Introduction to Linear Algebra
- Gilbert Strang

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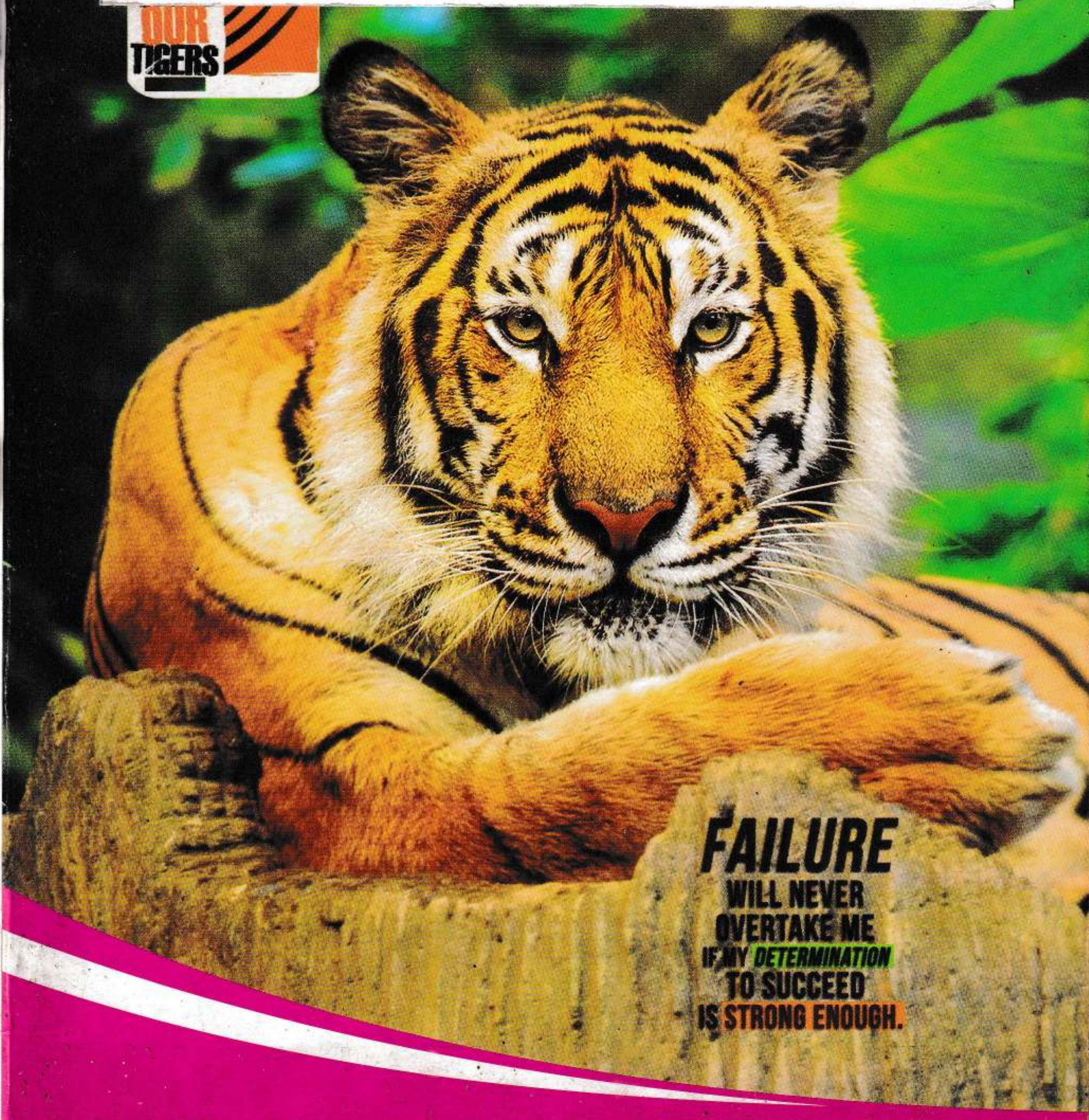
Linear Transformations

(Laguerre's Polynomials
Bessel's functions)



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		INTRODUCTION TO LINEAR ALGEBRA - Gilbert Strang, MIT (5th Edition)		

Laguerre's Polynomials

Laguerre's differential equation is a linear second order ODE :

$$\boxed{x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \nu\, y(x) = 0}$$

where, ν is a parameter.

$x=0$ is a regular singular point.

$$y(x) = \sum_{j=0}^{\infty} a_j\, x^{\alpha+j} = x^{\alpha}\sum_{j=0}^{\infty} a_j\, x^{j}, \quad a_0 \neq 0$$

$$y'(x) = \sum_{j=0}^{\infty} a_j(\alpha+j)\, x^{\alpha+j-1}$$

$$y''(x) = \sum_{j=0}^{\infty} a_j(\alpha+j)(\alpha+j-1)\, x^{\alpha+j-2}$$

$$x \sum_{j=0}^{\infty} a_j(\alpha+j)(\alpha+j-1)\, x^{\alpha+j-2} + (1-x)\sum_{j=0}^{\infty} a_j(\alpha+j)\, x^{\alpha+j-1}$$

$$+ \nu \sum_{j=0}^{\infty} a_j\, x^{\alpha+j} = 0$$

$$\sum_{j=0}^{\infty} a_j(\alpha+j)(\alpha+j-1)\, x^{\alpha+j-1} + \sum_{j=0}^{\infty} a_j(\alpha+j)\, x^{\alpha+j-1}$$

$$- \sum_{j=0}^{\infty} a_j(\alpha+j)\, x^{\alpha+j} + \nu \sum_{j=0}^{\infty} a_j\, x^{\alpha+j} = 0$$

The lowest power of x is $j+\alpha-1$ when $j=0$.

$$a_0\alpha\left[\alpha-1+1\right] = a_0\alpha^2 = 0$$

$$\Longrightarrow \alpha = 0$$

$$\sum_{j=0}^{\infty} a_j\, j(j-1)x^{j-1} + \sum_{j=0}^{\infty} j a_j\, x^{j-1} - \sum_{j=0}^{\infty} j\, a_j x^j + \nu\sum_{j=0}^{\infty} a_j x^j = 0$$

Equating the coeff. of x^j to zero,

$$a_{j+1}\left[j(j+1)+(j+1)\right] + (\nu - j)a_j = 0.$$

$$\boxed{a_{j+1} = \frac{-(\nu-j)}{(j+1)^2}\, a_j}$$

$\Longrightarrow$ recurrence relation b/w a_{j+1} and a_j.

$$y(x) = a_0 \left[1 - \frac{\nu}{1^2} x + \frac{\nu(\nu-1)}{1^2 \times 2^2} + \cdots \cdots + (-1)^j \frac{\nu(\nu-1) \cdots\cdots (\nu-(j-1))}{1^2 \times 2^2 \times \cdots\cdots \times j^2} \right]$$

If ν is not a +ve integer or zero, this will remain an infinite series.

For the particular case $\nu = n$, where n is a +ve integer or zero, the series will terminate at the $(n+1)^{th}$ term, and reduce to an n^{th} degree polynomial in x.

We'll further choose $a_0 = 1$, the resultant expression defines the Laguerre polynomial, $L_n(x)$.

$$L_n(x) = 1 - \frac{n}{(1!)^2}x + \frac{n(n-1)}{(2!)^2}x^2 + \cdots + (-1)^n \frac{n(n-1)\cdots 1}{(n!)^2}x^n$$

$$= 1 - {}^nC_1 \frac{x}{1!} + {}^nC_2 \frac{x^2}{2!} - \cdots + (-1)^n \, {}^nC_n \frac{x^n}{n!}$$

