Introduction to Linear Algebra
- Gilbert Strang

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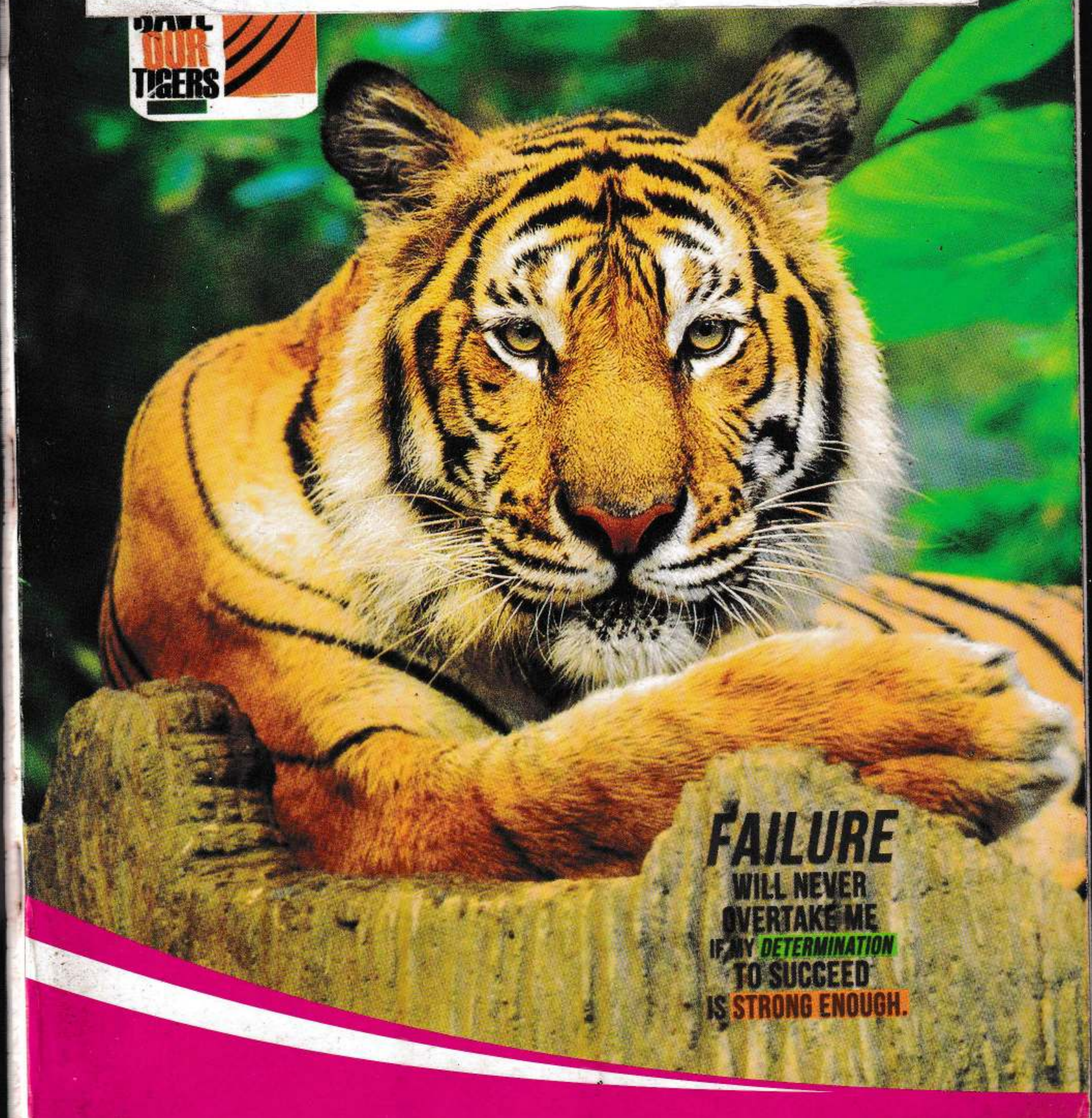
Linear Transformations
(Legendre and Hermite Polynomials)



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		LINEAR ALGEBRA		
		- Gilbert Strang, MIT		
		(5^{th} Edition)		

Legendre Polynomials

Any polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$ of degree n is a linear combination of $\{1, x, x^2, \ldots, x^n\}$ which form a basis of the space of all polynomials of degree not higher than n.

These basis polynomials can be orthogonalized by the Gram-Schmidt process, which converts a set of independent functions $q_0(x), q_1(x), \ldots, q_n(x)$ into a set of orthogonal functions $P_0(x), P_1(x), \ldots \ldots P_n(x)$:

$$\boxed{P_k(x) = q_k(x) - \sum_{i=1}^{k-1} \frac{\langle q_k(x), P_i(x)\rangle}{\langle P_i(x), P_i(x)\rangle} P_i(x)}$$

so that $\langle P_i(x), P_j(x)\rangle = 0$ for all $i \neq j$.

Here, $\langle P_i(x), P_j(x)\rangle$ is the inner product of $P_i(x)$ and $P_j(x)$ defined as:

$$\boxed{\langle P_i(x), P_j(x)\rangle = \int_{-1}^{1} P_i(x) P_j(x)\, dx}$$

Legendre Polynomials

Given $q_0(x) = 1,\ q_1(x) = x,\ q_2(x) = x^2,\ q_3(x) = x^3,$

$\ldots\ldots,\ q_n(x) = x^n.$

$P_0(x) = q_0(x) = 1$

$$P_1(x) = x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} = x$$

$$P_2(x) = x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^2\cdot x\,dx}{\int_{-1}^{1} x\cdot x\,dx}\,x$$

$$= x^2 - \frac{\frac{1}{3}\left[x^3\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}}x - 0 = x^2 - \frac{1}{3}\times\frac{2}{2}$$

$$= x^2 - \frac{1}{3} = \frac{1}{3}\left(3x^2 - 1\right)$$

$$P_3(x) = x^3 - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3\cdot x\,dx}{\int_{-1}^{1} x\cdot x\,dx}\,x - \frac{\int_{-1}^{1} x^3\left(x^2-\frac{1}{3}\right)dx}{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)\left(x^2-\frac{1}{3}\right)dx}\left(x^2-\frac{1}{3}\right)$$

$$= x^3 - 0 - \frac{\frac{1}{5}\left[x^5\right]_{-1}^{1}}{\frac{1}{3}\cdot\left[x^3\right]_{-1}^{1}}x - 0$$

$$= x^3 - \frac{3}{5}\frac{2}{2}x = x^3 - \frac{3}{5}x = \frac{1}{5}\left[5x^3 - 3x\right]$$

$$P_4(x) = x^4 - \frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^4\cdot x\,dx}{\int_{-1}^{1} x\cdot x\,dx}\,x - \frac{\int_{-1}^{1} x^4\left(x^2-\frac{1}{3}\right)dx}{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx}\left(x^2-\frac{1}{3}\right)$$

$$- \frac{\int_{-1}^{1} x^4\left(x^3-\frac{3}{5}x\right)dx}{\int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)^2 dx}\left(x^3-\frac{3}{5}x\right)$$

$$= x^4 - \frac{\frac{1}{5}\left[x^5\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}} - 0 - \frac{\frac{1}{7}\left[x^7\right]_{-1}^{1} - \frac{1}{3\cdot 5}\left[x^5\right]_{-1}^{1}}{\frac{1}{5}\left[x^5\right]_{-1}^{1} - \frac{2}{3\cdot 3}\left[x^3\right]_{-1}^{1} + \frac{1}{9}\left[x\right]_{-1}^{1}}\left(x^2-\frac{1}{3}\right)$$

$$- 0$$

$$= x^4 - \frac{1}{5} - \frac{\frac{2}{7} - \frac{2}{15}}{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}}\left(x^2-\frac{1}{3}\right) \Longrightarrow \frac{2}{5} - \frac{2}{9}$$

$$= x^4 - \frac{1}{5} - \frac{30-14}{105}\times\frac{45}{8}\left(x^2-\frac{1}{3}\right)$$

$$= x^4 - \frac{1}{5} - \frac{\cancel{16}^{\,8}}{\cancel{105}_{\,21\;7}}\times\frac{\cancel{45}^{\,\cancel{9}\;3}}{\cancel{8}}\left(x^2-\frac{1}{3}\right)$$

$$= x^4 - \frac{1}{5} - \frac{\cancel{16}^{\,2}}{\cancel{105}_{\,21\;7}}\times\frac{\cancel{45}}{\cancel{8}}\left(x^2-\frac{1}{3}\right)$$

$$= x^4 - \frac{1}{5} - \frac{6}{7}x^2 + \frac{2}{7} = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$$= \frac{1}{35}\left(35x^4 - 30x^2 + 3\right) \quad //$$

The orthogonal polynomials $P_0(x), P_1(x), \ldots, P_n(x)$ are the Legendre polynomials.

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$

$P_3(x) = x^3 - \frac{3}{5}x = \frac{1}{5}(5x^3 - 3x)$

$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35} = \frac{1}{35}(35x^4 - 30x^2 + 3)$

⋮

* $P_n(x)$ is an even function if n is even (or) an odd function if n is odd.

$P_n(-x) = (-1)^n P_n(x)$

Instead of normalization, the orthogonal Legendre polynomials are subject to standardization.

