\Rightarrow$ whole $3^{rd}$ row is zero

$\Rightarrow$ No $3^{rd}$ pivot

10. Find the inverses.

(a) $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$

Ans: $A = P_{14} P_{23} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$A^{-1} = P_{14} P_{23} \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1/5 & 0 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} P_{23} P_{14} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$$

(b) $B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix} = \mathrm{diag}(A_1, A_2) = A_1 \oplus A_2$

Ans: $$B^{-1} = A_1^{-1} \oplus A_2^{-1} = \frac{\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}^T}{1} \oplus \frac{\begin{bmatrix} 6 & -7 \\ -5 & 6 \end{bmatrix}^T}{1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \oplus \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$$

12. If $C = AB$ is invertible ($A, B$ square), then $A$ itself is invertible. Find a formula for $A^{-1}$ that involves $C^{-1}$ and $B$.

Ans: $C = AB \Rightarrow A^{-1}C = B$

$\Rightarrow A^{-1} = BC^{-1}$

13. If $M = ABC$ of 3 sq. matrices is invertible, then $B$ is invertible. So are $A$ & $C$. Find a formula for $B^{-1}$ that involves $M^{-1}$ and $A$ & $C$.

Ans: $M = ABC \Rightarrow M^{-1} = C^{-1}B^{-1}A^{-1}$

$B^{-1} = CM^{-1}A$

15. Prove that a matrix with a column of zeros can't have an inverse.

Ans: "A" has column of zeros $\longrightarrow$ $AB$ must also have

$\longrightarrow$ $AB = I$, is impossible

$\longrightarrow$ There is no $A^{-1}$.

21. There are sixteen 2x2 matrices whose entries are 1's and 0's. How many of them are invertible.

Ans: Invertible $\Rightarrow |A| \neq 0$

$|A| = 1, -1$

$$\left.\begin{matrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}\\ \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}\end{matrix}\right\} \# = 6$$

24. Use Gauss-Jordan elimination on $[U \ I]$ to find the upper triangular $U^{-1}$

$$UU^{-1} = I \Longrightarrow \begin{bmatrix}1&a&b\\0&1&c\\0&0&1\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Ans: $$\left[\begin{array}{ccc|ccc}1&a&0&1&0&-b\\0&1&0&0&1&-c\\0&0&1&0&0&1\end{array}\right] \to \left[\begin{array}{ccc|ccc}1&0&0&1&-a&ac-b\\0&1&0&0&1&-c\\0&0&1&0&0&1\end{array}\right]$$

$$\underbrace{\begin{matrix}1&-a&ac-b\\0&1&-c\\0&0&1\end{matrix}}_{U^{-1}}$$

26. What 3 matrices $E_{21}$ & $E_{12}$ and $D^{-1}$ reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix. Multiply $D^{-1} E_{12} E_{21}$ to find $A^{-1}$

Ans: $E_{21} A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$DE_{21}A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad E_{12} E_{21} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$E_{12} D E_{21} =$ $\quad D E_{12} E_{21} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^{-1} = D E_{12} E_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}$$

32. Suppose, $P$ & $Q$ have the same rows as $I$ but in any order. They are "permutation matrices." Show that $P-Q$ is singular by solving $(P-Q)x=0$

Ans: For $x = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$,

$$x = Px = Qx \Longrightarrow (P-Q)x = 0.$$

Non-zero solution.

$\therefore$ $(P-Q)$ is singular

33. Find & check inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}, \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Ans:

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D-CA^{-1}B \end{bmatrix}$$

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & S = D-CA^{-1}B \end{array}\right]$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|c} I & 0 \\ \hline CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & 0 \\ \hline 0 & S = D-CA^{-1}B \end{array}\right]\left[\begin{array}{c|c} I & A^{-1}B \\ \hline 0 & I \end{array}\right]$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} I & -A^{-1}B \\ \hline O & I \end{array}\right] \left[\begin{array}{c|c} A^{-1} & O \\ \hline O & S^{-1} \end{array}\right] \left[\begin{array}{c|c} I & O \\ \hline -CA^{-1} & I \end{array}\right]$$

$$= \cancel{\left[\begin{array}{c|c} A^{-1}+A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ \hline -S^{-1}CA^{-1} & S^{-1} \end{array}\right]}$$

Schur complement

$\underline{\underline{S = D - CA^{-1}B}}$

$$= \left[\begin{array}{c|c} A^{-1} & -A^{-1}BS^{-1} \\ \hline O & S^{-1} \end{array}\right] \left[\begin{array}{c|c} I & O \\ \hline -CA^{-1} & I \end{array}\right]$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A^{-1}+A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ \hline -S^{-1}CA^{-1} & S^{-1} \end{array}\right]$$