$$L_0(x) = 1$$

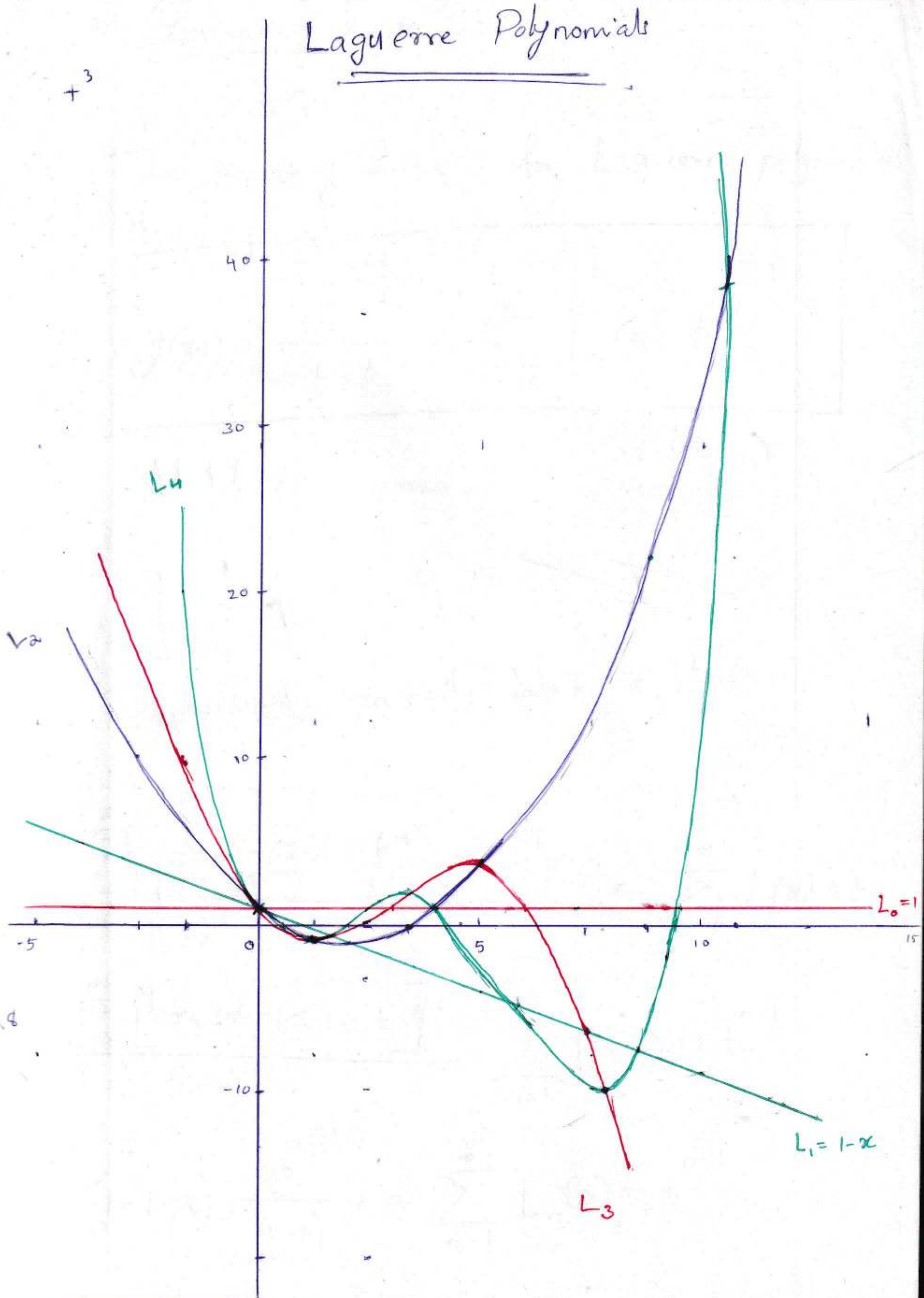
$$L_1(x) = 1 - x$$

$$L_2(x) = 1 - 2x + \frac{x^2}{2}$$

$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{x^3}{6}$$

$$L_4(x) = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{x^4}{24}$$

Laguerre Polynomials



Generating function

The generating function for Laguerre polynomials is

$$\boxed{g(x,t) = \frac{e^{\frac{-xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n,}$$

$|t|<1$

Differentiating partially w.r.t t,

$$\frac{e^{-\frac{xt}{1-t}}\left[\frac{(1-t)(-x)-(-xt)(-1)}{(1-t)^2}\right]\cancel{(1-t)} - e^{-\frac{xt}{1-t}}(-1)}{(1-t)^2} = \sum_{n=1}^{\infty} L_n(x)\, n t^{n-1}$$

$$\frac{e^{-\frac{xt}{1-t}}\left[-x+\cancel{tx}-\cancel{tx}+1-t\right]}{(1-t)^3} = \sum_{n=1}^{\infty} L_n(x)\, n\, t^{n-1}$$

$$(1-t-x)\,\frac{e^{-\frac{xt}{1-t}}}{(1-t)^3} = \sum_{n=1}^{\infty} L_n(x)\, n\, t^{n-1}$$

$$\frac{(1-t-x)}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)t^n = \sum_{n=1}^{\infty} L_n(x)\, n\, t^{n-1}$$

$$\boxed{(1-t-x)\sum_{n=0}^{\infty} L_n(x)\, t^n = (1-t)^2\sum_{n=1}^{\infty} L_n(x)\, n\, t^{n-1}}$$

Equating the coeff. of t^{n+1} from the 2 sides and rearranging terms,

$(1-t)^2 = 1-2t+t^2$

$$\cancel{(1-t-x) L_{n+1}(x) = (1-t)^2}$$

$$(1-x) L_{n+1}(x) - L_n(x) = (n+2) L_{n+2}(x) - 2(n+1) L_{n+1}(x) + n L_n(x)$$

$$\boxed{(n+2) L_{n+2}(x) = (2n+3-x) L_{n+1}(x) - (n+1) L_n(x)}$$

$$g(x,t) = \left[1 - \frac{xt}{1-t} + \frac{1}{2!}\left(\frac{xt}{1-t}\right)^2 - \frac{1}{3!}\left(\frac{xt}{1-t}\right)^3 + \cdots\right]\left(\frac{1}{1-t}\right)$$

$$= \left[1 - xt\left\{1 + t + \frac{t^2}{2!} + \cdots\right\} + \cdots\cdots\right]\left(1 + t + \frac{t^2}{2!} + \cdots\right)$$

$$= \left[1 - xt - xt^2 - \frac{xt^3}{2!} + \cdots\right]\left((1+)t + \frac{t^2}{2!} + \cdots\right)$$

$$= 1 - xt - xt^2 - \frac{xt^3}{2!} + \cdots + t - xt^2 + xt^3 - \cdots$$

Coeff.

Coeff. of t^0 is $1 = L_0(x)$

Coeff. of t^1 is $1-x = L_1(x)$

substitute in the recurrence relation,

$L_2(x) = 1 - 2x + \frac{x^2}{2}$

$$g(x,t) = \frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n,$$

$|t|<0$

Differentiating w.r.t x,

$$\frac{-t}{(1-t)^2} e^{\frac{-xt}{1-t}} = \sum_{n=0}^{\infty} \frac{dL_n}{dx} t^n$$

$$\frac{-t}{1-t} \sum_{n=0}^{\infty} L_n(x)\, t^n = \sum_{n=0}^{\infty} \frac{dL_n}{dx} t^n$$

$$-t \sum_{n=0}^{\infty} L_n(x)\, t^n = (1-t) \sum_{n=0}^{\infty} \frac{dL_n}{dx} t^n$$

Equating the coeff. of t^{n+1} on both sides,

$$-L_n(x) = \frac{dL_{n+1}(x)}{dx} - \frac{dL_n}{dx}$$

$$\Longrightarrow \boxed{\frac{dL_{n+1}(x)}{dx} = \frac{dL_n(x)}{dx} - L_n(x)}$$

Rodrigue's formula

Rodrigues' formula for the Hermite polynomials,

$$\boxed{L_n(x) = \frac{e^{x}}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)}$$

Proof

Lets define a coefficient extractor,

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$$x^n = [u^0] : \frac{u^{-n}}{1-xu} = u^{-n}\left[1+xu+(xu)^2+\cdots+(xu)^n+\cdots\right.$$

$$\sum_{n=0}^{\infty} L_n(x)\,t^n = e^{x}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

$$= e^{x}\sum_{n=0}^{\infty}\left(\frac{t}{u}\right)^n\frac{1}{n!}\frac{d^n}{dx^n}\left((xu)^n e^{-x}\right)$$

$$= [u^0] : e^{x}\sum_{n=0}^{\infty}\left(\frac{t}{u}\right)^n\frac{1}{n!}\frac{d^n}{dx^n}\frac{e^{-x}}{1-xu}$$

$$f(x+a) = f(x) + a\frac{d}{dx}f(x) + \frac{a^2}{2!}\frac{d^2}{dx^2}f(x) + \cdots$$

Taylor Series

$$= \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n}f(x)$$

$$a = \frac{t}{u}, \quad f(x) = \frac{e^{-x}}{1-xu}$$

$$\sum_{n=0}^{\infty} \frac{(t/u)^n}{n!}\frac{d^n}{dx^n}\left[\frac{e^{-x}}{1-xu}\right] = \frac{e^{-(x+t/u)}}{1-u(x+t/u)}$$

$$\sum_{n=0}^{\infty} L_n(x)\, t^n = [u^0] : e^{x}\,\frac{e^{-(x+t/u)}}{1-u(x+t/u)}$$

$$= [u^0] : e^{x}\,\frac{e^{-x}\cdot e^{-t/u}}{1-t-ux}$$

$$= \frac{1}{1-t}\,[u^0] : \frac{e^{-t/u}}{1-\frac{ux}{1-t}}$$

$$= \frac{1}{1-t}[u^0] : \left(\sum_{i=0}^{\infty} \frac{(-t/u)^i}{i!}\right)\left(\sum_{j=0}^{\infty}\left(\frac{+ux}{1-t}\right)^j\right)$$

$$= \frac{1}{1-t}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-xt}{1-t}\right)^k$$

$$= \frac{e^{\frac{-xt}{1-t}}}{1-t} = g(x,t).$$

Orthogonality

Diff. eqn. satisfied by Laguerre polynomials of degree n and k are :-

$$x\frac{d^2L_n}{dx^2} + (1-x)\frac{dL_n}{dx} + nL_n(x) = 0 \quad \text{——— (1)}$$

$$x\frac{d^2L_k}{dx^2} + (1-x)\frac{dL_k}{dx} + kL_k(x) = 0 \quad \text{——— (2)}$$

xing (1) by $e^{-x}L_k(x)$ and (2) by $e^{-x}L_n(x)$ and subtract

$$xe^{-x}\left(L_k(x)\frac{d^2L_n}{dx^2} - L_n(x)\frac{d^2L_k}{dx^2}\right) +$$

$$+ (1-x)e^{-x}\left(L_k(x)\frac{dL_n}{dx} - L_n(x)\frac{dL_k}{dx}\right)$$

$$+ (n-k)e^{-x}L_n(x)L_k(x) = 0$$

$$\frac{d}{dx}(xe^{-x}) = e^{-x} + xe^{-x} = e^{-x}(1+x)$$

$$\frac{d}{dx}\left[xe^{-x}\left\{L_k(x)\frac{dL_n}{dx} - L_n(x)\frac{dL_k}{dx}\right\}\right] +$$

$$(n-k)\,e^{-x}L_n(x)\,L_k(x) = 0$$

$\int$ing from 0 to ∞ w.r.t x,

$$\cancel{xe^{-x}\left[L_k(x)\frac{dL_n}{dx} - L_n(x)\frac{dL_k}{dx}\right]_0^\infty} +$$

$$(n-k)\int_0^\infty e^{-x}L_n(x)\,L_k(x)\,dx = 0$$

$$(n-k)\int_0^\infty e^{-x}L_n(x)\,L_k(x)\,dx = 0$$

If $n \neq k$,

$$\int_0^\infty e^{-x} L_n(x) L_k(x)\, dx = 0$$

$\Longrightarrow$ the Laguerre polynomials of different degrees are orthogonal to each other on the interval $(0,\infty)$ with weight factor e^{-x}.

$$g(x,t) = \frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n, \quad |t|<1$$