The Legendre polynomials are standardized by dividing by $P_n(1)$, so that $P_n(1) = 1$.

$$P_0(x) = 1$$

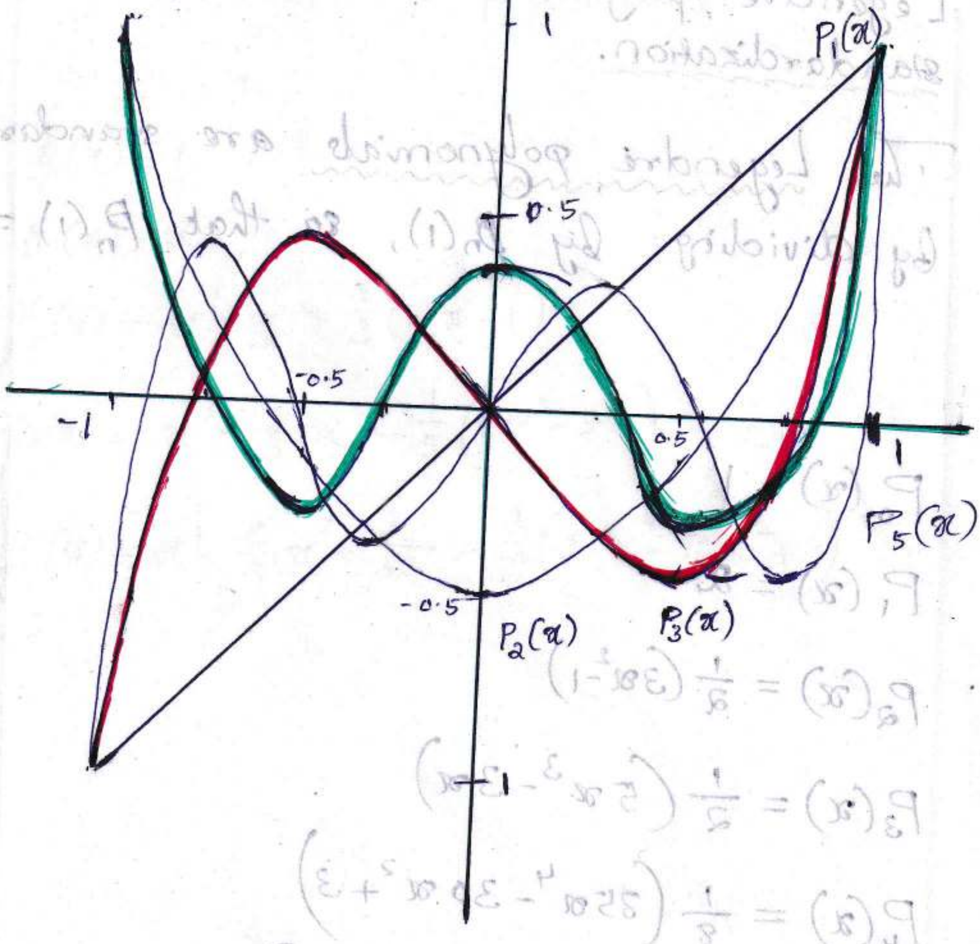
$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$



Legendre Polynomials

Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

n is constant

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

Assume that the differential equation has a solution, which can be written as the Maclaurin series :

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{y^{(m)}(0)}{m!} x^m$$

$$= \sum_{m=0}^{\infty} a_m x^m$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$+ k \sum_{m=0}^{\infty} a_m x^m = 0$$

$$k = n(n+1)$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m$$

$$- \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0$$

Put $s = m-2$, $\Longrightarrow$ $m = s+2$.

$$\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^{s} - \sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^{s+2}$$

$$-\sum_{s=-1}^{\infty}2(s+2)a_{s+2}x^{s+2} + \sum_{s=-2} k\,a_{s+2}x^{s+2} = 0$$

x^0 terms: $2a_2 + n(n+1)a_0 = 0 \longrightarrow \boxed{a_2 = \frac{-n(n+1)}{2}a_0}$

x^1 terms: $3.2\,a_3 + \left[-2 + n(n+1)\right]a_1 = 0$

$$\Longrightarrow \boxed{a_3 = \frac{-(n-1)(n+2)}{3.2}a_1}$$

x^s terms:

$$(s+2)(s+1)a_{s+2} + \left[-s(s-1) - 2s + n(n+1)\right]a_s = 0$$

$$\boxed{a_{s+2} = \frac{-(n-s)(n+s+1)}{(s+2)(s+1)}a_s}$$

$s = 0, 1, \ldots\ldots$

$\rightarrow$ recurrence relation (or) recursion formula

$$a_2 = \frac{-n(n+1)}{2!} a_0$$

$$a_4 = \frac{-(n-2)(n+3)}{4\cdot 3} a_2$$

$$= \frac{(n-2)\,n(n+1)(n+3)}{4!} a_0$$

$$a_3 = \frac{-(n-1)(n+2)}{3!} a_1$$

$$a_5 = \frac{-(n-3)(n+4)}{5\cdot 4} a_3$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

where,

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-1)n(n+1)(n+3)}{4!} x^4 - \cdots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots$$

These series converge for $|x| < 1$.

$y_1(x)$ contains even powers of x only, while $y_2(x)$ contains odd powers of x only.

$\Longrightarrow$ y_1 and y_2 are independent

$\therefore$ $y(x) = a_0 y_1(x) + a_1 y_2(x)$ is a general solution of the Legendre's differential equation on the interval $-1 < x < 1$.

The Wronskian of y_1 and y_2 at 0 is

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

ILA ⑦

$\Rightarrow$ y_1 and y_2 are independent

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* Let f & g be differentiable on $[a,b]$.

ⓐ If $W(f,g)(t_0) \neq 0$ for some $t_0 \in [a,b]$ then $f(t)$ and $g(t)$ are linearly independent on $[a,b]$

ⓑ If f and g are linearly dependent on $[a,b]$, then $W(f,g)(t) = 0$ for all $t \in [a,b]$

Polynomial solutions

The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions. For special functions arising as solutions of ODEs this happens quite frequently leading to various important families of polynomials.

For Legendre's equation this happens when the parameter 'n' is a non-negative integer, then the RHS becomes zero in the ~~recurrence~~ relation. ~~so~~ for $s = n$.

$\therefore$

$$a_{n+2} = 0, \quad a_{n+4} = 0, \quad a_{n+6} = 0, \ \ldots\ldots\ldots$$

Hence,

If n is even, $y_1(x)$ reduces to a polynomial of degree 'n'.

If n is odd, $y_2(x)$ reduces to a polynomial of degree n.

These polynomials, multiplied by some constants are called Legendre polynomials, $P_n(x)$.

stants

$$\boxed{a_{s+2} = \frac{-(n-s)(n+s+1)}{(s+2)(s+1)} a_s}$$

$$\Downarrow$$

$$\boxed{a_s = \frac{-(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2}}$$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(-1)^2\, n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} a_n$$

$$a_{n-6} = \frac{-(n-4)(n-5)}{6(2n-5)} a_{n-4} = \frac{(-1)^3 \cdot n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6\cdot (2n-1)(2n-3)(2n-5)} a_n$$

$$\vdots$$

$$a_{n-2m} = (-1)^m \frac{n(n-1)(n-2)\cdots(n-(2m-1))}{2\cdot 4\cdot 6\cdots 2m\cdot(2n-1)(2n-3)\cdots(2n-(2m-1))} a_n$$

This yields the polynomial solution:

$$y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + \cdots$$

$$= a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} - \cdots \right.$$

The Legendre polynomials $P_n(x)$ are defined by choosing

$$a_n = \frac{(2n-1)(2n-3) \cdots 5 \cdot 3 \cdot 1}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

such that, $P_n(1) = 1$

$$P_n(x) = \cancel{\frac{(2n)!}{2^n (n!)^2}} \left[\cancel{x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2}} + \cancel{\frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4}} - \cdots \right]$$

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2}\left[x^n - \sum_{l=1}^{M_n} \frac{(-1)^l n(n-1)(n-2)\cdots(n-2k+1)}{2^k l!\,(2n-1)(2n-3)\cdots(2n-2k+1)} x^{n-2k}\right]$$

$$\boxed{P_n(x) = \sum_{l=0}^{M_n} (-1)^l \frac{(2n-2l)!}{2^n \cdot l!\,(n-l)!\,(n-2l)!} x^{n-2l}}$$

is called the Legendre polynomial of degree n.

$$M_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$P_n(x) = \sum_{k=0}^{M_n} (-1)^k \frac{(2n-2k)!}{2^n\, k!\, (n-k)!\, (n-2k)!}\, x^{n-2k}$$

where $M_n = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

$$P_0(x) = 1$$

$$P_1(x) = \frac{2!}{2.}\, x^1 = x$$

$$P_2(x) = \frac{4!\,3}{4 \times 2! \times 2!}\, x^2 - \frac{2!}{4.}$$

$$= \frac{3}{2} x^2 - \frac{1}{2} = \frac{1}{2}\left(3x^2 - 1\right)$$

$$a_n = \frac{(2n-1)(2n-3)\cdots\cdots 5\cdot 3\cdot 1}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

Proof
10/12/2020.

Generating Function

Consider the function,

$$g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Expand $(1-2xt+t^2)^{-1/2} = (1-(2x-t)t)^{-1/2}$ in powers of t for $|t|<1$ using binomial expansion:

$$\cancel{\left[1-t(2x-t)\right]^{-1/2} = 1+\frac{1}{2}t(2x-t)+\dots}$$

For $|x|<1$,

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\frac{m(m-1)(m-2)}{3!}x^3+\dots$$

$m \notin W$,

m : -ve (or) fraction : infinite # of terms.

$$\left[1-t(2x-t)\right]^{-1/2} = 1+\left(\frac{-1}{2}\right)\left(-t(2x-t)\right)+\frac{1}{2!}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)t^2(2x-t)^2$$

$$+\frac{1}{3!}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)t^3(2x-t)^3+\cdots$$

$$=1+\frac{1}{2}t(2x-t)+\frac{1\times 3}{2\times 4}t^2(2x-t)^2$$

$$+\frac{1\times 3\times 5}{2\times 4\times 6}t^2(2x-t)^3+\cdots$$

$$=1+\frac{1}{2}t(2x-t)+\frac{3}{8}t^2(2x-t)^2+\frac{5}{16}t^3(2x-t)^3+\cdots$$

$$=1+xt-\frac{t^2}{2}+\frac{3}{8}t^2(4x^2+t^2-4xt)+$$

$$\frac{5}{16}t^3(8x^3-t^3-12x^2t+16xt^2)$$

$$=1+xt-\frac{t^2}{2}+\frac{3}{2}x^2t^2+\frac{3}{8}t^4-\frac{3}{2}xt^3$$

$$+\frac{5}{2}x^3t^3-\frac{5}{16}t^6-\frac{15}{4}t^4x^2+\cdots$$

$$=1+xt+\frac{1}{2}(3x^2-1)t^2+\frac{1}{2}(5x^3-3x)t^3+\cdots$$

Equating the coefficients of t^n for $n = 0, 1, 2,$ and 3. with $g(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$ we find the 1st few Legendre polynomials are given by :

$$P_0(x) = 1$$

$$P_1(x) = x$$

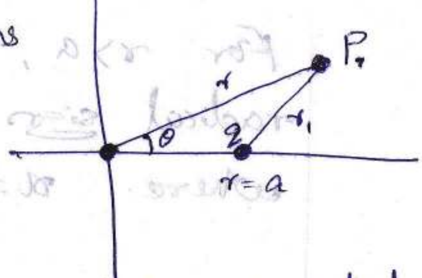
$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$\Rightarrow \boxed{g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \left(1-2xt+t^2\right)^{-1/2}}$$

is called the generating function for Legendre polynomials.