ⓐ $M = \left[\begin{array}{c|c} I & O \\ \hline C & I \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

$$M^{-1} = \left[\begin{array}{c|c} I & O \\ \hline -C & I \end{array}\right]$$

(b) $M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$

$$M^{-1} = \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline -D^{-1}CA & D^{-1} \end{array}\right]$$

(c) $M = \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$

$$\cancel{M^{-1} = \left[\begin{array}{c|c} 0 & 0 \\ \hline -D^{-1}CA^{-1} & \end{array}\right]}$$

Define, $M = \begin{bmatrix} 0 & I \\ I & D \end{bmatrix} = P_{12} \begin{bmatrix} I & D \\ 0 & I \end{bmatrix}$

$$M^{-1} = \begin{bmatrix} I & D \\ 0 & I \end{bmatrix}^{-1} P_{12} = \left[\begin{array}{c|c} I & -ID \\ \hline 0 & I \end{array}\right] P_{12}$$

$$= \left[\begin{array}{c|c} I & -D \\ \hline 0 & I \end{array}\right] P_{12} = \left[\begin{array}{c|c} -D & I \\ \hline I & 0 \end{array}\right]$$

34. Could a 4x4 matrix A be invertible, if every row contains the # 0, 1, 2, 3 in some order?

What if every row of B contains 0, 1, 2, -3 in some order?

Ans: A can be invertible

Ex:- $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 0 & -2 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 2 & 0 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 6 & 6 \\ 2 & -4 & -3 \end{vmatrix}$$

$$= -18 + 24 \neq 0$$

$B = [\vec{b_1}\ \vec{b_2}\ \vec{b_3}\ \vec{b_4}]$

$\vec{b_1} + \vec{b_2} + \vec{b_3} + \vec{b_4} = 0 \Longrightarrow$ $\vec{b_4}$ is a linear comb of other 3 vectors

$\therefore |B| = 0$ : B is singular

(OR) For $x = (1,1,1,1)$

$Bx = 0 \Rightarrow$ Non-zero solution

$\therefore$ B singular

36. Hilbert matrices,

$$H_{ij} = \frac{1}{i+j-1} = \int_0^1 x^{i+j-2} \, dx$$

Ex:-

$$H_{n\times n} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots & \frac{1}{n} \\ 1/2 & 1/3 & 1/4 & 1/5 & \cdots & \frac{1}{n+1} \\ 1/3 & 1/4 & 1/5 & 1/6 & \cdots & \frac{1}{n+2} \\ 1/4 & 1/5 & 1/6 & 1/7 & \cdots & \frac{1}{n+3} \\ \vdots & \vdots & \vdots & \vdots & & \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

40. Suppose $E_1, E_2, E_3$ are $4\times4$ identity matrices, except $E_1$ has $a, b, c$ in column 1, $E_2$ has $d, e$ in column 2 & $E_3$ has $f$ in column 3 (below in 1's).

$L = E_1 E_2 E_3 = ?$

Ans: $$E_1 E_2 E_3 = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & d & 1 & \\ 0 & e & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & f & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & f & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & f & 1 \end{bmatrix}$$

44. How does the identity $A(I+BA)=(I+AB)A$ connect the inverses of $I+BA$ and $I+AB$?

Ans: $(I+BA)^{-1}A^{-1} = A^{-1}(I+AB)^{-1}$

$\Rightarrow |A||I+BA| \neq 0$ & $|I+AB||A| \neq 0$.

If $|A| \neq 0$, $|I+BA| = 0 \Longrightarrow |I+AB| = 0$

$|I+BA| \neq 0 \Longrightarrow |I+BA| \neq 0$

AB & BA have same non-zero eigenvalues

2.6

2. $\boxed{\begin{aligned} x+y&=5 \\ x+2y&=7 \end{aligned}}$

Write down the $2\times 2$ triangular systems $Lc=b$ and $Ux=c$

Ans: $Ax=b \Longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$L=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $U=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$Lc=b$ : $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 5 \\ 7 \end{bmatrix} \Rightarrow \left.\begin{aligned} c_1&=5 \\ c_2&=2 \end{aligned}\right\} c=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$Ux=c$ : $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow x=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

By Back substitution.

4. $x+y+z=5$
$x+2y+3z=7$
$x+3y+6z=11$

Find $Lc=b$ & $Ux=c$

Ans: $Ax=b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Lc = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

$$Ux = c : \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

8. $A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$

(a) $E_{32} E_{31} E_{21} = E = ? \quad \forall \; EA = I$

(b) $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = L = E^{-1} = ?$

(a)

Ans: $$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

→ The multipliers $a, b, c$ are mixed up in $E$ but perfect in $L$.