~~For n=k~~,

$$\frac{e^{-\frac{2xt}{1-t}}}{(1-t)^2} = \sum_{n=0}^{\infty} L_n(x)\, t^n \sum_{k=0}^{\infty} L_k(x)\, t^k$$

xing by e^{-x} and integrate from 0 to ∞.

$$\frac{1}{(1-t)^2}\int_0^\infty e^{-\frac{2xt}{1-t}-x}\, dx = \frac{1}{(1-t)^2}\int_0^\infty e^{\frac{-xt-x}{1-t}}\, dx = \frac{1}{(1-t)^2}\int_0^\infty e^{-\frac{1+t}{1-t}x}\, dx$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^n t^k \int_0^\infty e^{-x} L_n(x) L_k(x)\, dx$$

For $n=k$,

$$RHS = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} t^n t^k \int_0^{\infty} e^{-x} L_n(x) L_k(x)\, dx = \sum_{n=0}^{\infty} t^{2n} \int_0^{\infty} e^{-x} L_n^2(x)\, dx$$

$$LHS = \frac{1}{(1-t)^2} \times (-1) \left(\frac{1-t}{1+t}\right) e^{-\frac{1+t}{1-t}x} \Big]_0^{\infty} = \frac{-1}{1-t^2} e^{\frac{-(1+t)}{(1-t)}x} \Big]_0^{\infty}$$

$$= \frac{-1}{1-t^2}(0-1) = \frac{1}{1-t^2}$$

For $t<<1$,

$$\frac{1}{1-t^2} = 1+t^2+t^4+\cdots = \sum_{n=0}^{\infty} t^{2n}$$

$\therefore$

$$\sum_{n=0}^{\infty} t^{2n} = \sum_{n=0}^{\infty} t^{2n} \int_0^{\infty} e^{-x} L_n^2(x)\, dx$$

Comparing the coeff. of t^{2n} for all n,

$$\int_0^\infty e^{-x} L_n^2(x)\, dx = 1$$

Orthogonality relations for Laguerre polynomials as :

$$\boxed{\int_0^\infty e^{-x} L_n(x)\, L_k(x)\, dx = \delta_{nk}}$$

Bessel Functions

Bessel's differential equation is an ordinary differential eqn of order 2 and degree 1.

$$\boxed{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)\, y(x) = 0}$$

where, the parameter 'm' is real & non-negative.

$x = 0$ is a regular singular point.

Frobenius method

Assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

Substituting,

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} m^2 a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + (n+r) - m^2\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Equating the coeff. of the lowest power of x to zero, indicial equation is obtained.

when $n=0$:

$$a_0 \left[r(r-1) + r - m^2\right] = 0$$

Since, $a_0 \neq 0$ we have

$$r^2 - m^2 = 0 \longrightarrow r = \pm m$$

Depending on the value of m, the solutions can differ vastly,

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \quad \text{and}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-m}$$

To find y_1,

Equating the coeff. of each power of x to zero,

Coef. of x^r : $(r^2 - m^2)a_0 = 0$

Coef. of x^{r+1} : $\left[(r+1)^2 - m^2\right]a_1 = 0$

Coef. of x^{r+2} : $\left[(r+2)^2 - m^2\right]a_2 - a_0 = 0$

$$\Longrightarrow a_2 = \frac{-1}{(r+2)^2 - m^2} a_0$$

Coef. of x^{r+n} : $\left[(n+r)^2 - m^2\right]a_n + a_{n-2} = 0$

$$\Longrightarrow \boxed{a_n = \frac{-1}{(n+r)^2 - m^2} a_{n-2}}$$

For $r = m$,

$$\left[(r+1)^2 - m^2\right] a_1 = \left[2m+1\right] a_1 = 0$$

$$\Rightarrow a_1 = 0.$$

$$\therefore a_3 = a_5 = a_7 = \cdots = 0$$

$$a_2 = \frac{-a_0}{(m+2)^2 - m^2} = \frac{-a_0}{(\cancel{m}+2-\cancel{m})(m+2+m)}$$

$$= \frac{-a_0}{2^2(m+1)}$$

$$a_4 = \frac{-a_2}{2^2 \times 2(m+2)} = \frac{a_0}{2^4 \times 2(m+1)(m+2)}$$

$$\boxed{a_6 = \frac{-a_4}{2^2 \times 3(m+3)}} = \frac{-a_0}{2^6 \times 3 \times 2(m+1)(m+2)(m+3)}$$

$$\boxed{a_{2n} = \frac{(-1)^n a_0}{2^{2n} \times n! \times (m+1)(m+2)\cdots\cdots(m+n)}}$$

$n = 0, 1, 2, \ldots$

The solution corresp. to $r = -m$ is found by simply replacing m by $-m$, provided m is not an integer.

$$y_1(x) = a_0\left[x^m - \frac{x^{m+2}}{2^2(m+1)} + \frac{x^{m+4}}{2^4 * 2!\,(m+1)(m+2)} - \cdots (-1)^k \frac{x^{2k}}{2^{2k} * k!\,(m+1)(m+2)\cdots(m+k)} + \cdots\right]$$

$$= a_0 \sum_{k=0}^{\infty} (-1)^k * \frac{x^{2k+m}}{2^{2k} * k!\,(m+1)(m+2)\cdots(m+k)}$$

$\Gamma(n+1) = n!$

$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$

$\Gamma(1) = 1$

$\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

$\boxed{\Gamma(m+1) = m\,\Gamma(m)}$

By choosing, $a_0 = \frac{1}{2^m\,\Gamma(m+1)}$

$$\boxed{y_1(x) = J_m(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma(m+k+1)!} \left(\frac{x}{2}\right)^{2k+m}}$$

$\Rightarrow$ Bessel function of the 1st kind of order m.

When the value of m is non-integral, the other linearly independent solution of the Bessel's diffⁱ eqⁿ is obtained by replacing m by $-m$:

$$y_2(x) = J_{-m}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma(-m+k+1)} \left(\frac{x}{2}\right)^{2k-m}$$

Bessel function of the 1st kind and order $-m$.

Q. Show that $J_{1/2}(x) = \sqrt{2/\pi}\, x^{-1/2} \sin x$ and

$J_{-1/2}(x) = \sqrt{2/\pi}\, x^{-1/2} \cos x$.

Ans:

$$J_{1/2}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma\left(k+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k+\frac{1}{2}}$$

$$\cancel{\Gamma\left(k+\frac{3}{2}\right) = \left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)} \ldots \quad \Gamma(k) = \int_0^{\infty} e^{-x} x^{k-1}\, dx$$

$$\boxed{\Gamma(k+1) = k\,\Gamma(k)} \quad \& \quad \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

$$\Gamma\left(k+\frac{3}{2}\right) = \left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)$$

$$= \left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)$$

$$= \left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)\cdots\frac{1}{2}\Gamma(1/2)$$

$$= \frac{(2k+1)(2k-1)(2k-3)\cdots 1}{2^k}\sqrt{\pi} \quad \cancel{= \frac{(2k+1)!\sqrt{\pi}}{2^k}}$$

$$\cancel{= \frac{(2k)!}{}} \quad \frac{(2k+1)!}{2^{k+1}\,(2k+2)(2k+4)\cdots 2}\sqrt{\pi}$$

$$= \cancel{\frac{(2k+1)!\sqrt{\pi}}{2^{k+1}(k+1)!}} \quad \frac{(2k+1)!}{2^{2k+1} \times k!}\sqrt{\pi}$$

$1, \ldots, 2k-2$

$a = 1,\ d = -2$

$1 + (n-1)(-2) = 2k-2$

$1 + (n-1)(-2) = 2k-3$

$-2n + 2 = 2k$

$2n - 2 = -2k$

when $x + (n-1)(2) = 2k + x$

$2n - 2 = 2k$

$n - 1 = k$

$n = k+1$

$$J_{1/2}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} \cancel{k!}}{\cancel{k!} (2k+1)! \sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \times \sqrt{\frac{2}{x}}$$

$$= \frac{1}{\sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right] \times \sqrt{2/x}$$

$$= \frac{1}{\sqrt{\pi}} \sin x \times \sqrt{2/x} = \sqrt{\frac{2}{\pi}}\, x^{-1/2} \sin x$$

$$J_{-1/2}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma\left(k+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2k-\frac{1}{2}}$$

$$\Gamma\left(k+\tfrac{1}{2}\right) = \left(k-\tfrac{1}{2}\right) \Gamma\left(k-\tfrac{1}{2}\right)$$

$$= \left(k-\tfrac{1}{2}\right)\left(k-\tfrac{3}{2}\right) \Gamma\left(k-\tfrac{3}{2}\right)$$

$$= \left(k-\tfrac{1}{2}\right)\left(k-\tfrac{3}{2}\right) \cdots \tfrac{1}{2}\, \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{(2k-1)(2k-3)\cdots 1}{2^k} \Gamma(1/2)$$

$$= \frac{(2k-1)!}{2^k \times 2^{k-1}(k-1)!} \sqrt{\pi} = \frac{(2k-1)!\,\sqrt{\pi}}{2^{2k-1}(k-1)!} = \frac{(2k)!}{2^{2k-1}\cdot k!}\sqrt{\pi}$$

$$J_{-1/2}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k-1} \times \cancel{k!}}{\cancel{k!} \times (2k)! \times \sqrt{\pi}} \left(\frac{x}{2}\right)^{2k-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \sqrt{\frac{2}{x}}$$

$$= \sqrt{\frac{2}{\pi}}\; x^{-1/2} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi}}\; x^{-1/2} \cos x$$