Ex:- The generating function is useful in solving physical problems involving the potential associated with any inverse square force.



* Electrostatic potential due to a charge 'q' displaced from origin.

Consider an electric charge q placed on the z-axis at $z = a$.

The electrostatic potential at a non-axial point due to this charge is given by

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_1}$$

$$r_1 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$$

[Law of cosines.

$$V = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$$

$$= \frac{q}{4\pi\varepsilon_0 r} \cdot \frac{1}{\sqrt{1 + \left(\frac{a}{r}\right)^2 + 2\left(\frac{a}{r}\right)\cos\theta}}$$

For $r > a$, the expression under the radical sign may be written as $(1-2xt+t^2)^{-1/2}$ where $x = \cos\theta$ and $t = \frac{a}{r}$; $|t| < 1$.

$$V = \frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n$$

Recurrence relations

The generating function for Legendre polynomials is:

$$g(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$\frac{\partial g}{\partial t} = \frac{-1}{2}\,\frac{-2x+2t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\boxed{\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)}$$

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$(x-t)\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$(1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x) + (t-x)\sum_{n=0}^{\infty} P_n(x)\, t^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)P_n(x)t^{n+1}-(2n+1)xP_n(x)t^n+nP_n(x)t^{n-1}\right]=0$$

To collect the coeff. of t^n, we replace n by $n-1$ in the 1st term and by $n+1$ in the last term. Then equating the resultant expression to zero, we get:

$$nP_{n-1}(x)-(2n+1)xP_n(x)+(n+1)P_{n+1}(x)=0$$

$$\boxed{(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)}$$

$n=1$,

$$3x\,P_1(x) = 2P_2(x) + P_0(x)$$

$$P_0(x)=1,\quad P_1(x)=x,\quad \cancel{P_2(x)=\tfrac{1}{2}(3x^2-1)}$$

$$3x^2 = 2P_2(x)+1 \Longrightarrow \underline{P_2(x)=\tfrac{1}{2}(3x^2+1)}$$

$$g(x,t) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$\frac{\partial g}{\partial x} = \frac{\left(\frac{-1}{2}\right)(-2t)}{\left(1-2xt+t^2\right)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$\frac{t}{\left(1-2xt+t^2\right)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$\frac{t}{\sqrt{1-2xt+t^2}} = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$t\sum_{n=0}^{\infty} P_n(x)\, t^n = \left(1-2xt+t^2\right)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$\sum_{n=0}^{\infty}\left[2x\,P_n'(x) + P_n(x)\right] t^{n+1} = \sum_{n=0}^{\infty}\left(1+t^2\right) P_n'(x)\, t^n$$

$$= \sum_{n=0}^{\infty} t^n P_n'(x) + \sum_{n=0}^{\infty} P_n'(x)\, t^{n+2}$$

Collecting coefficients of t^{n+1} from both sides and equating them.

$$\boxed{2x P_n'(x) + P_n(x) = P_{n+1}'(x) + P_{n-1}'(x)}$$

1^{st} recurrence relation,

$$(2n+1)x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

Differentiating w.r.t x and multiply by 2 the result

$$(2n+1) P_n(x) + (2n+1)x P_n'(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x)$$

$$2(2n+1) x P_n'(x) = 2(n+1) P_{n+1}'(x) + 2n P_{n-1}'(x) - 2(2n+1) P_n(x)$$

2^{nd} recurrence relation is multiplied by $2n+1$,

$$\cancel{2(2n+1)x P_n'(x) + P_n(x) = P_{n+1}'(x) + P_{n-1}'(x)}$$

$$2(2n+1)x P_n'(x) + (2n+1) P_n(x) = (2n+1) P_{n+1}'(x) + (2n+1) P_{n-1}'(x)$$

Substituting for $2(2n+1)x P_n'(x)$,

$$2(2n+1) \underline{\underline{P_{n+1}'(x)}} + 2n P_{n-1}'(x) - 2(2n+1) P_n(x) + 2(2n+1) P_n(x) =$$
$$= (2n+1) \underline{\underline{P_{n+1}'(x)}} + (2n+1) P_{n-1}'(x)$$

$$\boxed{P_{n+1}'(x) - P_{n-1}'(x) = (2n+1) P_n(x)}$$

Orthogonality Relations

Legendre polynomials are orthogonal. This enables us to express a given function defined on the interval $(-1,1)$ in a series of Legendre polynomials.

The m^{th} and n^{th} order Legendre polynomials $P_m(x)$ and $P_n(x)$ resp. satisfy the equations.

$$(1-x^2)P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0$$

$$(1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

Xing the 1st eqn. by $P_n(x)$ and the 2nd equation by $P_m(x)$, and subtract

$$(1-x^2)\left[P_n P_m'' - P_m P_n''\right] - 2x\left[P_n P_m' - P_m P_n'\right] = \left[n(n+1) - m(m+1)\right] P_m P_n$$

$$\frac{d}{dx}\left[(1-x^2)\left(P_n P_m' - P_m P_n'\right)\right] = \left[n(n+1) - m(m+1)\right] P_m P_n$$

$\int$ing over x from $x=-1$ to $x=+1$,

$$\left[n(n+1)-m(m+1)\right]\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = (1-x^2)\left[P_n P_m' - P_m P_n'\right]_{-1}^{+1}$$

$(1-x^2)$ vanishes for both the limits $(x=\pm 1)$

$\longrightarrow$ RHS $= 0$

$$\left[n(n+1)-m(m+1)\right]\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0$$

when $m \neq n$,

$$\boxed{\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0}$$

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$$\frac{1}{\sqrt{1-2sx+s^2}} = \sum_{m=0}^{\infty} P_m(x)\, s^m$$

Xing these equations & integrating w.r.t x, between $x=-1$ and $x=+1$,

$$\int_{-1}^{+1} \frac{dx}{\left(\sqrt{1-2tx+t^2}\right)\left(\sqrt{1-2sx+s^2}\right)} =$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \underbrace{\int_{-1}^{+1}\left\{P_m(x)\, P_n(x)\, dx\right\}}\, s^m t^n$$

RHS survives for terms with $m=n$ only.

$$\int_{-1}^{+1} \frac{dx}{1-2tx+t^2} = \sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} \left[P_n(x)\right]^2 dx \right\} t^{2n}$$

Substitute $1-2tx+t^2 = u \Longrightarrow -2t\,dx = du$

$$dx = -\frac{du}{2t}$$

$x=-1$: $u = 1+2t+t^2 = (1+t)^2$

$x=+1$: $u = 1-2t+t^2 = (1-t)^2$

$$I = \int_{-1}^{+1} \frac{dx}{1-2tx+t^2} = \frac{1}{2t} \int_{(1-t)^2}^{(1+t)^2} \frac{du}{u}$$

$$= \frac{1}{2t} \left[\log|u| \right]_{(1-t)^2}^{(1+t)^2}$$

$$= \frac{1}{2t} \ln \left(\frac{1+t}{1-t}\right)^2 = \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right)$$

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

$$\log(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots$$

$$\frac{1}{t}\log\left(\frac{1+t}{1-t}\right) = \frac{2}{t}\left[t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right]$$

$$= 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots\right]$$

$$= 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}$$

$$\therefore \sum_{n=0}^{\infty}\left\{\int_{-1}^{+1}\left[P_n(x)\right]^2 dx\right\} t^{2n} = \sum_{n=0}^{\infty}\left(\frac{2}{2n+1}\right) t^{2n}$$

Equating the coeffs of t^{2n},

$$\boxed{\int_{-1}^{+1}\left[P_n(x)\right]^2 dx = \frac{2}{2n+1}}$$

$$\boxed{\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1}\, \delta_{nm}}$$

where,

the Kronecker's delta,

$$\delta_{nm} = \begin{cases} 0 & \text{when } n \neq m \\ 1 & \text{when } n = m \end{cases}$$

$\Rightarrow$ Legendre polynomials of different orders are orthogonal.

Completeness of Legendre polynomials

We define a set of orthogonal functions is complete if there is no other function orthogonal to all of them.

We'll expand a function in a series of Legendre polynomials, which form a complete orthogonal set on $(-1,1)$. This completeness means that any well behaved function $f(x)$ can be approximated to any desired accuracy by a series of $P_k(x)$ through the relation:

$$\boxed{f(x) = \sum_{k=0}^{\infty} A_k \, P_k(x) \quad , -1 \le x \le 1}$$

$$\int_{-1}^{+1} P_m(x)\, f(x)\, dx = \sum_{k=0}^{\infty} A_k \underbrace{\int_{-1}^{+1} P_m(x)\, P_k(x)\, dx}_{\frac{2}{2m+1}\delta_{km}}$$

$$= \frac{2}{2m+1} \sum_{k=0}^{\infty} A_k\, \delta_{km}$$

$$= \frac{2A_m}{2m+1}$$

$$\boxed{A_m = \frac{2m+1}{2} \int_{-1}^{+1} P_m(x)\, f(x)\, dx}$$

(or)

$$A_k = \frac{2k+1}{2} \int_{-1}^{+1} P_k(x)\, f(x)\, dx$$

Ex.