12. $A$ & $B$ are symmetric across the diagonal. Find their triple factorization $LDU$ and say how $U$ is related to $L$.

Symmetric $$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

Ans: $$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= LDU = LDL^T$$

$$B: \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= LDU = LDL^T$$

13. Compute L & U for the symmetric matrix A

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find 4 conditions on a, b, c, d to get A = LU with 4 pivots.

Ans: $A \longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U$$

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$a \neq 0,\ b \neq a,\ c \neq b,\ d \neq c$

18. * If $A = LDU$ & also $A = L_1 D_1 U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$ ie., the 3 factors are unique.

Derive the eq$^n$. $L_1^{-1} L D = D_1 U_1 U^{-1}$.
Are the 2 sides triangular or diagonal?
Deduce $L = L_1$, $U = U_1$. Then $D = D_1$.

Ans $LDU = L_1 D_1 U_1 \longrightarrow L_1^{-1} LDU = D_1 U_1$

(a) $\underline{L_1^{-1} L D = D_1 U_1 U^{-1}}$

LHS is lower triangular & right side is

(b) upper triangular $\Longrightarrow$ both sides are diagonal

(c)

19. Tridiagonal matrices have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A=LU$ & $A=LDL^T$

$$A=\begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix},\quad A=\begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix}$$

Ans:

$$A=\begin{bmatrix}1&1&0\\1&2&1\\0&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=LIU=LIL^T$$

$$A=\begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}a&a&0\\0&b&b\\0&0&c\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = L \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} U$$

⇒ A tridiagonal matrix $A$ has bidiagonal factors $L$ and $U$.

22. Eliminate upwards:

Upper times lower $A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Ans: $\begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$

$$U^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \boxed{A = UL}$$

Q4. Which invertible matrices allow $A = LU$ (elimination without row exchanges)?

Ans:

* $A = LU$ is possible only if all upper left $k \times k$ submatrices $A_k$ must be invertible (sizes $k = 1, 2, \ldots, n$).

The upper left blocks all factor at the same time as 'A': $A_k$ is $L_k U_k$

$$LU = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}$$

2.7

4. Show that $A^2=0$ is possible but $A^TA=0$ is not possible (unless $A=0$).

Ans: $\cancel{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$

$b=0,\ d=0,\ a=0$

$$\therefore \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Longrightarrow \underline{A^2=0}$$

$$A^TA = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} =$$

$A^TA$ has ~~many~~ dot products of columns of A with themselves.

zero dot products $\Longrightarrow$ zero columns $\Longrightarrow$ $\underline{A=0}$.

6. $M^T = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$

M is a Block symmetric matrix if

$A^T = A$, $B^T = C$, $D^T = D$.

13. (a) Find a 3x3 permutation matrix with $P^3 = I$ (but not $P = I$).

Ans: ~~Cyclic permutation (or) its transpose~~

$$P = P_{31} P_{23} P_{12} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & A \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

cyclic permutation (or) its transpose matrix

$$P = P_{23} P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$P^3 = I$ :

$(1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$

(6) Find a $4 \times 4$ permutation $\hat{P}$ with ($\hat{P}^4 \neq I$)

Ans: $\hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same above $P$.

has $\hat{P}^4 = P \neq I$

14. If 'P' has 1's on the antidiagonal from $(1,n)$ to $(n,1)$, describe $PAP$. Note $P = P^T$.

Ans:

$$P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{bmatrix} \quad \& \quad \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \end{bmatrix} P = \begin{bmatrix} x_n & x_{n-1} & \cdots & x_2 & x_1 \end{bmatrix}$$

$$(PAP)_{ij} = A_{(n+1)-i,\,(n+1)-j}$$

15.

(b) Find a 4x4 example with $P^T = P$ that moves all 4 rows

Ans: $1 \to 2$ & $3 \to 4$

$$P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \quad \text{where} \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

17. Find 2x2 symmetric matrices $S = S^T$ with these properties:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \to \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

(a) S is not invertible

Ans: $ad = b^2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(b) S is invertible but can't be factored into LU (row exchanges needed).

Ans: $$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

(c) S can be factored into $LDL^T$ but not into $LL^T$ (because of -ve D).

Ans: $S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

18.

(a) How many entries of S can be chosen independently if $S = S^T$ is $5 \times 5$?

Ans: * The upper (or lower) part of a symmetric matrix completely determines the other half.

$$\begin{bmatrix} a_1 & a_2 \\ * & a_3 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ * & a_4 & a_5 \\ * & * & a_6 \end{bmatrix}, \dots$$

$5+4+3+2+1 = 15$ independent entries.