Bessel Functions of the 1st kind

If m is zero or a +ve integer, say n,

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k+n}$$

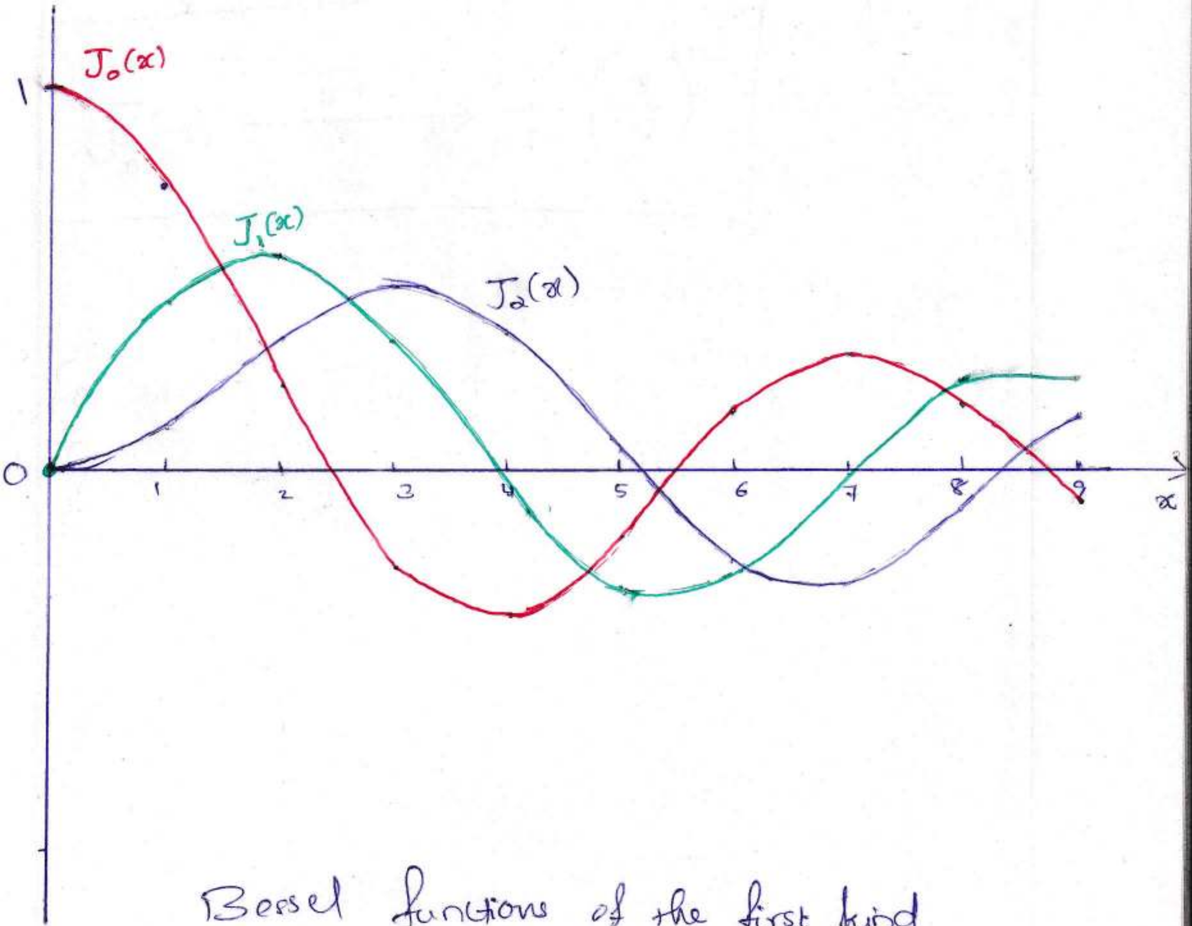
$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 \times (2!)^2} - \frac{x^6}{2^6 \times (3!)^2} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3 \times (1!) \times (2!)} + \frac{x^5}{2^5 \times (2!) \times (3!)} - \frac{x^7}{2^7 \times (3!) \times (4!)} + \cdots$$

$$J_2(x) = \frac{x^2}{2^2 \times (2!)} - \frac{1}{3!}\,\frac{x^4}{2^4} + \frac{1}{(2!) \times (4!)} \times \frac{x^6}{2^6} - \frac{1}{(3!) \times (5!)} \times \frac{x^8}{2^8} + \cdots$$

- These functions exhibit oscillatory behaviour

- becomes zero for a # of values of x



Bessel functions of the first kind

Mathematica: N[BesselJ[n, x] 50]

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times (n+k)!} \left(\frac{x}{2}\right)^{2k+n}$$

$$\boxed{J_n(x) \xrightarrow{x \to 0} \frac{1}{n!} \left(\frac{x}{2}\right)^n}$$

For $n = 0$, $J_0(0) = 1$

$$y_2(x) = J_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \, \Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k-n}$$

It is not possible to use this eq^n. to obtain the Bessel function of 1st kind for -ve integral orders $-n$ $(n = 1, 2, 3, \ldots)$, since the gamma function occuring in the denominator will become infinite for $k \leq (n-1)$.

$\therefore$ The 1st n terms in the series will be zero.

Put $p = k - n$

$$J_{-n}(x) = \sum_{p=0}^{\infty} (-1)^{p+n} \frac{1}{(p+n)!\,\Gamma(p+1)} \left(\frac{x}{2}\right)^{2p+n}$$

$$= (-1)^n \sum_{p=0}^{\infty} (-1)^p \frac{1}{p!\,(n+p)!} \left(\frac{x}{2}\right)^{2p+n}$$

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k+n}$$

$$\Longrightarrow \boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

When n is an even integer, $J_{-n}(x) = J_n(x)$.

When n is odd integer, $J_{-n}(x) = -J_n(x)$

$\therefore$ When n is integral $J_n(x)$ and $J_{-n}(x)$ are solutions of Bessel's differential equation, but these solutions are not linearly independent.

When we wish to calculate the numerical value of Bessel's functions for different values of x, we resort to the series in eq^n.

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n}$$

For any value of x, we get a sufficiently accurate value of $J_n(x)$ by including terms of this series till the next higher term makes a negligible contribution, and summing the contributions of all significant terms.

For small values of x, only a few terms will contribute, but for large x, many terms in the series will have non-negligible contributions.

It may be inconvenient to use the series to compute the values of the Bessel function for large values of x.

$\Longrightarrow$ recurrence relations

Recurrence Relations

$$J_m(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k+m}$$

Differentiating $x^m J_m(x)$ w.r.t x,

$$\frac{d}{dx}\left[x^m J_m(x)\right] = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2m)}{k! \, \Gamma(m+k+1)} \frac{x^{2k+2m-1}}{2^{2k+m}}$$

$$m x^{m-1} J_m(x) + x^m \frac{d J_m}{dx} = x^m \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma(m+k)} \frac{x^{2k+m-1}}{2^{2k+m-1}}$$

$$= x^m \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! \times \Gamma((m-1)+k+1)} \left(\frac{x}{2}\right)^{2k+(m-1)}$$

$$= x^m J_{m-1}(x)$$

$\div$ing by x^m,

$$\boxed{\frac{m}{x} J_m(x) + \frac{dJ_m}{dx} = J_{m-1}(x)}$$

Differentiate $x^{-m} J_m(x)$ w.r.t x,

$$\frac{d}{dx}\left[x^{-m} J_m(x)\right] = \sum_{k=0}^{\infty} (-1)^k \frac{2k}{k!\,\Gamma(m+k+1)} \frac{x^{2k-1}}{2^{2k+m}}$$

$$-m x^{-m-1} J_m(x) + x^{-m} \frac{dJ_m(x)}{dx} = \sum_{k=0}^{\infty} (-1)^k \frac{2k}{k!\,\Gamma(m+k+1)} \frac{x^{2k+m-1}}{2^{2k+m}}$$

$$\cancel{-m x^{-m-1} J_m(x) + x^{-m} \frac{dJ_m(x)}{dx} = x^{-m} \sum_{k=0}^{\infty} (-1)^k \frac{2k(m+k+1)}{k!\,(m+k+1)\,\Gamma(m+k+1)} \frac{x^{2k+m}}{2^{2k+m}}}$$

$$\cancel{= x^{-m} \sum_{k=0}^{\infty} (-1)^k \frac{2k(m+k+1)}{k!\,\Gamma((m+1)+k+1)} \frac{x^{2k+m+1-2}}{2^{2k+m+1-1}}}$$

Set $t = k-1 \Rightarrow k = t+1$

$k: 1 - \infty \Longrightarrow t: 0 \to \infty$

$$-m x^{-m-1} J_m(x) + x^{-m} \frac{d J_m(x)}{dx} =$$

$$= \sum_{t=0}^{\infty} (-1)^{t+1} \frac{2(t+1)}{(t+1)!\,\Gamma(m+t+2)} \frac{x^{2t+1}}{2^{2t+m+2}}$$

$$= \sum_{t=0}^{\infty} (-1)^t \frac{1}{t!\,\Gamma((m+1)+t+1)} \frac{x^{2t+m+1} \times x^{-m}}{2^{2t+m+1}}$$

$$= -x^{-m} \sum_{t=0}^{\infty} (-1)^t \frac{1}{t!\times\Gamma((m+1)+t+1)} \frac{x^{2t+m+1}}{2^{2t+m+1}}$$

$$= -x^{-m} J_{m+1}(x)$$

xing by x^m,

$$-\frac{m}{x} J_m(x) + \frac{d J_m(x)}{dx} = -J_{m+1}(x)$$

$$\boxed{\frac{m}{x} J_m(x) - \frac{d J_m(x)}{dx} = J_{m+1}(x)}$$

$$\cancel{J_{m-1}(x) + J_{m+1}(x) = \frac{2x}{m}}$$

$$\boxed{J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)}$$

$$\boxed{J_{m-1}(x) - J_{m+1}(x) = 2 J'_m(x)}$$

* The derivations of the recurrence relations given above holds for Bessel functions of the 1st kind whose orders may be integral or non-integral.

Generating function

The generating function for the Bessel functions of the 1^{st} kind and integral order is:

$$\boxed{g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n}$$

Differentiate partially w.r.t t, keeping x fixed.