Expand the function, $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases}$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$

Ans:

$$A_k = \frac{2k+1}{2} \int_{-1}^{+1} P_k(x) f(x)\, dx$$

$$= \frac{2k+1}{2} \left[\int_{-1}^{0} P_k(x) f(x)\, dx + \int_{0}^{1} P_k(x) f(x)\, dx \right]$$

$$= \frac{2k+1}{2} \int_{0}^{1} P_k(x)\, dx$$

$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$A_0 = \frac{1}{2} \int_0^1 P_0(x)\, dx = \frac{1}{2} \int_0^1 dx = \frac{1}{2}$$

$$A_1 = \frac{3}{2} \int_0^1 P_1(x)\, dx = \frac{3}{2} \int_0^1 x\, dx = \frac{3}{4}$$

$$A_2 = \frac{5}{2} \int_0^1 P_2(x)\, dx = \frac{5}{2} \int_0^1 \frac{1}{2}(3x^2 - 1)\, dx = \frac{5}{4} \left[x^3 - x \right]_0^1 = 0$$

$$A_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{4}\int_0^1 (5x^3 - 3x)\,dx$$

$$= \frac{7}{4}\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]_0^1 = \frac{7}{4}\left[\frac{5}{4} - \frac{3}{2}\right]$$

$$= \frac{7}{4} \times \frac{-1}{4} = -\frac{7}{16}$$

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots\cdots$$

Rodrigues' formula

Consider the function,

$$v = (x^2-1)^n$$

$$\frac{dv}{dx} = n(x^2-1)^{n-1} * 2x = 2nx(x^2-1)^{n-1}$$

$$(x^2-1)\frac{dv}{dx} = 2nx(x^2-1)^n = 2nxv$$

ie; v satisfies the differential equation

$$(1-x^2)\frac{dv}{dx} + 2nxv = 0$$

Differentiating again w.r.t x,

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + 2nx\frac{dv}{dx} + 2nv = 0$$

$$(1-x^2)\frac{d^2v}{dx^2} + 2(n-1)\frac{dv}{dx} + 2nv = 0$$

Differentiating again w.r.t x,

$$(1-x^2)\frac{d^{2+1}v}{dx^{2+1}} - 2x\frac{d^{1+1}v}{dx^{1+1}} + 2(n-1)\frac{dv}{dx} + 2(n-1)x\frac{d^{1+1}v}{dx^{1+1}}$$

$$+ 2n\frac{dv}{dx} = 0$$

$$(1-x^2)\frac{d^{2+1}v}{dx^{2+1}} + 2x(n-1-1)\frac{d^{1+1}v}{dx^{1+1}} + (1+1)(2n-1)\frac{dv}{dx} = 0$$

$\vdots$

$$(1-x^2)\frac{d^{2+r}v}{dx^{2+r}} + 2x(n-r-1)\frac{d^{1+r}v}{dx^{1+r}} + (r+1)(2n-r)\frac{d^{r}v}{dx^{r}} = 0$$

When $r=n$,

$$(1-x^2)\frac{d^{n+2}v}{dx^{n+2}} - 2x\frac{d^{n+1}v}{dx^{n+1}} + n(n+1)\frac{d^{n}v}{dx^{n}} = 0$$

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^n v}{dx^n}\right) - 2x\frac{d}{dx}\left(\frac{d^n v}{dx^n}\right) + n(n+1)\frac{d^n v}{dx^n} = 0$$

Comparing it with Legendre's differential equation :

$$(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$\Rightarrow \frac{d^n v}{dx^n}$ satisfies the Legendres' equation

$\therefore$

$$P_n(x) = C\frac{d^n v}{dx^n} \quad , \quad C : \text{constant}$$

$$\boxed{P_n(x) = C\frac{d^n}{dx^n}(x^2-1)^n} \Longleftarrow$$

To determine this constant C, we have to consider terms with the highest power of x on both sides.

The term with the highest power of x in the expression for $P_n(x)$ is $\frac{(2n)!}{2^n (n!)^2} x^n$.

$$\frac{(2n)!}{2^n (n!)^2} x^n = C \cdot \frac{d^n}{dx^n} x^{2n}$$

$$= C \cdot 2n(2n-1)(2n-2) \cdots [2n-(n-1)] x^n$$

$$= C \cdot \frac{(2n)!}{n!} x^n$$

$$\Longrightarrow \quad C = \frac{1}{2^n \cdot n!}$$

Hence,

$$\boxed{P_n(x) = \frac{1}{(2^n)\,n!}\frac{d^n}{dx^n}(x^2-1)^n}$$

→ Rodrigues' formula for $P_n(x)$

Ex:- Obtain the value of $P_3(x)$ using Rodrigues' formula

Ans:
$$P_n(x) = \frac{1}{(2^n)\, n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$P_3(x) = \frac{1}{(2^3)\, 3!} \frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3}\left(x^6 - 3x^4 + 3x^2 - 1\right)$$

$$= \frac{1}{48} \frac{d^2}{dx^2}\left(6x^5 - 12x^3 + 6x\right)$$

$$= \frac{1}{48} \frac{d}{dx}\left(30x^4 - 36x^2 + 6\right)$$

$$= \frac{1}{48}\left(120x^3 - 72x\right)$$

$$= \frac{1}{48} \times 24\left[5x^3 - 3x\right] = \frac{1}{2}\left(5x^3 - 3x\right)$$

Hermite Polynomials

Hermite's differential equation,

$$\boxed{\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\nu\, y(x) = 0}$$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j\, a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1)\, a_j x^{j-2}$$

$$\sum_{j=2}^{\infty} j(j-1)\, a_j\, x^{j-2} - 2x \sum_{j=1} j\, a_j\, x^{j-1}$$

$$+ 2\nu \sum_{j=0}^{\infty} a_j\, x^j = 0$$

$$\sum_{j=2}^{\infty} j(j-1)\, a_j\, x^{j-2} - \sum_{j=1}^{\infty} 2j\, a_j\, x^j$$

$$+ \sum_{j=0}^{\infty} 2\nu\, a_j\, x^j = 0$$

$$\boxed{\sum_{j=2}^{\infty} j(j-1)\, a_j\, x^{j-2} + \sum_{j=1}^{\infty} 2(\nu - j)\, a_j\, x^j + 2\nu\, a_0\, x^0 = 0}$$

x^0 terms : $2a_2 + 2\nu a_0 = 0$

$$\boxed{a_2 = \frac{-2\nu}{2\times 1} a_0}$$

x^1 terms : $\cancel{3\times 2\, a_3 - 2a_1 x} +$

$$3\times 2\, a_3 + 2(\nu - 1)\, a_1 = 0$$

$$\boxed{a_3 = \frac{-2(\nu-1)}{3\times 2} a_1}$$

x^2 terms: $4 \times 3\, a_4 + 2(\nu - 2)\, a_2 = 0$

$$\boxed{a_4 = \frac{-2(\nu-2)}{4 \times 3}\, a_2 = \frac{(-2)^2\, \nu(\nu-2)}{4!}\, a_0}$$

x^3 terms: $5 \times 4\, a_5 + 2(\nu - 3)\, a_3 = 0$

$$\boxed{a_5 = \frac{-2(\nu-3)}{5 \times 4}\, a_3 = \frac{(-2)^2\, (\nu-1)(\nu-3)}{5!}\, a_1}$$

x^4 terms: $6 \times 5\, a_6 + 2(\nu - 4)\, a_4 = 0$

$$\boxed{a_6 = \frac{-2(\nu-4)}{6 \times 5}\, a_4 = \frac{(-2)^3\, \nu(\nu-2)(\nu-4)}{6!}\, a_0}$$

with $$\boxed{a_{j+2} = \frac{-2(\nu - j)}{(j+1)(j+2)}\, a_j}$$

The general solution of Hermite's differential equation is :

$$y(x) = \sum_{j=0}^{\infty} a_j x^j$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

where,

$$y_1(x) = 1 + \frac{(-2)\nu}{2!} x^2 + \frac{(-2)^2 \nu(\nu-2)}{4!} x^4 + \frac{(-2)^3 \nu(\nu-2)(\nu-4)}{6!} x^6 + \cdots$$

$$y_2(x) = x + \frac{(-2)(\nu+1)}{3!} x^3 + \frac{(-2)^2(\nu-1)(\nu-3)}{5!} x^5 + \cdots$$

If ν is a non-zero negative integer $y(x)$ series will be an infinite series

The Wronskian of y_1 and y_2 at 0 is :

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\Longrightarrow$ y_1 and y_2 are independent

When $\nu = 2l$, $l = 0, 1, 2, \dots$

then the series corresp to $y_1(x)$ will terminate at the term:

$$\frac{(-2)^l (2l)(2l-2)\cdots(2l-(2l-2))}{(2l)!} x^{2l}$$

$$= \left[\frac{(-2)^l (2l)(2l-2)\cdots\cdots 2}{(2l)!}\right] \times x^{2l}$$

$$= \frac{(-1)^l \, l!}{(2l)!} (2x)^{2l}$$

all subsequent terms will have their numerators equal to zero.

The series corresp. to $y_2(x)$ will however remain an infinite series.