(b) How do L & D give the same # of choices in $LDL^T$.

Ans: $S = S^T \longrightarrow S = LDL^T$

L has 10 & D has 5

$\longrightarrow$ total 15 in $LDL^T$.

(c) How many entries can be chosen if 'A' is skew-symmetric.

Ans: $A^T = -A \Longrightarrow$ zero diagonal.

$$\begin{bmatrix} 0 & a_1 & a_2 \\ * & 0 & a_3 \\ * & * & 0 \end{bmatrix}$$

$4+3+2+1 = 10$ choices.

19. A is rectangular $(m \times n)$ and S is symmetric $(m \times m)$.

(a) Transpose of $A^T S A$ to show its symmetry. What shape is this matrix?

Ans: $(A^T S A)^T = A^T S^T A = A^T S A$ is $n \times n$.

(b) Show why $A^T A$ has no -ve # on its diagonal.

Ans: $A^T A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

$(A^T A)_{ii} = |\text{column } i \text{ of } A|^2 \geq 0.$

---

22. Find the $PA = LU$ factorization

(a) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Ans: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$PA = LU$ : $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

Ans: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU$

Q4. Factor the following matrix into $PA = LU$.

Factor it also into $A = L_1 P_1 U_1$.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}$$

Ans: $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 8 \\ 0 & 0 & -2/3 \end{bmatrix}$$

$PA = LU$

If we wait to exchange as $a_{12}$ is the pivot.

$$L_1^{-1} A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$$

$$A = L_1 \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} = L_1 P_{13} P_{23} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= L_1 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= L_1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\rightarrow \quad A = L_1 P_1 U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

25. Identity matrix can not be the product of 3 row exchanges (or five). It can be the product of 2 exchanges (or four).

26 (a) Choose $E_{21}$ to remove the 3 below the 1st pivot. Then multiply $E_{21} S E_{21}^T$ to remove both 3's.

(b) Choose $E_{32}$ to remove the 4 below the 2nd pivot. Then $S$ is reduced to $D$ by $E_{32} E_{21} S E_{21}^T E_{32}^T = D$. Invert the $E$'s to find $L$ in $S = LDL^T$.

Ans $$S = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 11 & 4 \\ 0 & 4 & 9 \end{bmatrix} \longrightarrow D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans: $$E_{21} S E_{21}^T = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} S \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix} \underset{E_{12}}{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

$$E_{32}E_{21}SE_{21}^TE_{32}^T = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&4\\0&4&9\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-2\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&2&4\\0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}1&0&0\\0&1&-2\\0&0&1\end{bmatrix}}_{E_{23}} = \begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix} = D$$

$$S = LDL^T = (E_{32}E_{21})^T D (E_{32}E_{21})^T$$

27. If every row of a 4×4 matrix contains the numbers 0, 1, 2, 3 in some order, can the matrix be symmetric?

Ans:
$$A = \begin{bmatrix}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{bmatrix} = A^T$$

29. Wires go b/w Boston, Chicago, & Seattle. Those cities are at voltages $x_B, x_C, x_S$.

With unit resistance b/w cities, the currents b/w the cities are in $y$:

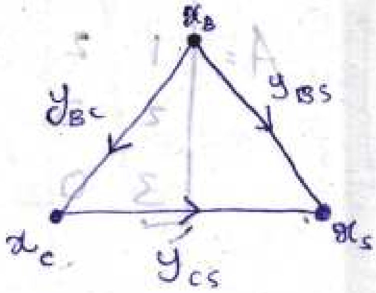
$$y = Ax \quad : \quad \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}$$

Check $\boxed{3.5}$ $\boxed{Ex:3}$ Kirchhoff's law

(a) Find the total currents $A^T y$ out of the 3 cities

(b) Verify that $(Ax)^T y = x^T (A^T y)$

Incidence matrix : matrix that shows the relationship b/w 2 classes of objects.




direction going away from the node is +ve.

$$\partial = \begin{array}{c c} \begin{array}{ccc} y_{BC} & y_{CS} & y_{BS} \end{array} & \\ \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} & \begin{array}{c} x_B \\ x_C \\ x_S \end{array} \end{array} = A^T$$

Total currents are :

$$A^T y = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$$

30. Producing $x_1$ trucks & $x_2$ planes needs $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. If the unit costs $y_1, y_2, y_3$ are 700/- per ton, 3/- per pound, and 3000/- per month, what are the values of one truck and one plane? Those are the components of $A^T y$.