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)\left(\frac{x}{2}\left(1+\frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{+\infty} J_n(x)\; n t^{n-1}$$

$$\frac{x}{2}\;\frac{t^2+1}{t^2}\;\underbrace{\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]}_{g(x,t)} = \sum_{n=-\infty}^{+\infty} J_n(x)\; n\, t^{n-1}$$

$$\frac{x}{2}(t^2+1)\sum_{n=-\infty}^{+\infty} J_n(x)\,t^n = t^2\sum_{n=-\infty}^{+\infty} J_n(x)\,n\,t^{n-1}$$

$$= \sum_{n=-\infty}^{+\infty} J_n(x)\,n\,t^{n+1}$$

Equating the coef. of like powers of 't' on the 2 sides.

Coef. of t^{n+1}:

$$\frac{x}{2}J_{n-1}(x) + \frac{x}{2}J_{n+1}(x) = n\,J_n(x)$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

Ex:- Starting from $g(x,t)$ show that

$$J_0(x) + 2J_2(x) + 2J_4(x) + 2J_6(x) + \cdots + 2J_{2k}(x) + \cdots = 1$$

Ans: $$g(x,t) = \exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n$$

Set $t=1$,

$$LHS = e^0 = 1$$

$$RHS = \sum_{n=-\infty}^{+\infty} J_n(x) = J_0(x) + \left[J_1(x) + J_{-1}(x)\right] + \left[J_2(x) + J_{-2}(x)\right] + \left[J_3(x) + J_{-3}(x)\right] + \left[J_4(x) + J_{-4}(x)\right] + \cdots$$

$$J_{-n}(x) = (-1)^n J_n(x).$$

Terms involving Bessel functions of odd integral order vanishes.

$$\boxed{J_0(x) + 2J_2(x) + 2J_4(x) + \cdots + 2J_{2k}(x) + \cdots = 1}$$

Integral Representation of Bessel Functions

$$g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n$$

Put $t = e^{i\theta}$, then

$$t - \frac{1}{t} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{+\infty} J_n(x)\, e^{in\theta}$$

$$= J_0(x) + J_1(x)\, e^{i\theta} + J_{-1}(x)\, e^{-i\theta} + J_2(x)\, e^{2i\theta} + J_{-2}(x)\, e^{-2i\theta} + \cdots$$

$$= J_0(x) + J_1(x)\left[e^{i\theta} - e^{-i\theta}\right] + J_2(x)\left[e^{2i\theta} + e^{-2i\theta}\right] + \cdots$$

$$= J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right]$$

$$+ 2i\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right]$$

Equating the real & Imaginary parts of the 2 sides,

$$\cos(x \sin\theta) = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right.$$

$$= J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta)$$

$$\sin(x\sin\theta) = 2\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right]$$

$$= 2\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\theta$$

the

$+\theta+\cdots$

If $\theta = \pi/2$,

$$\cos x = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(k\pi)$$

$$\boxed{\begin{aligned}\cos x &= J_0(x) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x)\\ \sin x &= 2\sum_{k=1}^{\infty}(-1)^{k+1} J_{2k-1}(x)\end{aligned}}$$

$$\int_0^{\pi} \cos(x\sin\theta)\, d\theta = \int_0^{\pi} \left[J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta) \right] d\theta$$

$$= J_0(x)\int_0^{\pi} d\theta + 2\sum_{k=1}^{\infty} J_{2k}(x) \int_0^{\pi} \cos(2k\theta)\, d\theta$$

$$= \pi J_0(x)$$

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta$$

rLA (12) $$\int_0^{\pi} \cos(2k\theta)\cos(n\theta)\, d\theta = \begin{cases} 0, & \text{if } n \neq 2k \text{ (odd)} \\ \pi/2, & n = 2k \text{ (even)} \end{cases}$$

$$\int_0^{\pi} \cos(n\theta)\, d\theta = 0 \quad , \ n \neq 0$$

$$\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = \int_0^{\pi}\left[J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x)\cos(2k\theta)\right]\cos n\theta \, d\theta$$

$$= \int_0^{\pi} J_0(x)\cos n\theta \, d\theta + 2\sum_{k=1}^{\infty} J_{2k}(x)\int_0^{\pi}\cos(2k\theta)\cos(n\theta)\,d\theta$$

$$= 0 + 2\sum_{k=1}^{\infty} J_{2k}(x) \times \begin{cases} 0 & \text{if } n \neq 2k \\ \pi/2 & \text{if } n = 2k \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \neq 2k \\ \pi J_n(x) & \end{cases}$$

~~Similarly~~,

$$\int_0^{\pi} \sin\left[(2k-1)\theta\right]\sin(n\theta)\,d\theta = \begin{cases} 0 & \text{if } n \neq 2k-1 \text{ (even)} \\ \pi/2 & \text{if } n = 2k-1 \text{ (odd)} \end{cases}$$

Similarly,

$$\int_0^{\pi} \sin(x\sin n\theta)\sin(n\theta)\,d\theta = \int_0^{\pi}\left[2\sum_{k=1}^{\infty} J_{2k-1}(x)\sin(2k-1)\theta\right]\sin(n\theta)\,d\theta$$

$$= 2\sum_{k=1}^{\infty} J_{2k-1}(x)\int_0^{\pi}\sin[(2k-1)\theta]\sin(n\theta)\,d\theta$$

$$= 2\sum_{k=1}^{\infty} J_{2k-1}(x)\cdot\begin{cases}0 & \text{if } n\neq 2k-1\\ \pi/2 & \text{if } n=2k-1\end{cases}$$

$$=\begin{cases}0 & \text{if } n\neq 2k-1 \text{ (even)}\\ \pi J_n(x) & \text{if } n=2k-1 \text{ (odd)}\end{cases}$$

Combining,

$$\int_0^{\pi}\left[\cos(x\sin\theta)\cos(n\theta)+\sin(x\sin\theta)\sin(n\theta)\right]d\theta = \pi J_n(x)$$

$$\int \cos(n\theta - x\sin\theta)\, d\theta = \pi J_n(x)$$

$\therefore$ For all +ve integral values of n, including zero,

$$\boxed{J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta, \quad n = 0, 1, 2, \ldots}$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{e^{i(n\theta - x\sin\theta)} + e^{-i(n\theta - x\sin\theta)}}{2}\, d\theta$$

$$\theta = -\phi \Rightarrow d\theta = -d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi} e^{i(n\theta - x\sin\theta)} - \frac{1}{2\pi}\int_0^{-\pi} e^{i(n\phi - x\sin\phi)}\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi} e^{i(n\theta - x\sin\theta)} + \frac{1}{2\pi}\int_0^{\pi} e^{i(n\theta - x\sin\theta)}\, d\theta$$

$$\boxed{J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n\theta - x\sin\theta)}\, d\theta}$$

Ex: Show that $\boxed{J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\, d\theta}$

Ans:

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} e^{ix\sin\theta}\, d\theta + \frac{1}{2\pi}\int_{-\pi/2}^{\pi} e^{ix\sin\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} e^{ix\cos\theta}\, d\theta + \frac{1}{2\pi}\int_{-\pi/2}^{\pi} e^{ix\cos\theta}\, d\theta$$

$\sin(-\pi - \frac{\pi}{2} - \theta) =$
$= +\sin(\theta + \pi/2) = \cos\theta$
$\sin(\frac{\pi}{2} - \theta) = \cos\theta$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} e^{ix\cos\theta}\, d\theta + \frac{1}{2\pi}\int_{0}^{\pi} e^{ix\cos\theta}\, d\theta$$

$$= \frac{1}{2\pi}\times 2\int_0^{\pi} e^{ix\cos\theta}\, d\theta = \cancel{\frac{1}{\pi}\int_0^{\pi} e^{ix\cos\theta}}\, d\theta$$

$\alpha = 2\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\, d\theta + \frac{1}{2\pi}\int_0^{\pi} e^{ix\cos(\pi-\theta)}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\, d\theta$$

Chebyshev Polynomials

A Chebyshev polynomial expansion is merely a Fourier cosine series in disguise.
Every theorem, every identity of Chebyshev polynomials has its Fourier counterpart.

ie; A Chebyshev series is just a Fourier cosine expansion with a change of variable.

The mapping is: $x = \cos(\theta)$

and then $T_n(x) = \cos(n\theta)$

The nth degree Chebyshev polynomial $T_n(x)$ converts to Fourier's,

$$\cos(n\theta) = T_n(\cos\theta)$$

The following 2 series are equivalent under the transformation:

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta)$$

ie;

the coeff. of $f(x)$ as a Chebyshev series are identical with the Fourier cosine coeff. of $f(\cos\theta)$.