For the series to appear more systematic, we choose

$$a_0 = \frac{(-1)^l (2l)!}{l!}$$

The polynomial thus obtained is known as the Hermite polynomial of degree $2l$, $H_{2l}(x)$.

$$H_{2l}(x) = a_0 \left[1 + \frac{(-2)(2l)}{2!} x^2 + \frac{(-2)^2 (2l)(2l-2)}{4!} x^4 + \frac{(-2)^3 (2l)(2l-2)(2l-4)}{6!} x^6 \right.$$
$$+ \cdots\cdots + \frac{(-2)^{l-1} (2l)(2l-2)\cdots 4}{(2l-2)!} x^{2l-2}$$
$$\left. + \frac{(-2)^l (2l)(2l-2)\cdots 2}{(2l)!} x^{2l} \right]$$

$$= \frac{(-1)^l (2l)!}{l!}\left[1 + \frac{(-1)(2l)}{2!}(2x)^2 + \frac{(-1)^2 l(l-1)}{4!}(2x)^4 + \frac{(-1)^3 l(l-1)(l-2)}{6!}(2x)^6 + \cdots + \frac{(-1)^{l-1} l!}{(2l-2)!}\times(2x)^{2l-2} + \frac{(-1)^l l!}{(2l)!}(2x)^{2l}\right]$$

$$H_{2l}(x) = \frac{(-1)^l (2l)!}{l!} + \frac{(-1)^{l-1}(2l)!}{2!\,(l-1)!} + \cdots \cdots + \frac{(-1)(2l)!}{(2l-2)!}(2x)^{2l-2} + (2x)^{2l}$$

ic,

as

$H_{2l}(x)$.

$\ldots(2l-4)x^6$

x^{2l}

When ν is a +ve odd integer.

$\nu = 2l+1 \; ; \; l = 0, 1, 2, \ldots$

then the series corresp. to $y_2(x)$ will terminate at the term

$$\frac{(-2)^{l}(2l)(2l-2)\cdots(2l-(2l-1))\, x^{2l+1}}{(2l+1)!}$$

$$\frac{(-1)^{l} \cdot 2^{l} \cdot 2^{l} \cdot l! \times x^{2l+1}}{(2l+1)!}$$

$$\frac{(-1)^{l} \; l!}{(2l+1)!} \times \frac{(2x)^{2l+1}}{2}$$

Choosing, $a_1 = 2\dfrac{(-1)^l\,(2l+1)!}{l!}$

This leads to Hermite's polynomial of degree $(2l+1)$:

$$H_{2l+1}(x) = a_1\Big[x + \frac{(-2)(2l)}{3!}x^3 + \frac{(-2)^2(2l)(2l-2)}{5!}x^5 + \frac{(-2)^3(2l)(2l-2)(2l-4)}{7!}x^7 + \cdots + \frac{(-2)^{l-1}(2l)(2l-2)\cdots 4}{(2l-1)!}x^{2l-1} + \frac{(-2)^l(2l)(2l-2)\cdots 2}{(2l+1)!}x^{2l+1}$$

$$= 2\frac{(-1)^l(2l+1)!}{l!}\Big[x + \frac{(-2)(2l)}{3!}x^3 + \frac{(-2)^2(2l)(2l-2)}{5!}x^5 + \frac{(-2)^3(2l)(2l-2)(2l-4)}{7!}x^7 + \cdots + \frac{(-2)^{l-1}(2l)(2l-2)\cdots 4}{(2l-1)!}x^{2l-1} + \frac{(-2)^l(2l)(2l-2)\cdots 2}{(2l+1)!}x^{2l+1}\Big]$$

$$H_{2l+1}(x) = \frac{(-1)^{l}(2l+1)!}{l!}\,2x + \frac{(-1)^{l-1}(2l+1)!}{3!\,(l-1)!}(2x)^3 + \cdots$$

$$+ \frac{(-1)\,(2l+1)!}{(2l-1)!}(2x)^{2l-1} + (2x)^{2l+1}$$

* The constants a_0 and a_1 are taken in this particular manner, because Hermite polynomials of degree n is defined in such a way that the term containing the highest power of x is $(2x)^n$.

Taking $\nu = n$, where n is any +ve integer including zero, and define the Hermite polynomial of degree n as:

n is even

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$+ \cdots\cdots + \frac{(-1)^{n/2}\, n!}{(n/2)!}$$

n is odd

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$+ \cdots\cdots + \frac{(-1)^{\frac{n-1}{2}}\, n!}{\left(\frac{n-1}{2}\right)!}(2x)$$

First few Hermite polynomials are:

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

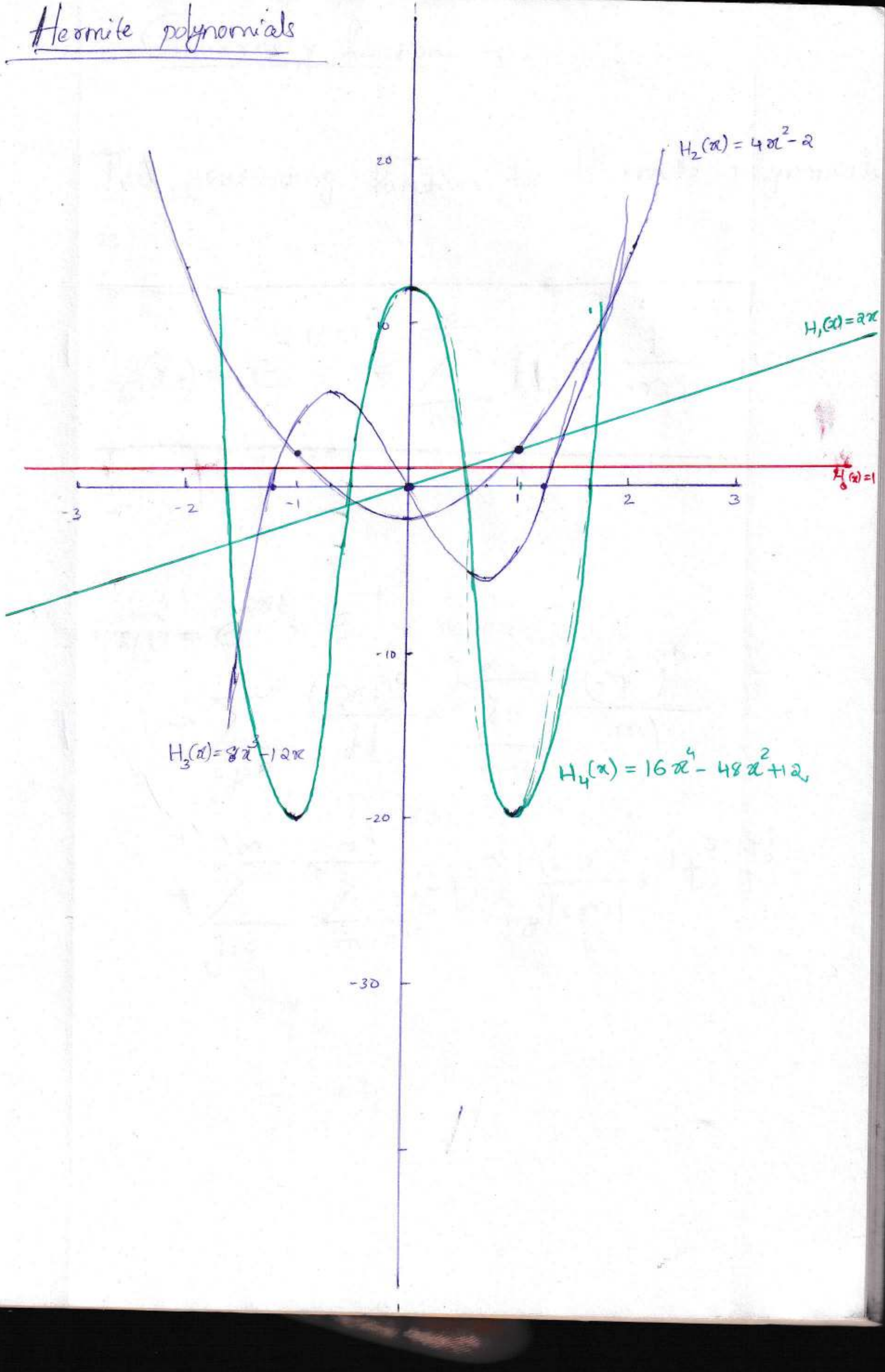
$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

Hermite polynomials



Generating function

The generating function for Hermite polynomials is :

$$\boxed{g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}}$$

$$g(x,t) = e^{2xt} * e^{-t^2}$$

$$= \sum_{j=0}^{\infty} \frac{(2xt)^j}{j!} \times \sum_{m=0}^{\infty} \frac{(-t^2)^m}{m!}$$

$$= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(2x)^j}{j! * m!} \times t^{2m+j}$$

Put $j+2m=n$,

this equality can be satisfied for various combinations of the values of j and m.

$j=n,\ m=0$;

$j=n-2,\ m=1$;

$j=n-4,\ m=2$; and so on.

If n is even,

$j=0,\ m=\frac{n}{2}$

If n is odd,

$j=1,\ m=\frac{n-1}{2}$

ious

The coeff. of t^n is :

$$\frac{(2x)^n}{n! \times 0!} - \frac{(2x)^{n-2}}{(n-2)! \times 1!} + \frac{(2x)^{n-4}}{(n-4)! \times 2!} - \cdots$$

$$\cdots \begin{cases} \dfrac{(-1)^{n/2}}{0!\left(\frac{n}{2}\right)!} & \text{for even } n \\[2ex] \dfrac{(-1)^{\frac{n-1}{2}}}{1! \times \left(\frac{n-1}{2}\right)!}(2x) & \text{for odd } n \end{cases}$$

$\therefore$

Coeff. of $\frac{t^n}{n!}$, i.e., $H_n(x)$ will be obtained by multiplying the above expression by $n!$.