Ans: $$Ax = \begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 50x_2 \\ 40x_1 + 1000x_2 \\ 2x_1 + 50x_2 \end{bmatrix} = \begin{bmatrix} \text{tons of steel} \\ \text{pounds of rubber} \\ \text{months of labor} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{\# of trucks} \\ \text{\# of planes} \end{bmatrix}$$

A took a vector in truck-plane space into a vector in steel-rubber-labor space.

$$A^T y = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix} = \begin{bmatrix} \text{cost } \cancel{\text{per}} \text{ one truck} \\ \text{cost } \cancel{\text{per}} \text{ one plane} \end{bmatrix}$$

$$y = \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} \text{unit cost per steel} \\ \text{" rubber} \\ \text{" labor} \end{bmatrix}$$

$A^T$ sends ~~trucks-planes~~ steel-rubber-labor vector into ~~steel-rubber-labor vector~~ truck-plane: [now in rupee, but it's just a change of units].

31. $Ax$ : amounts of steel, rubber, labor for produce $x$

then $Ax \cdot y$ is the __________ i/p.

$x \cdot A^T y$ is the value of __________

Ans: $Ax \cdot y$ : cost of i/p

$x \cdot A^T y$ : value of outputs

32. The matrix P that multiplies $(x, y, z)$ to give $(z, x, y)$ is also a rotation matrix.

$P, P^3 = ?$

The rotation axis $a = (1,1,1)$ doesn't move, it equals $Pa$.

What is the angle of rotation from $v = (2,3,-5)$ to $Pv = (-5,2,3)$ ?

Ans: $P = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{23}} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P_{13}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$P^3 = I \Longrightarrow$ 3 rotations for $360^\circ$

~~P rotates 180°~~

$P \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow$ P rotates $120^\circ$ around $(1,1,1)$.

33. Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product ES of an elementary row operation matrix E and a symmetric matrix S

Ans:

$$E_{12} \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \text{ is symmetric.}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

(OR)

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$$

$$U^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow (U^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = LS$$

34. Factorization : $\boxed{A = LS}$ (triangular with 1's times symmetric)

$$\boxed{A = LDU = L(U^T)^{-1} \times U^T DU = LS}$$

$L(U^T)^{-1}$ : lower triangular times lower triangular
$\Longrightarrow$ lower triangular

$(U^T DU)^T = U^T DU$ : $U^T DU$ is symmetric

35. A group of matrices includes $AB$ & $A^{-1}$ if it includes $A$ and $B$. "Products and inverses stay in the group".

check OM(23) for groups

Which of the following sets are groups?

Lower triangular matrices $L$ with 1's on the diagonal, symmetric matrices $S$, positive matrices $M$, diagonal invertible matrices $D$, permutation matrices $P$, matrices with $Q^T = Q^{-1}$

if

ses

$Q^T = Q^{-1}$

36. A square northwest matrix $B$ is zero in the southeast corner, below the antidiagonal that connects $(1,n)$ to $(n,1)$.

Will $B^T$, $B^2$ be northwest matrices ?

Will $B^{-1}$ be northwest or southeast ?

What's the shape of $BC$ = northwest × southeast

Ans:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1,n-1} & b_{1,n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2,n-1} & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & 0 & \cdots & 0 & 0 \\ b_{n,1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} & \cdots & b_{n-1,1} & b_{n,1} \\ b_{12} & b_{22} & b_{32} & \cdots & b_{n-1,2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{1,n-1} & b_{2,n-2} & 0 & \cdots & 0 & 0 \\ b_{1,n} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

→ transpose of a northwest matrix is northwest matrix.

$$B^2 = \begin{bmatrix} \sum b_{1i} b_{i1} & \sum b_{1i} b_{i2} & \cdots & \sum b_{1i} b_{in} \\ \sum b_{2i} b_{i1} & \sum b_{2i} b_{i2} & \cdots & \sum b_{2i} b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum b_{n-1,i} b_{i,1} & \sum b_{n-1,i} b_{i2} & \cdots & \sum b_{n-1,i} b_{in} \\ \sum b_{ni} b_{i,1} & \sum b_{n,i} b_{i,2} & \cdots & \sum b_{ni} b_{in} \end{bmatrix}$$

$\Longrightarrow$ Square of a northwest matrix is not northwest matrix

$\Longrightarrow$ Inverse of northwest matrix is not northwest but southeast.

$$BC = \begin{bmatrix} b_{1n}c_{1n} & b_{1n}c_{2n}+b_{1,n-1}c_{2,n-1} & \cdots & \sum b_{1i}c_{in} \\ 0 & b_{2,n-1}c_{2,n-1} & & \sum b_{2i}c_{in} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \sum b_{n-1,i}c_{in} \\ 0 & 0 & & \sum b_{ni}c_{in} \end{bmatrix}$$

$\Rightarrow$ Product of northwest and southeast matrix is upper triangular matrix

37. If you take powers of a permutation (a) matrix, why is some $P^k$ eventually equal to $I$?

Ans: There are $n!$ permutation matrices

Eventually 2 powers of $P$ must be the same permutation.