Chebyshev to Fourier

$T_0(x) = 1$

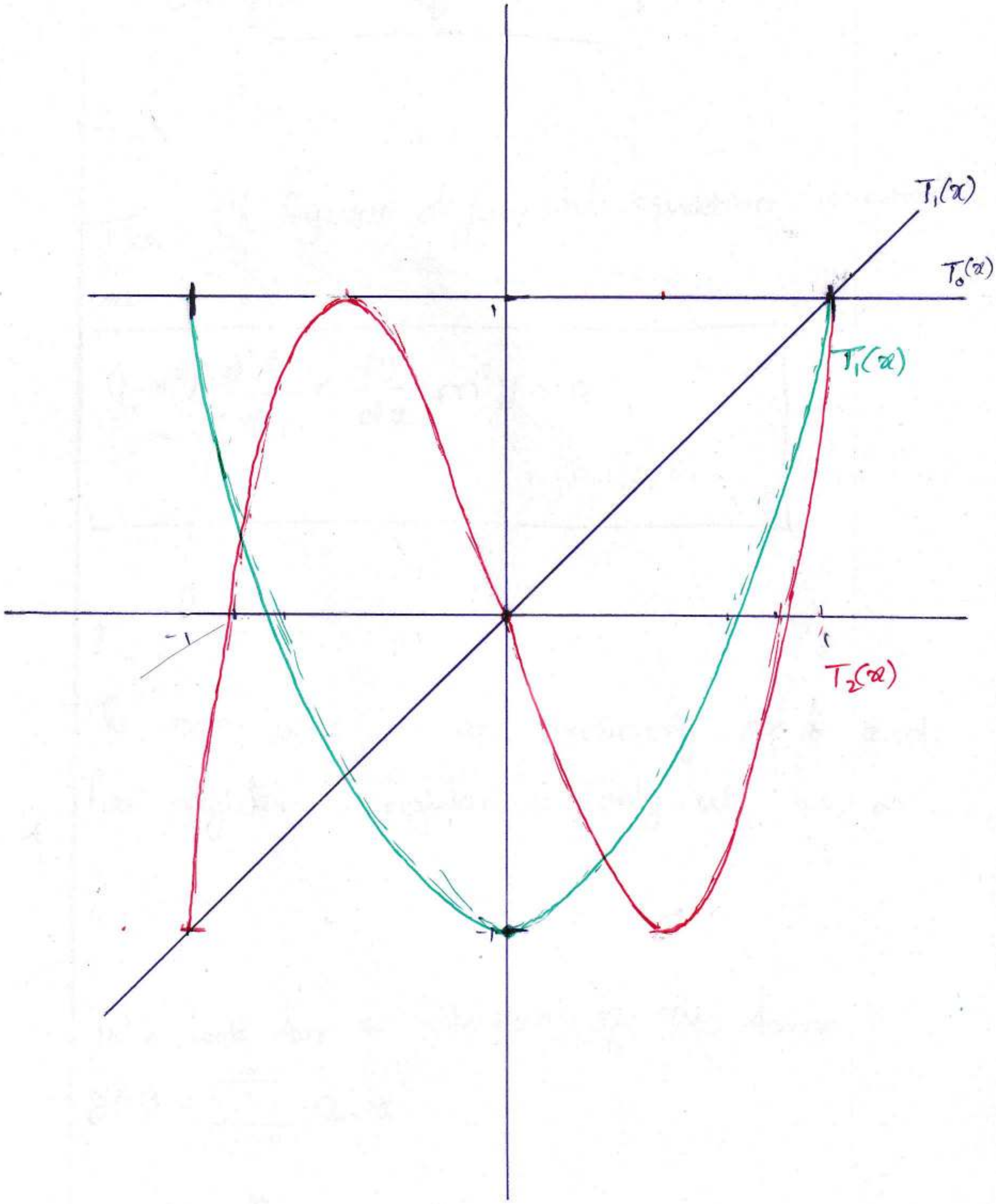
$T_1(x) = x = \cos\theta$

$T_2(x) = 2x^2 - 1 = 2\cos^2\theta - 1 = \cos 2\theta$

$T_3(x) = 4x^3 - 3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta$

$T_4(x) = 8x^4 - 8x^2 + 1 = 8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta$

$T_5(x) = 16x^5 - 20x^3 + 5x = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos 5\theta$



Chebyshev polynomials

Chebyshev differential equation

The Chebyshev differential equation is written as

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0, \quad n = 0, 1, 2, \cdots}$$

The point $x = 0$ is an ordinary point and has regular singularities only at $-1, 1, \infty$.

We look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - x\sum_{n=1}^{\infty} a_n n x^{n-1} + m^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\cancel{a_n n(n-1) x^{n-2} - a_n n(n-1) x^{n} - a_n n x^n} +$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=1}^{\infty} n a_n x^n$$

$$+ \sum_{n=0}^{\infty} m^2 a_n x^n = 0$$

$$\boxed{a_{n+2} = \frac{(n^2-m^2)\, a_n}{(n+2)(n+1)}}, \ \forall \ \cancel{n \in \mathbb{N}}$$

coeff. of x^0: $2a_2 + m^2 a_0 = 0 \implies \boxed{a_2 = \frac{-m^2 a_0}{2!}}$

coeff. of x^1: $3\cdot 2 a_3 - a_1 + m^2 a_1 = 0 \implies \boxed{a_3 = \frac{(1^2 - m^2)\, a_1}{3!}}$

coeff. x^2: $4\cdot 3 a_4 - 2a_2 - 2a_2 + m^2 a_2 = 0$

$4\cdot 3\, a_4 - (2^2 - m^2) a_2 = 0 \implies$

$$\boxed{a_4 = \frac{(2^2 - m^2) a_2}{4 * 3} = \frac{-m^2(2^2 - m^2) a_0}{4!}}$$

coeff. x^3: $5\cdot 4\, a_5 - 3\cdot 2\, a_3 - 3a_3 + m^2 a_3 = 0$

$$\boxed{a_5 = \frac{(3^2 - m^2) a_3}{5 * 4} = \frac{(1^2 - m^2)(3^2 - m^2)}{5!} a_1}$$

Coeff. of x^4: $6\cdot 5\, a_6 - 4\cdot 3\, a_4 - 4a_4 + m^2 a_4 = 0$

$$\boxed{a_6 = \frac{(4^2 - m^2) a_4}{6 * 5} = \frac{-m^2(2^2 - m^2)(4^2 - m^2) a_0}{6!}}$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where,

$$y_1(x) = 1 - \frac{m^2x^2}{2!} + \frac{m^2(2^2-m^2)}{4!}x^4 - \frac{m^2(2^2-m^2)(4^2-m^2)}{6!}x^6$$

$$+ \cdots + \frac{m^2(2^2-m^2)\cdots((2n-2)^2-m^2)}{(2n)!}x^{2n} + \cdots$$

$$y_2(x) = x + \frac{(1^2-m^2)}{3!}x^3 + \frac{(1^2-m^2)(3^2-m^2)}{5!}x^5 + \frac{(1^2-m^2)(3^2-m^2)(5^2-m^2)}{7!}x^7$$

$$+ \cdots + \frac{(1^2-m^2)(3^2-m^2)\cdots((2n-1)^2-m^2)}{(2n+1)!}x^{2m+1} + \cdots$$

The Wronskian of y_1 and y_2 at 0 is:

$$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\Longrightarrow$ y_1 and y_2 are independent

$y(x) = a_0 y_1(x) + a_1 y_2(x)$ is a general solution of the Chebyshev differential equation.

These series converge for $|x| < 1$.

When the parameter 'n' is a non-negative integer, ~~then the~~ then the RHS becomes zero in the recurrence relation for $m = n$. and there is a polynomial solution of degree n.

$$a_{n+2} = a_{n+4} = a_{n+6} = \cdots = 0$$

- $m=0: \; y(x) = y_0(x) = 1$

$m=1: \; y(x) = y_1(x) = x$

$m=2: \; y(x) = y_0(x) = 1-2x^2$

$m=3: \; y(x) = y_1(x) = x - \frac{4}{3}x^3$

(OR)

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

Put $x=\cos t \implies dx=-\sin t\,dt$

$$\frac{dt}{dx}=\frac{-1}{\sin t}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{-1}{\sin t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\times\frac{dt}{dx}\left(\frac{dy}{dx}\right)=\frac{-1}{\sin t}\times\frac{d}{dt}\left[\frac{-1}{\sin t}\frac{dy}{dt}\right]$$

$$=\frac{-1}{\sin t}\left[\frac{d}{dt}\left(\frac{-1}{\sin t}\right)\frac{dy}{dt}-\frac{1}{\sin t}\frac{d^2y}{dt^2}\right]$$

$$=\frac{1}{\sin t}\left[\frac{-\cos t}{\sin^2 t}\frac{dy}{dt}+\frac{1}{\sin t}\frac{d^2y}{dt^2}\right]$$

$$=\frac{1}{\sin^2 t}\left[\frac{-\cos t}{\sin t}\frac{dy}{dt}+\frac{d^2y}{dt^2}\right]$$

$$\cancel{(1-\cos^2 t)}\,\frac{1}{\cancel{\sin^2 t}}\left[\frac{-\cos t}{\sin t}\frac{dy}{dt}+\frac{d^2y}{dt^2}\right]-\cos t\left[\frac{-1}{\sin t}\frac{dy}{dt}\right]+n^2y=0$$

$$\cancel{\frac{-\cos t}{\sin t}\frac{dy}{dt}}+\frac{d^2y}{dt^2}+\cancel{\frac{\cos t}{\sin t}\frac{dy}{dt}}+n^2y=0$$

$$\boxed{\frac{d^2y}{dt^2}+n^2y=0}$$

$$y=e^{\lambda t},$$

$$\lambda^2+n^2=0 \implies \lambda=\pm in$$

$$y(t)=k_1e^{int}+k_2e^{-int}$$

$$=k_1(\cos nt+i\sin nt)+k_2(\cos nt-i\sin nt)$$

$$=(k_1+k_2)\cos(nt)+i(k_1-k_2)\sin(nt)$$

$$=C_1\cos(nt)+C_2\sin(nt)$$

$$=C_1T_n(x)+C_2U_n(x)$$

$$=C_1\cos\left[n\cos^{-1}x\right]+C_2\cos\left[n\cos^{-1}t\right]$$

where,

$$T_n(x) = \cos\left[n \cos^{-1} x\right]$$ is called the

Chebyshev polynomial of the 1st kind.

$$T_0(x) = \cos(0) = 1$$

$$T_1(x) = \cos\left[\cos^{-1} x\right] = x$$

$$T_2(x) = \cos\left[2\cos^{-1} x\right] = 2\cos^2\left[\cos^{-1} x\right] - 1 = 2x^2 - 1$$

Recurrence formula

Set $x = \cos t$,

$$T_n(x) = T_n(\cos t) = \cos\left[n \cos^{-1} x\right] = \cos\left[n \cos^{-1}(\cos t)\right]$$
$$= \underline{\underline{\cos nt}}$$

$$T_{n+1}(x) = T_{n+1}(\cos t)$$
$$= \cos\left[(n+1)t\right] = \cos(nt)\cos t - \sin(nt)\sin t$$

$$T_{n-1}(x) = T_{n-1}(\cos t)$$
$$= \cos\left[(n-1)t\right] = \cos(nt)\cos t + \sin(nt)\sin t$$