Ex:- $H_n(x)$ and $H_n(-x)$

$\cancel{e^{2xt-t^2} =}$

We'll change x to $-x$, and t to $-t$.

$$e^{2xt-t^2} = \sum_n H_n(-x)\frac{(-t)^n}{n!}$$

$$= \sum_n (-1)^n H_n(-x)\frac{t^n}{n!}$$

$$e^{2xt-t^2} = \sum_n H_n(x)\frac{t^n}{n!}$$

$$\Longrightarrow \boxed{\begin{array}{c} H_n(x) = (-1)^n H_n(-x) \\ H_n(-x) = (-1)^n H_n(x) \end{array}}$$

Recurrence Relation

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

Differentiating w.r.t t,

$$(2x-2t)\, e^{2xt-t^2} = \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!}$$

$$(2x-2t) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!}$$

Equating coeff. of t^{n+1} from the 2 sides,

$$2x \frac{H_{n+1}(x)}{(n+1)!} - 2 \frac{H_n(x)}{n!} = \frac{H_{n+2}(x)}{(n+1)!}$$

$$\boxed{H_{n+2}(x) = 2x H_{n+1}(x) - 2(n+1) H_n(x)} \quad \text{— (r1)}$$

Ex:-

$H_0(x) = 1$ & $H_1(x) = 2x$

Ans: $H_{n+2}(x) = 2x\, H_{n+1}(x) - 2(n+1) H_n(x)$

$n=0$,

$$H_2(x) = 2x\, H_1(x) - 2\, H_0(x)$$

$$= 2x(2x) - 2 = 4x^2 - 2$$

$$H_3(x) = 2x\, H_2(x) - 4H_1(x) = 2x(4x^2-2) - 4(2x)$$

$$= 8x^3 - 12x$$

$$H_4(x) = 2x\, H_3(x) - 6H_2(x) = 16x^4 - 48x^2 + 12$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

Partially differentiating w.r.t x,

$$2t\, e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

Equating the coeff. of t^{n+1},

$$\frac{2 H_n(x)}{n!} = \frac{H_{n+1}'(x)}{(n+1)!}$$

$$2 H_n(x) = \frac{H_{n+1}'(x)}{n+1} \Longrightarrow$$

$$\boxed{H_{n+1}'(x) = 2(n+1) H_n(x)} \quad \text{——— } (r2)$$

Combining eqⁿs.

$$H_{n+2}(x) = 2x\,H_{n+1}(x) - 2(n+1)\,H_n(x)$$

$$H'_{n+1}(x) = 2(n+1)\,H_n(x)$$

$$\boxed{H_{n+2}(x) = 2x\,H_{n+1}(x) - H'_{n+1}(x)}$$

$n+1 \longrightarrow n$

$$\boxed{H_{n+1}(x) = 2x\,H_n(x) - H'_n(x)}$$

Differentiating w.r.t x,

$$H'_{n+1}(x) = 2H_n(x) + 2x\,H'_n(x) - H''_n(x) = 2(n+1)\,H_n(x)$$

$$\boxed{H''_n(x) - 2x\,H'_n(x) + 2n\,H_n(x) = 0}$$

It signifies ~~compare with~~ Hermite's differential equation (for +ve integral or 0 values of n) for $H_n(x)$.

It means that if the polynomials $H_n(x)$ satisfying the recurrence relations (r1) and (r2) they must satisfy Hermite's differential equation.

Rodrigues' formula

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$= 1 + tH_1(x) + \frac{t^2}{2!}H_2(x) + \cdots + \frac{t^n}{n!}H_n(x) + \cdots$$

Partial differentiation w.r.t 't' yields,

$$\frac{\partial}{\partial t}\left(e^{2xt-t^2}\right) = H_1(x) + \frac{2t}{2!}H_2(x) + \cdots + \frac{nt^{n-1}}{n!}H_n(x) + \cdots$$

$$\frac{\partial^2}{\partial t^2}\left(e^{2xt-t^2}\right) = H_2(x) + \frac{3\times 2}{3!}H_3(x) + \cdots + \frac{n(n-1)t^{n-2}}{n!}H_n(x) + \cdots$$

$$\vdots$$

$$\frac{\partial^n}{\partial t^n}\left(e^{2xt-t^2}\right) = H_n(x) + \frac{(n+1)n(n-1)\cdots 2}{(n+1)!}tH_{n+1}(x) + \cdots$$

At $t=0$,

$$H_n(x) = \left[\frac{\partial^n}{\partial t^n}\left(e^{2xt-t^2}\right)\right]_{t=0}$$

$$\cancel{e^{2xt-t^2}}\;\cancel{e^{2xt}\times e^{-t^2}}, \quad e^{2xt-t^2} = e^{x^2} \times e^{-(x-t)^2}$$

$$H_n(x) = e^{x^2}\left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0}$$

$e^{-(x-t)^2}$ is a function of $(x-t)$, and for such a function the partial derivative w.r.t 't' can be obtained from the partial derivative w.r.t 'x' by just changing the sign.

So, for the n^{th} order partial derivative the sign will change 'n' times.

$$\therefore \boxed{\begin{aligned} H_n(x) &= e^{x^2}(-1)^n\left[\frac{\partial^n}{\partial x^n} e^{-(x-t)^2}\right]_{t=0} \\ &= (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) \end{aligned}}$$

$\Longrightarrow$ Rodrigues' formula for the Hermite polynomials.

Orthogonality relations

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

change t to u & express the expansion as a power series in u:

$$e^{2xu-u^2} = \sum_{m=0}^{\infty} H_m(x)\frac{u^m}{m!}$$

Xing these ~~resultant~~ equations and the resultant eqn by e^{-x^2} gives:

$$e^{-x^2+2xt-t^2+2xu-u^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-x^2} H_n(x) H_m(x)\frac{t^n u^m}{n!\,m!}$$

$$e^{2tu} \times e^{-(x^2+t^2+u^2-2xt-2xu-2tu)} = e^{2tu}\cdot e^{-(x-t-u)^2}$$

-ting w.r.t x from $-\infty$ to $+\infty$ and interchanging the order of the summation and integration.

$$e^{2tu}\int_{-\infty}^{+\infty} e^{-(x-t-u)^2}dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)\,dx\,\frac{t^n u^m}{n!\,m!}$$

Put $z = x-t-u \implies dz = dx$

$$\int_{-\infty}^{+\infty} e^{-z^2}dz = \Gamma(1/2) = \sqrt{\pi}$$

check on (22) (17)

Set $z^2 = k \implies 2z\,dz = dk$

$$\int_{-\infty}^{+\infty} e^{-z^2}dz = 2\int_0^{\infty} e^{-z^2}dz = \cancel{2}\int_0^{\infty} e^{-k}\cdot\frac{dk}{\cancel{2}\sqrt{k}}$$

$$= \int_0^{\infty} k^{-1/2}e^{-k}\,dk = \Gamma(1/2)$$

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\,dx = n!$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \times \Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}$$

Let $m = n = 0$,

$$\int_0^{\pi/2} dx = \frac{\pi}{2} = \frac{\Gamma\frac{1}{2} \times \Gamma\frac{1}{2}}{2\,\Gamma 1} = \frac{\left(\Gamma\frac{1}{2}\right)^2}{2 \times 0!} = \frac{\left(\Gamma\frac{1}{2}\right)^2}{2}$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\Longrightarrow \int_{-\infty}^{\infty} e^{-z^2} dz = \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\Longrightarrow \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) \, dx \, \frac{t^n u^m}{n!\, m!} = \sqrt{\pi}\, e^{2tu}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2tu)^n}{n!} = \sqrt{\pi}\left[1 + \frac{2tu}{1!} + \frac{(2tu)^2}{2!} + \cdots\right]$$

Equating the coeff. of $t^n u^m$,

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0 \quad \text{if } n \neq m$$

and

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n!\, \pi^{1/2}$$

Combining the results,

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n!\, \pi^{1/2}\, \delta_{nm}}$$

δ_{nm} : Kronecker delta

The Hermite polynomials of different degrees are orthogonal to each other on the interval $(-\infty, +\infty)$ with weight function e^{-x^2}.

For
unity
from

For Legendre polynomials the weight function is unity and the range of integration varies from -1 to $+1$.

Normalized Hermite functions

Many formulas are simpler when expressed in terms of unnormalized Hermite polynomials. However, the unnormalized functions increase very rapidly with 'N', so we shall use only the normalized Hermite functions ψ_n:

$$\boxed{\psi_n(x) = e^{-x^2/2} \cdot H_n(x) \frac{1}{\pi^{1/4} \cdot 2^{n/2} \sqrt{n!}}}$$

The normalized Hermite functions $\psi_n(x)$ have the orthogonality property that

$$\boxed{\int_{-\infty}^{+\infty} \psi_n(x)\, \psi_m(x)\, dx = \delta_{mn}}$$

$$\psi_0(x) = \pi^{-1/4} \cdot e^{-x^2/2}$$

$$\psi_1(x) = \sqrt{2}\,\pi^{-1/4}\, x\, e^{-x^2/2}$$

$$\psi_2(x) = \left(\sqrt{2}\, x^2 - \frac{1}{\sqrt{2}}\right) \pi^{-1/4}\, e^{-x^2/2}$$

$$\psi_3(x) = \left(\sqrt{4/3}\; x^3 - \sqrt{3}\, x\right) \pi^{-1/4}\, e^{-x^2/2}$$

$$\psi_4(x) = \left(\sqrt{2/3}\, x^4 - \sqrt{6}\, x^2 + \frac{\sqrt{6}}{4}\right) \pi^{-1/4}\, e^{-x^2/2}$$

$$\psi_5(x) = \left(\frac{2}{15}\sqrt{15}\, x^{15} - \frac{2}{3}\sqrt{15}\, x^3 + \frac{\sqrt{15}}{2} x\right) \pi^{-1/4}\, e^{-x^2/2}$$