If $P^r = P^s$, then $P^{r-s} = I$, $r-s \leq n!$

(b) Find a $5 \times 5$ permutation $P$ so that the smallest power to equal $I$ is $P^6$

Ans: $$P = \begin{bmatrix} P_2 & 0 \\ 0 & P_3 \end{bmatrix} = P_2 \oplus P_3 = \text{diag}(P_2, P_3)$$

where, $$P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} P_2^6 & 0 \\ 0 & P_3^6 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & I_3 \end{bmatrix} = I_5$$

# 3 VECTOR SPACES & SUBSPACES

Matrices : Numbers $\longrightarrow$ Vectors $\longrightarrow$ Spaces of vectors.

We are here looking inside the calculations to find the mathematics !

* The space $\mathbb{R}^n$ consists of all column vectors $\vec{v}$ with $n$ components. The components of $\vec{v}$ are real numbers, which is the reason for the letter $\mathbb{R}$.
* A vector whose $n$ components are complex numbers lies in the space $\mathbb{C}^n$.

The one-dimensional space $\mathbb{R}^1$ is a line.
$\mathbb{R}^2$ is represented by the usual xy-plane.
Each vector gives the x and y coordinates of a point in the plane: $\vec{v} = (x, y)$.

The vectors in $\mathbb{R}^3$ correspond to points $(x, y, z)$ in 3D space.

Ex:- $\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2$, $(1,1,0,1,1) \in \mathbb{R}^5$,

$\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \in \mathbb{C}^2$

* A vector space consists of a set $V$ (elements of $V$ are called vectors), a field $F$ (elements of $F$ are called scalars), and 2 operations:

- An operation called vector addition that takes 2 vectors $\vec{v}, \vec{w} \in V$, and produces a 3rd vector, written as $\vec{v}+\vec{w} \in V$

- An operation called scalar multiplication that takes a scalar $c \in \mathbb{F}$ and a vector $\vec{v} \in V$, and produces a new vector, written $c\vec{v} \in V$.

which satisfy the following conditions:

- Associativity of vector addition:

$$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w}) \quad \forall\ \vec{u}, \vec{v}, \vec{w} \in V$$

- Existence of a zero vector:

There exists a vector $\vec{0} \in V$ such that $\vec{u}+\vec{0} = \vec{u}\ \ \forall\ \vec{u} \in V$, and $\vec{0}$ is called the zero vector

- Existence of -ves : (additive inverse)

  For every $\vec{v} \in V$ there exists an additive inverse $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$.

- Associativity of multiplication :

  $$ab(\vec{v}) = a(b\vec{v}) \quad \forall \; a, b \in \mathbb{F} \text{ and } \vec{v} \in V$$

- Distributivity :

  $$(a+b)\vec{v} = a\vec{v} + b\vec{v} \text{ and } a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v}$$

  $$\forall \; a, b \in \mathbb{F} \text{ and } \vec{u}, \vec{v} \in V$$

- Unitarity :

  $$1\vec{v} = \vec{v} \quad \forall \; \vec{v} \in V$$

→ The above 8 conditions are required for every vector space.

A real vector space is a set of vectors together with rules for vector addition & for multiplication by real numbers.

Vector spaces other than $\mathbb{R}^n$ :