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(nt)\cos t = 2T_n(x)\, x$$

$$\boxed{T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)}$$

Generating function

The generating function for Chebyshev polynomials $T_n(x)$ is :

$$\boxed{g(x,z) = \frac{1-zx}{1-2zx+z^2} = \sum_{n=0}^{\infty} T_n(x) z^n}$$

Proof $\quad x = \cos t$

$$\sum_{n=0}^{\infty} T_n(x) z^n = \sum_{n=0}^{\infty} \cos(nt) z^n = 1 + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} z^{|n|} e^{int}$$

$$= 1 + \frac{1}{2}\left[\sum_{n=1}^{\infty} z^n e^{int} + \sum_{n=1}^{\infty} z^n e^{-int}\right]$$

$$= 1 + \frac{1}{2}\left[\frac{ze^{it}}{1-ze^{it}} + \frac{ze^{-it}}{1-ze^{-it}}\right]$$

$$= 1 + \frac{1}{2}\left[\frac{ze^{it} - z^2 + ze^{-it} - z^2}{1 - ze^{it} - ze^{-it} + z^2}\right]$$

$$= 1 + \frac{1}{2}\,\frac{2z\cos t - 2z^2}{1 - 2z\cos t + z^2}$$

$$= 1 + \frac{z\cos t - z^2}{1 - 2z\cos t + z^2}$$

$$\boxed{= 1 + \frac{xz - z^2}{1 - 2xz + z^2} = \cancel{1 - xz}}$$

$$\boxed{= \frac{1 - zx}{1 - 2zx + z^2}}$$

(OR)

$$\sum_{n=0}^{\infty} T_n(x) z^n = \sum_{n=0}^{\infty} \cos(nt) z^n$$

$$= \mathbb{R}\left(\sum_{n=0}^{\infty} e^{int} z^n\right)$$

$$= \mathbb{R}\left(\sum_{n=0}^{\infty} \left(e^{it} z\right)^n\right)$$

$$= \mathbb{R}\left(\frac{1}{1 - e^{it} z}\right)$$

$$= \mathbb{R}\left(\frac{1 - e^{-it} z}{(1 - e^{it} z)(1 - e^{-it} z)}\right)$$

$$= \mathbb{R}\left(\frac{1-e^{-it}z}{1-2\cos \cancel{t}\, z+z^{2}}\right)$$

$$= \frac{1-(\cos t)\,z}{1-2(\cos t)\,z+z^{2}} = \frac{1-zx}{1-2zx+z^{2}}$$

Rodrigues' formula

$$T_n(x) = (-1)^n \, 2^n \frac{n!}{(2n)!} \sqrt{1-x^2} \, \frac{d^n}{dx^n}\left(1-x^2\right)^{n-\frac{1}{2}}$$

$$= \frac{\Gamma(1/2)}{(-2)^n \, \Gamma\left(n+\frac{1}{2}\right)} \frac{d^n}{dx^n}\left(\left[1-x^2\right]^{n-\frac{1}{2}}\right)$$

2 ▫ Orthogonality

Fourier basis is orthogonal

$$\int_0^{\pi} \cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases} 0, & \text{for } m \neq n \\ \pi/2, & \text{for } m = n \end{cases}$$

Set $x = \cos\theta \implies dx = -\sin\theta\, d\theta$

$$d\theta = \frac{-1}{\sin\theta}dx = \frac{-dx}{\sqrt{1-x^2}}$$

$$\int_{1}^{-1} T_m(x)\, T_n(x) \frac{-dx}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{for } m \neq n \\ \pi/2 & \text{for } m = n \end{cases}$$

$$\boxed{\int_{-1}^{1} T_m(x)\, T_n(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n = 0 \\ \pi/2 & \text{for } m = n = 1, 2, \dots \end{cases}}$$

An arbitrary function $f(x)$ which is continuous and single valued, defined over the interval $-1 \leq x \leq 1$, can be expanded as a series of Chebyshev polynomials:

$$f(x) = A_0 T_0(x) + A_1 T_1(x) + A_2 T_2(x) + \cdots$$

$$= \sum_{n=0}^{\infty} A_n T_n(x)$$

where the coefficients A_n are given by

$$A_0 = \frac{1}{\pi} \int_{-1}^{+1} \frac{f(x)\,dx}{\sqrt{1-x^2}} \quad \text{and} \quad A_n = \frac{2}{\pi} \int_{-1}^{+1} \frac{f(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx$$

* A linear transformation or homomorphism $f: V \to V$ of a vector space V to itself, is called an endomorphism of V.

Isomorphism of vector spaces.

The idea of vector space can be extended to include objects that you would not initially consider to be ordinary vectors, ex: Matrix spaces.

Consider the set $M_{2\times3}(\mathbb{R})$ of 2 by 3 matrices with real entries. This set is closed under addition. When such a matrix is multiplied by a real scalar, the resulting matrix is in the set also.

$\therefore$ $M_{2\times3}(\mathbb{R})$ is closed under addition & scalar multiplication $\Rightarrow$ it is a real Euclidean vector space.

The objects in the space - the "vectors" - are now matrices.

Any 2 by 3 matrix is a unique linear combination of the following 6 matrices:

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $E_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

they span $M_{2\times 3}(\mathbb{R})$.

Given a 2×3 matrix, form a 6-vector by writing the entries in the 1st row of the matrix followed by entries in the 2nd row. Then, to every matrix in $M_{2\times3}(\mathbb{R})$ there corresponds a unique vector in $\mathbb{R}^6$, and vice versa. This one-to-one correspondence b/w $M_{2\times3}(\mathbb{R})$ and $\mathbb{R}^6$.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} (a, b, c, d, e, f) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$

is compatible with the vector space operations of addition & scalar multiplication.

$\phi(A+B) = \phi(A) + \phi(B)$

$\phi(kA) = k\,\phi(A)$

$\Longrightarrow$ The spaces $M_{2\times 3}(\mathbb{R})$ and $\mathbb{R}^6$ are structurally identical, ie., isomorphic, a fact which is denoted $M_{2\times 3}(\mathbb{R}) \cong \mathbb{R}^6$.

Each basis "vector" E_i given above for $M_{2\times 3}(\mathbb{R})$ corresponds to the standard basis vector e_i for $\mathbb{R}^6$.

* Two vector spaces V and W over the same field $\mathbb{F}$ are isomorphic if there is a bijection $T: V \longrightarrow W$ which preserves addition and scalar multiplication,

ie., for all vectors u and v in V, and all scalars $c \in \mathbb{F}$.

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v)$$

The correspondence T is called an isomorphism of vector space.

When $T: V \to W$ is an isomorphism we will write $T: V \xrightarrow{\cong} W$ if we want to emphasize that it is an isomorphism. When V and W are isomorphic, but the specific isomorphism is not named, we'll just write $V \cong W$.

The identity function $I_V: V \to V$ is an isomorphism.

Morphism — map

isomorphism — map expressing sameness

Ex:-

Space of two-wide row vectors and the space of two-tall column vectors.

$$(a_0 \;\; a_1) + (b_0 \;\; b_1) = (a_0+b_0 \;\; a_1+b_1) \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0+b_0 \\ a_1+b_1 \end{pmatrix}$$

$$r\cdot(a_0 \;\; a_1) = (ra_0 \;\; ra_1) \longleftrightarrow r\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} ra_0 \\ ra_1 \end{pmatrix}$$

Ex:-

P_2, the space of quadratic polynomials, and $\mathbb{R}^3$.

$$a_0 + a_1x + a_2x^2 \longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\begin{array}{r} a_0 + a_1x + a_2x^2 \\ + \; b_0 + b_1x + b_2x^2 \\ \hline (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 \end{array} \longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \end{bmatrix}$$

$$r(a_0 + a_1x + a_2x^2) = \longleftrightarrow r\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ra_0 \\ ra_1 \\ ra_2 \end{bmatrix}$$

$$= ra_0 + (ra_1)x + (ra_2)x^2$$

Ex:-

The vector space $G = \{c_1 \cos\theta + c_2 \sin\theta \mid c_1, c_2 \in \mathbb{R}\}$ of functions of θ is isomorphic to the vector space $\mathbb{R}^2$ under this map.

$$c_1 \cos\theta_1 + c_2 \cos\theta_2 \xrightarrow{f} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Condition 1:

$$f(a) = f(b) \implies f(a_1 \cos\theta + a_2 \sin\theta) = f(b_1 \cos\theta + b_2 \sin\theta)$$

then by the definition of f.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \implies a_1 = b_1 \text{ and } a_2 = b_2$$

$\therefore a = b$.

$\implies f$ is one-one

Any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, is the image under f

of $x\cos\theta + y\sin\theta \in G$.

$\Rightarrow f$ is onto

<u>condition 2:</u> f preserve structure?

$$f\left((a_1\cos\theta + a_2\sin\theta) + (b_1\cos\theta + b_2\sin\theta)\right) =$$

$$= f\left((a_1+b_1)\cos\theta + (a_2+b_2)\sin\theta\right)$$

$$= \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= f(a_1\cos\theta + a_2\sin\theta) + f(b_1\cos\theta + b_2\sin\theta)$$

$$f\left(r(a_1\cos\theta + a_2\sin\theta)\right) = f(ra_1\cos\theta + ra_2\sin\theta)$$

$$= \begin{bmatrix} ra_1 \\ ra_2 \end{bmatrix} = r\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= r\, f(a_1\cos\theta + a_2\sin\theta)$$

$\therefore$ <u>$G \cong \mathbb{R}^2$</u>

Ex:- Let V be the space $\{c_1 x + c_2 y + c_3 z \mid c_1, c_2, c_3 \in \mathbb{R}\}$ of linear combinations of 3 variables x, y, z under the natural addition and scalar multiplication operations. Then V is isomorphic to P_2, the space of quadratic polynomials.