Recursion relations

$$H'_n(x) = 2x\, H_n(x) - H_{n+1}(x)$$

$$\left[e^{x^2/2}\psi'_n(x) + x\, e^{x^2/2}\psi_n(x)\right]\cancel{2^{n/2}}\,\cancel{\sqrt{n!}}\,\cancel{\pi^{1/4}} = 2x\, \cancel{e^{x^2/2}}\, \psi_n(x)\cdot\cancel{2^{n/2}}\cancel{\sqrt{n!}}\,\cancel{\pi^{1/4}}$$

$$- \cancel{e^{x^2/2}}\,\psi_{n+1}(x)\cdot\cancel{2^{n/2}}\cdot 2^{1/2}\cdot\sqrt{(n+1)\cancel{n!}}\cdot\cancel{\pi^{1/4}}$$

$$\psi'_n(x) + x\,\psi_n(x) = 2x\,\psi_n(x) - \psi_{n+1}(x) \times \sqrt{2(n+1)}$$

$$\boxed{\psi'_n(x) = x\,\psi_n(x) - \sqrt{2(n+1)}\,\psi_{n+1}(x)}$$

$$H'_{n+1} = 2(n+1)H_n \Longrightarrow H'_n(x) = 2n\, H_{n-1}(x)$$

$$\left[\cancel{e^{x^2/2}}\psi'_n(x) + x\,\cancel{e^{x^2/2}}\psi_n(x)\right]\cancel{2^{n/2}}\,\sqrt{n!}\times\cancel{\pi^{1/4}} =$$

$$2n\,\cancel{e^{x^2/2}}\,\psi_{n-1}(x)\times\cancel{2^{n/2}}\times 2^{-1/2}\,\sqrt{(n-1)!}\times\cancel{\pi^{1/4}}$$

$$\psi'_n(x) + x\,\psi_n(x) = \sqrt{2n}\,\psi_{n-1}(x)$$

$$\boxed{\psi'_n(x) = -x\,\psi_n(x) + \sqrt{2n}\,\psi_{n-1}(x)}$$

$$\frac{d\psi_n}{dx} = x\psi_n(x) - \sqrt{2(n+1)}\,\psi_{n+1}(x)$$

$$\frac{d\psi_n}{dx} = -x\psi_n(x) + \sqrt{2n}\,\psi_{n-1}(x) \quad , \; n \geq 0$$

These must be initiated by the derivations of ψ_0 and ψ_1 which are:

$$\frac{d\psi_0}{dx} = \frac{-1}{\sqrt{2}}\psi_1 = -x\psi_0 \qquad \Big\} \; \psi_1 = \sqrt{2}\,x\psi_0$$

$$= -x\psi_0 = -x\,\pi^{-1/4} e^{-x^2/2}$$

$$\frac{d\psi_1}{dx} = -x\psi_1(x) + \sqrt{2}\,\psi_0(x)$$

$$= -\sqrt{2}\,x^2\psi_0(x) + \sqrt{2}\,\psi_0(x)$$

$$= \sqrt{2}\left[1 - x^2\right]\psi_0(x)$$

$$= \sqrt{2}\,\pi^{-1/4} e^{-x^2/2}(1 - x^2)$$

Raising & Lowering Operators

The raising operator $\mathcal{R}$, when applied to a Hermite function of degree n, gives a result which is proportional to the Hermite function of the next highest degree. Similarly, the lowering operator $\mathcal{L}$ reduces the degree of a Hermite function by one.

$$\mathcal{R} = \left(\frac{d}{dx} - x\right) \quad \text{- raising operator}$$

$$\mathcal{R}\psi_n = -\sqrt{2(n+1)}\,\psi_{n+1}$$

$$\mathcal{L} = \left(\frac{d}{dx} + x\right) \quad \text{- lowering operator}$$

$$\mathcal{L}\psi_n = \sqrt{2n}\,\psi_{n-1}$$

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$$

$$e^{x^2/2}\Psi_n'' + \cancel{x e^{x^2/2}\Psi_n'} + e^{x^2/2}\Psi_n(x) + x^2 e^{x^2/2}\Psi_n(x) + \cancel{x e^{x^2/2}\Psi_n}$$

$$- \cancel{2x e^{x^2/2}\Psi_n'} - 2x^2 e^{x^2/2}\Psi_n + 2n e^{x^2/2}\Psi_n = 0$$

$$\Psi_n''(x) - x^2\Psi_n(x) = -(2n+1)\Psi_n(x)$$

$$\frac{d^2\Psi_n}{dx^2} - x^2\Psi_n(x) = -(2n+1)\Psi_n(x)$$

$$\left(\frac{d^2}{dx^2} - x^2\right)\Psi_n(x) = \eta\,\Psi_n(x) = -(2n+1)\Psi_n(x)$$

The Hermite eigenoperator

$$\eta = \frac{d^2}{dx^2} - x^2 = \frac{1}{2}\left[\mathcal{R}\mathcal{L} + \mathcal{L}\mathcal{R}\right]$$

$$x\psi_n = \frac{1}{2}\left[\mathcal{L} - \mathcal{R}\right]\psi_n$$

$$= \sqrt{n/2}\ \psi_{n-1} + \sqrt{(n+1)/2}\ \psi_{n+1}$$

$$\frac{d}{d\cancel{x}}\psi_n = \frac{1}{2}\left[\mathcal{L} + \mathcal{R}\right]\psi_n$$

$$= \sqrt{n/2}\ \psi_{n-1} - \sqrt{(n+1)/2}\ \psi_{n+1}$$

Ex:- 1D Harmonic Oscillator

The time independent Schrodinger eq^n-

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x)$$

Conservative force: A conservative force may be defined as one for which the work done in moving b/w 2 points A and B is independent of the path taken b/w the 2 points.

The implication of "conservative" in this context is that, you could move it from A to B by one path and return to A by another path with no net loss of energy - any closed return path to A takes net zero work.

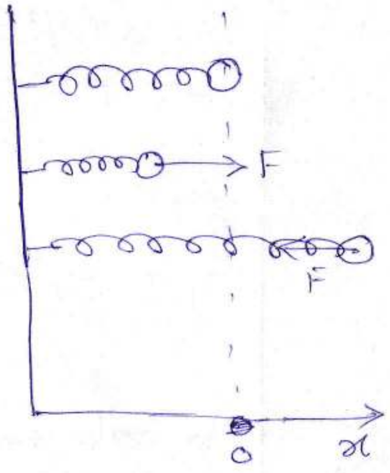
The energy of an object which is subject only to that conservative force is dependent upon its position and not upon the path by which it reached that position. This makes it possible to define a potential energy function which depends upon position only.

The potential energy U is equal to the work you must do to move an object from the $U=0$ reference point to the position r. The reference point at which you assign the value $U=0$ is arbitrary, so may be chosen for convenience; like choosing the origin of a coordinate system.

$$U = -\int_{x_0}^{x} F(x)\,dx$$

$$F(x) = -\frac{dU}{dx}$$

$F = -kx$



Hook's law

$$U = -\int_0^x -kx\,dx$$

$$= \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2 x^2 \quad , \quad \omega = \sqrt{\frac{k}{m}} \text{ : angular freq. of the oscillator}$$

The Hamiltonian of the particle is :

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$\hat{x}$: position operator

$\hat{p} = -i\hbar\frac{\partial}{\partial x}$: momentum operator

Time-independent Schrödinger equation,

$$H|\psi\rangle = E|\psi\rangle$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x)$$

We'll rewrite eqn. in a dimensionless form by introducing a new independent variable through the relation

$$\xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} x \qquad \text{and}$$

define a parameter λ as :

$$E = \left(\lambda+\frac{1}{2}\right)\hbar\omega$$

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi}\times\frac{d\xi}{dx} = \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{d\psi}{d\xi}$$

$$\frac{d}{dx}\left(\frac{d\psi}{dx}\right) = \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{d}{dx}\left(\frac{d\psi}{d\xi}\right) = \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{d^2\psi}{d\xi^2}\times\frac{d\xi}{dx}$$

$$= \left(\frac{m\omega}{\hbar}\right)^{\cancel{2}}\frac{d^2\psi}{d\xi^2}$$

$$\frac{-\hbar^2}{2\cancel{m}}\left(\frac{\cancel{m}\omega}{\cancel{\hbar}}\right)\frac{d^2\psi}{d\xi^2} + \frac{1}{2}\cancel{m}\omega^2\left(\frac{\hbar}{\cancel{m}\omega}\right)\xi^2\psi = \left(\lambda+\frac{1}{2}\right)\hbar\omega\,\psi$$

Xing by $\frac{-2}{\hbar\omega}$,

$$\frac{d^2\psi}{d\xi^2} + \left(2\lambda+1-\xi^2\right)\psi = 0$$

Lets take, $\psi(\xi) = e^{-\xi^2/2} f(\xi)$

$$\frac{d\psi}{d\xi} = e^{-\xi^2/2}\frac{df}{d\xi} - \xi\, e^{-\xi^2/2} f(\xi)$$

$$\frac{d^2\psi}{d\xi^2} = \cancel{e^{-\xi^2/2}}\frac{d^2 f}{d\xi^2} - \xi\,\cancel{e^{-\xi^2/2}}\frac{df}{d\xi} - \cancel{e^{-\xi^2/2}} f + \xi^2\,\cancel{e^{-\xi^2/2}} f - \xi\,\cancel{e^{-\xi^2/2}}\frac{df}{d\xi}$$

$$= \left[\frac{d^2 f}{d\xi^2} - 2\xi\frac{df}{d\xi} + (\xi^2-1) f\right] e^{-\xi^2/2}$$

Substituting,

$$\left[\frac{d^2 f}{d\xi^2} - 2\xi \frac{df}{d\xi} + (\xi^2 - 1) f + (2\lambda + 1 - \xi^2) f\right] e^{-\xi^2/2} = 0$$

$$\frac{d^2 f}{d\xi^2} - 2\xi \frac{df}{d\xi} + 2\lambda f(\xi) = 0$$

$\longrightarrow$ Hermite differential eqn. for ~~$n = \lambda$, where~~ 'λ' is 0 or +ve integer.

Whether 'λ' is an integer ?

Frobenius method

$$f(\xi_\varphi) = \xi_\varphi^{s} \sum_{r=0}^{\infty} a_r \xi_\varphi^{r} = \sum_{r=0}^{\infty} a_r \xi_\varphi^{r+s}$$

$$f'(\xi_\varphi) = \sum_{r=0}^{\infty} (r+s)\, a_r \xi_\varphi^{r+s-1}$$

$$f''(\xi_\varphi) = \sum_{r=0}^{\infty} (r+s)(r+s-1)\, a_r \xi_\varphi^{r+s-2}$$

Substituting

$$\sum_{r=0}^{\infty}\left[(r+s)(r+s-1)\, a_r \xi^{r+s-2} - 2\{(r+s)-\lambda\} a_r \xi^{r+s}\right] = 0$$

$$\sum_{r=0}^{\infty}\left[(r+s)(r+s-1) a_r - 2\xi^2\{r+s-\lambda\} a_r\right] \xi^{r+s-2} = 0$$

is valid for all ξ.