- $F$ : vector space of all real functions $f(x)$

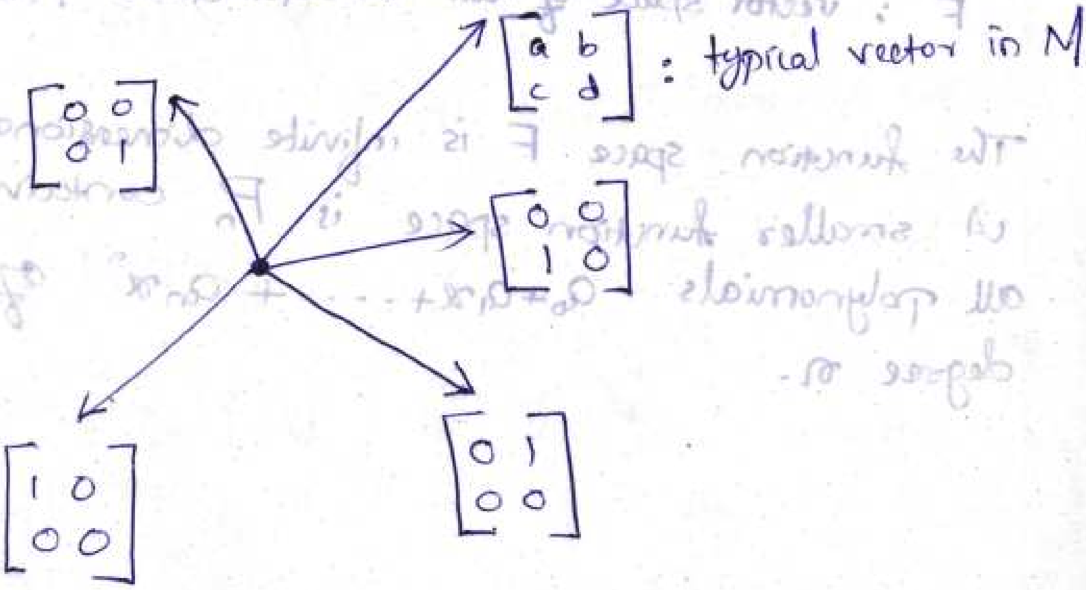
The function space $F$ is infinite dimensional. A smaller function space is $P_n$ containing all polynomials $a_0 + a_1 x + \dots + a_n x^n$ of degree $n$.

• $Z$ : vector space that consists of only a zero vector.

- The space $Z$ is zero-dimensional. & it is the smallest possible vector space.

- The vector space $Z$ contains exactly one vector (zero).

• $M$ : The vector space of all real $2 \times 2$ matrices.




"Four-dimensional matrix space $M$"

# Subspaces

* A subspace of a vector space is a set of vectors (including $\vec{0}$) that satisfies 2 requirements:

If $\vec{v}$ & $\vec{w}$ are vectors in the subspace and $c$ is any scalar, then

(i) $\vec{v}+\vec{w}$ is in the subspace

(ii) $c\vec{v}$ is in the subspace.

$\Longrightarrow$ All linear combinations stay in the subspace

* All these operations follow the rules of the host space, so the 8 required conditions are automatic. We just have to check the linear combinations requirements for a subspace.

* Every subspace contains the zero vector

* The whole vector space is a subspace of itself

Ex:-

List of all the possible subspaces of $\mathbb{R}^3$.

L : any line through $(0,0,0)$

P : any plane through $(0,0,0)$

Z : The single vector $(0,0,0)$

$\mathbb{R}^3$ : the whole space

Ex:1 Keep only the vectors $(x,y)$ whose components are +ve or zero (quarter plane).

Ans: $(2,3)$ is included

$-1(2,3) = (-2,-3)$ is not included

$\Longrightarrow$ rule ② is violated.

$\therefore$ The quarter plane is not a subspace.

Ex:2 Include also the vectors whose components are both -ve, i.e., 2 quarter-planes.

Ans: Rule ② is satisfied.

$\vec{v} = (2,3)$ and $\vec{w} = (-1,-4)$

$\vec{v} + \vec{w} = (1,-1)$ is outside the quarter-planes

$\therefore$ Two quarter-planes don't make a subspace.

Ex:3

Subspaces of the vector space M of all 2x2 matrices :

(i) U : all upper triangular matrices $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

(ii) D : all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

(iii) L : all lower triangular matrices $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$

Add any 2 matrices in U and the sum is in U.

All diagonal matrices, & the sum is diagonal.

- D is also a subspace of U.

Zero matrix is in these subspaces.

Z is always a subspace

Multiples of the identity matrix also form a subspace

ie.,

The matrices $cI$ form a "line of matrices" inside M and U and D.

$2I + 3I$ is in this subspace, and so is 3 times $4I$.

Note : The matrix $I$ by itself is not a subspace. Only the zero matrix is.

## The Column Space of A

$\boxed{Ax = b} \Rightarrow$

If 'A' is not invertible, the system is solvable for some 'b', and not solvable for other 'b'.

We want to describe the good right sides 'b' — the vectors that can be written as 'A' times some vector 'x'. Those b's from the "column space" of 'A'.

To solve $Ax = b \iff$ express 'b' as a combination of the columns.

* The column space consists of all linear combinations of the columns.

The combinations are all possible vectors $Ax$. They fill the column space $C(A)$.

* The system $Ax = b$ is solvable iff 'b' is in the column space of 'A'.

ie.,

When b is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution 'x' to the system $Ax = b$.

$A_{m \times n} \longrightarrow C(A)$ is a subspace of $R^m$.

Start with any set $S$ of vectors in a vector space $V$. To get a subspace $SS$ of $V$, we take all combinations of the vectors in that set:

$S$ = set of vectors in $V$ (probably not a subspace)

$\{v_1, v_2, \ldots, v_n\} \in V$.