More than one possibility:

$$c_1 x + c_2 y + c_3 z \xrightarrow{f_1} c_1 + c_2 x + c_3 x^2$$

$$c_1 x + c_2 y + c_3 z \xrightarrow{f_2} c_2 + c_3 x + c_1 x^2$$

$$c_1 x + c_2 y + c_3 z \xrightarrow{f_3} -c_1 + c_2 x - c_3 x^2$$

$$c_1 x + c_2 y + c_3 z \xrightarrow{f_4} c_1 + (c_1 + c_2) x + (c_1 + c_3) x^2$$

f_2 condition 1

$$f(c_1 x + c_2 y + c_3 z) = f(d_1 x + d_2 y + d_3 z)$$

$$\Rightarrow c_2 + c_3 x + c_1 x^2 = d_2 + d_3 x + d_1 x^2$$

$$\Rightarrow c_2 = d_2, \ c_3 = d_3, \ c_1 = d_1$$

$$\Rightarrow c_1 x + c_2 y + c_3 z = d_1 x + d_2 y + d_3 z$$

$\therefore$ f_2 is one-to-one

$\in \mathbb{R}\}$

y, z

ratic

any member $c_2 + c_3 x + c_1 x^2$ of the codomain is the image of some $c_1 x + c_2 y + c_3 z$.

Condition 2.

$$f_2\Big((c_1 x + c_2 y + c_3 z) + (d_1 x + d_2 y + d_3 z)\Big) = f_2\Big((c_1+d_1)x + (c_2+d_2)y + (c_3+d_3)z\Big)$$

$$= (c_2+d_2) + (c_3+d_3)x + (c_1+d_1)x^2$$

$$= (c_2 + c_3 x + c_1 x^2) + (d_2 + d_3 x + d_1 x^2)$$

$$= f(c_1 x + c_2 y + c_3 z) + f(d_1 x + d_2 y + d_3 z)$$

$$f_2\Big(r\,(c_1 x + c_2 y + c_3 z)\Big) = f_2(rc_1 x + rc_2 y + rc_3 z)$$

$$= rc_2 + rc_3 x + rc_1 x^2$$

$$= r\,(c_2 + c_3 x + c_1 x^2)$$

$$= r\, f_2(c_1 x + c_2 y + c_3 z)$$

$$\therefore \quad V \cong P_2$$

* If $T : V \xrightarrow{\simeq} W$ is an isomorphism of vector spaces, then its inverse $T^{-1} : W \xrightarrow{\simeq} V$ is also an isomorphism.

Proof

Since T is a bijection, T^{-1} exists as a function $W \to V$.

$$T^{-1}(w+x) = T^{-1}(w) + T^{-1}(x)$$

$$\Rightarrow w + x = T\left(T^{-1}(w) + T^{-1}(x)\right) = T\left(T^{-1}(w)\right) + T\left(T^{-1}(x)\right)$$

$$= w + x$$

true

$$T^{-1}(cw) = cT^{-1}(w)$$

$$cw = T\left(cT^{-1}(w)\right) = cT\left(T^{-1}(w)\right) = cw$$

true

$$\therefore \quad T^{-1} : W \xrightarrow{\simeq} V$$

* If $S : V \xrightarrow{\cong} W$ and $T : W \xrightarrow{\cong} X$ are both isomorphisms of vector spaces, then so is their composition, $T \circ S : V \xrightarrow{\cong} X$.

* If $T : V \longrightarrow W$ is an isomorphism, then T carries linearly independent sets to linearly independent sets, spanning sets to spanning sets, and bases to bases.

* Two finite dimensional vector spaces are isomorphic iff they have the same dimension.

* Isomorphism is an equivalence relation.

(1) Identity map $I_V : V \to V$ is an isomorphism.

$\therefore$ any vector is isomorphic to itself.

(2) If $f : V \to W$ is an isomorphism then so is its inverse $f^{-1} : W \to V$

$\therefore$ If V is isomorphic to W, then also W is isomorphic to V.

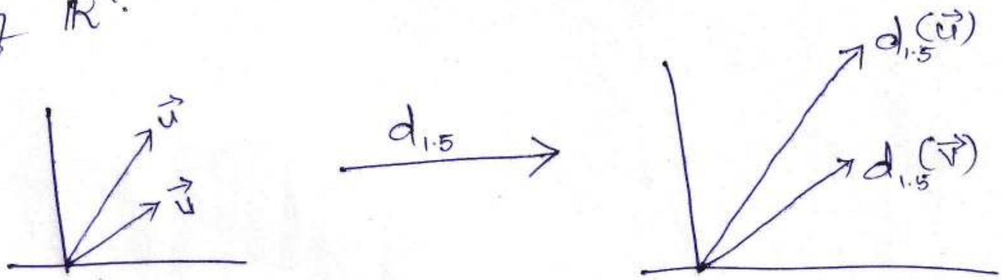
(3) If $f : V \to W$ is an isomorphism and $g : W \to U$ is an isomorphism, then so also $g \circ f : V \to U$.

$\therefore$ If V is isomorphic to W and W is isomorphic to U, then also V is isomorphic to U.

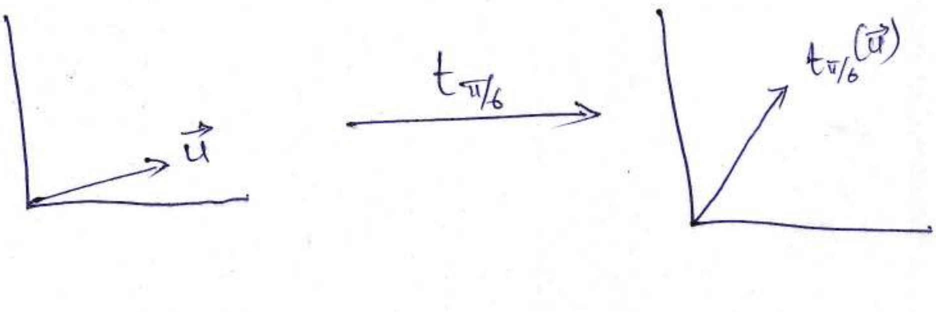
* An isomorphism of a vector space with itself, $f: A \xrightarrow{\cong} A$ is called an automorphism.

Ex: Identity map

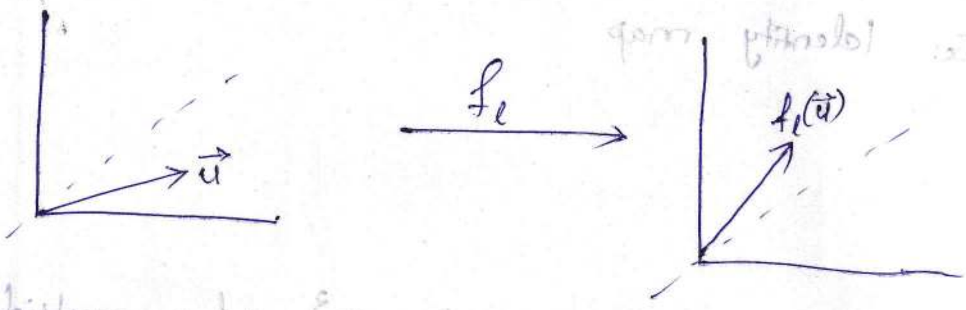
Ex:- A dilution map, $d_s: \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a non-zero scalar s is an automorphism of $\mathbb{R}^2$.



Ex:- A rotation or turning map $t_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors thro' an angle θ is an automorphism.



Ex:- Map $f_\ell : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that flips or reflects all vectors over a line ℓ thro' the origin is an automorphism of $\mathbb{R}^2$.



* A function b/w vector spaces $h: V \rightarrow W$ that preserves addition

if $\vec{v}_1, \vec{v}_2 \in V$ then $h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$

and scalar multiplication

if $\vec{v} \in V$ and $r \in \mathbb{R}$ then $h(r \cdot \vec{v}) = r \cdot h(\vec{v})$

is a homomorphism (or) linear map.

- Whereas, isomorphisms are bijections that preserve the algebraic structure, homomorphisms are simply functions that preserve the algebraic structure. In the case of vector spaces, the term linear transformation is used in preference to homomorphism.

Ex:- The projection map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\pi} \begin{bmatrix} x \\ y \end{bmatrix}$$ is a homomorphism.

$$\pi\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = \pi\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \pi\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + \pi\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

$$\pi\left(r\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = \pi\left(\begin{bmatrix} rx_1 \\ ry_1 \\ rz_1 \end{bmatrix}\right) = \begin{bmatrix} rx_1 \\ ry_1 \end{bmatrix}$$

$$= r\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = r\,\pi\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right)$$

This map is not ~~not~~ an isomorphism since it is not one-to-one.

Ex:- $f_1 : P_2 \longrightarrow P_3$ given by

$$a_0 + a_1 x + a_2 x^2 \longrightarrow a_0 x + \left(\frac{a_1}{2}\right) x^2 + \left(\frac{a_2}{3}\right) x^3$$

Ex:- $f_2 : M_{2\times 2} \longrightarrow \mathbb{R}$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow a + d$$

Ex:- B/w any 2 spaces there is a zero homomorphism mapping every vector in the domain to the zero vector in the codomain.

Ex:- The map $g : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{g} 3x + 2y - 4.5z$$ is linear.

ie., homomorphism

Ex:- The map $\hat{g} : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{\hat{g}} 3x + 2y - 4.5z + 1$$ is not linear.

$$\hat{g}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 4 \quad ; \quad \hat{g}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) + \hat{g}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 5$$

Ex:- The map $t_1 : \mathbb{R}^3 \to \mathbb{R}^2$ ~~is linea~~ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{t_1} \begin{bmatrix} 5x - 2y \\ x + y \end{bmatrix}$$ is linear

Ex:- The map $t_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{t_2} \begin{bmatrix} 5x - 2y \\ xy \end{bmatrix}$$ is not linear.

Ex:- The map $t_1 : \mathbb{R}^3 \to \mathbb{R}^2$ ~~is linear~~ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{t_1} \begin{bmatrix} 5x-2y \\ x+y \end{bmatrix}$$ is linear

Ex:- The map $t_2 : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{t_2} \begin{bmatrix} 5x-2y \\ xy \end{bmatrix}$$ is not linear.

* A linear transformation or homomorphism $f: V \to V$ of a vector space V to itself, is called an endomorphism of V.