~~Lowest power of~~

$\therefore$ Coeff. of each power of ξ can be equated to zero.

The lowest power of ξ is $r+s-2$ when $r=0$.

For $r=0$, and $r=1$ we get two indicial equations.

$$\left.\begin{aligned} s(s-1)\, a_0 &= 0 \\ s(s+1)\, a_1 &= 0 \end{aligned}\right\} \begin{aligned} a_0 \neq 0 &\implies s = 0 \text{ (or) } 1 \\ a_1 \neq 0 &\implies s = 0 \text{ (or) } -1 \end{aligned}$$

$S = -1$ is not acceptable condition since it introduces new singularity at $\xi = 0$.

$\xi = 0$ is not a singular point from the form of the Hermite differential eqn.

$\Longrightarrow \underline{S = 0 \text{ (or) } 1}$

In general for $\xi\varphi^{r+s}$,

$$(r+s+2)(r+s+1)\,a_{r+2} = 2\left[r+s-\lambda\right] a_r$$

$$\boxed{\frac{a_{r+2}}{a_r} = \frac{2(r+s-\lambda)}{(r+s+2)(r+s+1)}}$$

For $s=0$ starting with a_0 (a_1) generates all the even (odd) numbered coeff. that go with the even (odd) powers of $\xi\varphi$,

$$a_2 = \frac{-2\lambda}{2} a_0, \quad a_4 = \frac{-2(\lambda-2)}{12} = \frac{(-2)^2 \lambda(\lambda-2)}{24} a_0, \quad \ldots$$

$$a_3 = \frac{-2(\lambda-1)}{6} a_1, \quad a_5 = \frac{-2(\lambda-3)}{20} a_3 = \frac{(-2)^2(\lambda-1)(\lambda-3)}{120} a_1, \quad \ldots$$

$$f(\xi)_{even} = a_0 + a_2\xi^2 + a_4\xi^4 + \cdots$$

$$f(\xi)_{odd} = a_1\xi + a_3\xi^3 + a_5\xi^5 + \cdots$$

$$f(\xi) = \xi^s\left[\left(a_0 + a_2\xi^2 + a_4\xi^4 + \cdots\right) + \left(a_1\xi + a_3\xi^3 + a_5\xi^5 + \cdots\right)\right]$$

The full solutions $f(\xi)$ can be determined from 2 constants a_0 and a_1 which is expected for a 2nd order equation.

~~However~~, not all the solutions so obtained are normalizable!

At very large r, the recursion formula behaves as,

$$\frac{a_{r+2}}{a_r} = \frac{2\left[1 + \frac{s}{r} - \frac{\lambda}{r}\right]}{r\left[1 + \frac{s+2}{r}\right]\left[1 + \frac{s+1}{r}\right]} \xrightarrow{r\to\infty} \frac{2}{r}$$

For,

$$e^{\xi^2} = 1 + \xi^2 + \frac{\xi^4}{2!} + \cdots = \sum_{r=0,2,4,\dots} b_r \xi^r$$

where, $b_r = \frac{1}{(r/2)!}$

Ratio of two adjacent coeff. is :

$$\frac{b_{r+2}}{b_r} = \frac{(r/2)!}{\left(\frac{r+2}{2}\right)!} = \frac{1}{\frac{r}{2}+1} \xrightarrow{r\to\infty} \frac{2}{r}$$

Hence, asymptotically, $f(\xi)$ is behaving like e^{ξ^2} for large ξ.

ie; if λ is a -ve integer

$\psi = e^{-\xi^2/2} f(\xi)$ goes like $e^{+\xi^2/2}$.

The wavefunction ψ approaches infinity for $x \to \pm\infty$.

Since, $|\psi(x)|^2 dx$ defines the probability of finding the particle b/w x and $x+dx$, it follows that the probability of finding the oscillating particle away from the origin will be infinite. It is not physically admissible.

$\therefore$ A value of λ that is neither a +ve integer nor zero is not permitted.

For $\lambda = n$, is zero or a ~~+ve~~ integer.

we can get a particular polynomial solution of the Hermite differential equation - the Hermite polynomial of degree n.

$$f(\varepsilon_\varphi) = N' H_n(\varepsilon_\varphi)$$

$\therefore$

The solution of time-independent Schrödinger equation can be written as:

$$\psi = N e^{-\varepsilon_\varphi^2/2} H_n(\varepsilon_\varphi)$$

This form of the solution is well behaved; it tends to zero as $x \to \pm\infty$.

The allowed values of λ correspond to $n = 0, 1, 2, \dots$. This implies that the linear harmonic oscillator has a discrete set of equidistant energy levels:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \dots$$

The corresp. wavefunction is:

$$\psi_n(x) = N_n \, e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

n: vibrational quantum #

$$\int_{-\infty}^{+\infty} |\psi_n(x)|^2 dx = 1 \quad , \quad \xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$$

$$d\xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} dx$$

$$\Longrightarrow$$

$$N_n^2 \left(\frac{\hbar}{m\omega}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\xi^2} H_n(\xi) H_n(\xi) \, d\xi = 1$$

$$\int_{-\infty}^{+\infty} e^{-\xi^2} H_n(\xi) H_m(\xi) \, d\xi = 2^n n! \, \pi^{1/2} \delta_{nm}$$

$$N_n = \left[\left(\frac{m\omega}{\hbar}\right)^{1/2} \frac{1}{2^n n! \, \pi^{1/2}}\right]^{1/2}$$

The normalized wavefunctions are given by,

$$\psi_n(x) = \left[\left(\frac{m\omega}{\hbar}\right)^{1/2} \frac{1}{2^n n! \, \pi^{1/2}}\right]^{1/2} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

$$\Psi_n = N_n e^{-\xi^2/2} H_n(\xi)$$

$$\frac{d\Psi_n}{d\xi} = N_n e^{-\xi^2/2}\left[-\xi H_n(\xi) + \underbrace{\frac{dH_n}{d\xi}}_{2n H_{n-1}}\right]$$

$$= N_n e^{-\xi^2/2}\left[-\xi H_n(\xi) + 2n H_{n-1}\right]$$

$$= -\xi \Psi_n + N_n 2n e^{-\xi^2/2} H_{n-1}$$

$$N_n \, 2n \, e^{-\xi^2/2} H_{n-1} = \left[\left(\frac{m\omega}{\hbar}\right)^{1/2} \frac{1}{2^n n! \, \pi^{1/2}}\right]^{1/2} 2n \, e^{-\xi^2/2} H_{n-1}$$

$$= \left[\left(\frac{m\omega}{\hbar}\right)^{1/2} \frac{1}{2^{n-1}(n-1)! \cdot \pi^{1/2}}\right]^{1/2} \sqrt{2}\,\sqrt{n}\, e^{-\xi^2/2} H_{n-1}$$

$$= N_{n-1} e^{-\xi^2/2} H_{n-1} \cdot \sqrt{2n}$$

$$= \sqrt{2}.\ \sqrt{n}\ \psi_{n-1}$$

$$\frac{1}{\sqrt{2}}\left[\frac{d\psi_n}{d\xi} + \xi\psi_n\right] = \sqrt{n}\ \psi_{n-1}$$

$$\boxed{a^- \psi_n = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\psi_n = \sqrt{n}\ \psi_{n-1}}$$

$a^- = \xi + \frac{d}{d\xi}$: lowering (annihilation) operators.

$$\frac{d\psi_n}{d\xi} = N_n e^{-\xi^2/2}\left[-\xi H_n(\xi) + \frac{dH_n}{d\xi}\right]$$

$$\frac{dH_n}{dx} = 2x H_n(x) - H_{n+1}(x)$$

$$\frac{d\psi_n}{d\xi} = N_n e^{-\xi^2/2}\left[-\xi H_n(\xi) + 2\xi H_n(\xi) - H_{n+1}(\xi)\right]$$

$$= N_n e^{-\xi^2/2}\left[\xi H_n(\xi) - H_{n+1}(\xi)\right]$$

$$N_n e^{-\xi^2/2} H_{n+1} = \left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{1}{2^n n!\cdot\pi^{1/2}}\right]^{1/2} H_{n+1}(\xi)$$

$$= \left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{1}{2^{n+1}(n+1)!\,\pi^{1/2}}\right]^{1/2}\sqrt{2(n+1)}\,e^{-\xi^2/2}H_{n+1}(\xi)$$

$$= \sqrt{2(n+1)}\,\psi_{n+1}$$

$$\frac{d\psi_n}{d\xi} = \xi\psi_{n+1} \mp \sqrt{2(n+1)}\,\psi_{n+1}$$

$\Longrightarrow$

$$\cancel{\frac{1}{\sqrt{2}}\left[\frac{d\psi_n}{d\xi_\varphi} - \xi_\varphi\psi_n\right] = \sqrt{n+1}\,\psi_n}$$

$$\cancel{a^+\psi_n = \frac{1}{\sqrt{2}}\left[\frac{d}{d\xi_\varphi}\right]}$$

$$\frac{1}{\sqrt{2}}\left[\xi_\varphi\psi_n - \frac{d\psi_n}{d\xi_\varphi}\right] = \sqrt{n+1}\;\psi_{n+1}$$

$$\boxed{a^+\psi_n = \frac{1}{\sqrt{2}}\left[\xi_\varphi - \frac{d}{d\xi_\varphi}\right]\psi_n = \sqrt{n+1}\;\psi_{n+1}}$$

$a^+ = \xi_\varphi - \frac{d}{d\xi_\varphi}$: raising (creation) operators.

$$(aa^{+} - a^{+}a)\psi_n = a\sqrt{n+1}\,\psi_{n+1} - a^{+}\sqrt{n}\,\psi_{n-1}$$

$$= (n+1)\psi_n - n\psi_n = \psi_n$$