$SS$ = all combinations of vectors in $S$

= all $c_1 v_1 + \cdots + c_n v_n$

= Subspace of $V$ "spanned" by $S$.

The subspace $SS$ is the "span" of $S$, containing all combinations of vectors in $S$.

* $SS$ is the smallest subspace containing $S$.

Ex:- When $S$ is the set of columns, $SS$ is the column space.

i.e., the columns span the column space

When there is only one non-zero vector $\vec{v}$ in $S$, the subspace $SS$ is the line thro' $\vec{v}$.

⇒ This is a fundamental way to create subspaces.

Ex: 4 $\quad Ax = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$

$C(A)$ of all combinations of the 2 columns fill up a plane in $\mathbb{R}^3$.

The plane has zero thickness, so most right sides b in $\mathbb{R}^3$ are not in the column space.

ie., For most 'b' there is no solution to our 3 equations in 2 unknowns.

$(0,0,0)$ is in the column space. The Plane passes thro' the origin.

→ There is certainly a solution to $Ax = 0$

Ex: 5 Describe the column spaces (they are subspaces of $\mathbb{R}^2$) for

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \& \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

Ans: $C(I)$ is the whole space $\mathbb{R}^2$.

$$C(I) = \mathbb{R}^2$$

$C(A)$ is a line thro' $(1,2)$ & $(0,0)$

i.e., $Ax = b$ is only solvable when $b$ is on that line.

$C(B)$ is all of $\mathbb{R}^2$.

$$C(I) = C(B) = \mathbb{R}^2$$

But, now '$x$' has extra components & there are more solutions. — more combinations that give '$b$'.

3.1(A) We are given 3 diff. vectors $b_1, b_2, b_3$. Construct a matrix so that the eq's $Ax = b_1$ and $Ax = b_2$ are solvable., but $Ax = b_3$ is not solvable. How can you decide if this is possible? How could you construct $A$?

Ans: make $b_1, b_2$ the columns of $A$.

$$Ax = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_3$$

If $b_3$ is a combination of $b_1$ and $b_2$, then it is not possible to construct the required $A$., since $b_3$ would necessarily be in the column space & $Ax = b_3$ would necessarily be solvable.

If no, we have the desired matrix $A$.

$\boxed{3\cdot 1 \,(B)}$ Describe a subspace $S$ of each vector space $V$ and then a subspace $SS$ of $S$.

$V_1$ : all combinations of $(1,1,0,0)$ & $(1,1,1,0)$ & $(1,1,1,1)$

$V_2$ : all vectors $\perp$ to $u=(1,2,1)$, so $u\cdot v=0$

$V_3$ : all symmetric $2\times 2$ matrices (a subspace of $M$)

$V_4$ : all solutions to the equation $\frac{d^4y}{dx^4}=0$

(a subspace of $F$)

Describe each $V$ in 2 ways : "All combinations of ________", "all solutions of the equations ________"

Ans: $V_1$ : A subspace $S$ comes from all combinations of the 1st 2 vectors. $(1,1,0,0)$ & $(1,1,1,0)$.

A subspace $SS$ of $S$ comes from all multiples of the 1st vector $(1,1,0,0)$.

$V_1$ = all combinations of the 3 vectors

= all solutions of

$V_2 =$ ~~∀~~ all combinations of $(1,-1,1)$ & $(1,0,-1)$

$=$ all solutions of $u \cdot v = 0$

(?) subspace $S$ of $V_2$ is the line thro' $(1,-1,1)$, is $\perp$ to $u$. ~~The smallest~~ The subspace $SS$ of $S$ is $Z$.

$V_3 =$ all symmetric $2 \times 2$ matrices

$=$ all combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$=$ all solutions $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $b = c$

Note: only the upper (or lower) part of a symmetric matrix completely determines the other half.

i.e., $a, b, d$ matters

$\Longrightarrow$ 3 dimensional.

The diagonal matrices are a subspace $S$ of the symmetric matrices. The multiples $cI$ are a subspace ~~S~~$SS$ of the diagonal matrices.

$V_4$ = all solutions to $\frac{d^4y}{dx^4} = 0$.

$V_4$ contains all cubic polynomials
$y = a + bx + cx^2 + dx^3$, with $\frac{d^4y}{dx^4} = 0$.

$\rightarrow$ 4 dimensional.

$V_4$ = all combinations of $1, x, x^2, x^3$

The quadratic polynomials give a subspace $S$.
The linear polynomials are one choice of SS.
The constant could be SSS.

In all 4 parts, we could take $S = V$ itself
and SS = the zero subspace $Z$.