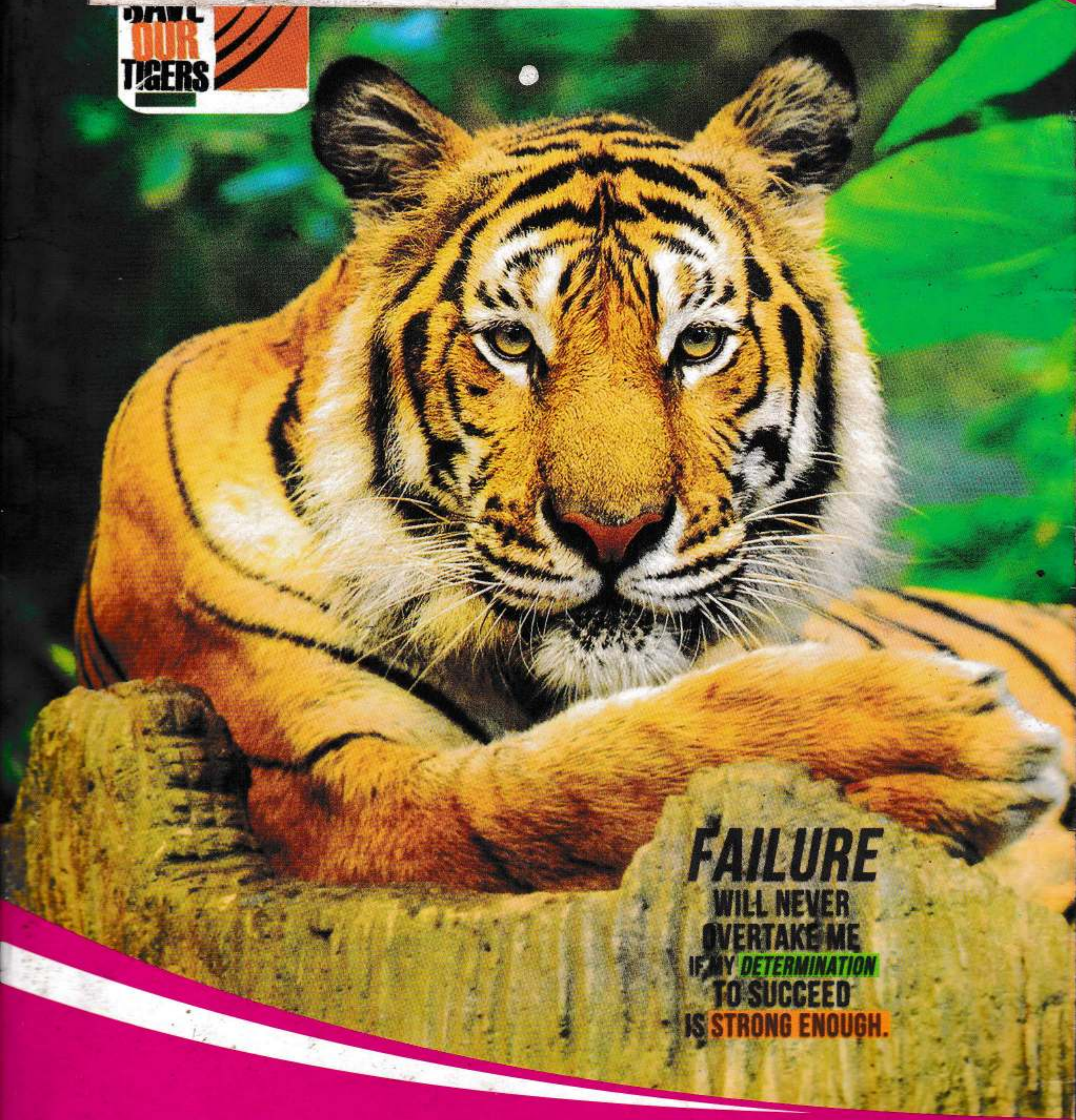
Introduction to Linear Algebra
- Gilbert Strang

12

Linear Transformations

(Fourier Series and Transform)



sooraj551729@gmail.com

+91-9400635788

(12)

INDEX

Name : SOORAJ.S Subject : ....

Std. : .... Div. : .... Roll No. : ....

School / College : ....

S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		INTRODUCTION TO		
		LINEAR ALGEBRA		
		- Gilbert Strang, MIT		
		(5th Edition)		

☐ Orthogonal Bases for Function Space

3 leading even-odd bases for theoretical and numerical computations.

⑤ The Fourier basis :

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots\ldots$$

⑥ The Legendre basis :

$$1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x, \ldots\ldots$$

The Chebyshev basis :

$$1, x, 2x^2 - 1, 4x^3 - 3x, \ldots\ldots$$

The Fourier basis functions (sines & cosines) are all periodic. They repeat over every 2π interval because $\cos(x+2\pi) = \cos x$ and $\sin(x+2\pi) = \sin x$. So this basis is especially good for functions $f(x)$ that are themselves periodic: $f(x+2\pi) = f(x)$..

The sine-cosine basis is also excellent for approximation. If we have a smooth periodic function $f(x)$, then a few sines and cosines (low frequencies) are all we need. Jumps in $f(x)$ and noise in the signal are seen in higher frequencies (larger n).

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad \text{for all } n, m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & \text{for } n \neq m \\ \pi & \text{for } n = m \end{cases}$$

$\Longrightarrow$ Fourier Basis is orthogonal.

~~Fourier series~~):

The Fourier series of a function $f(x)$ is its expansion into sines and cosines:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

⟹ Fourier series is just linear algebra in function space (infinite dimensional)

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) dx$$

$$= 2\pi \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos nx\,dx}_{0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \sin nx\,dx}_{0}$$

$$= \pi a_0$$

$$\Longrightarrow \boxed{\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx}$$

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right] \cos mx\,dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx\,dx$$

$$a_0 \int_{-\pi}^{\pi} \cos mx\,dx = 0$$

$$a_0 \int_{-\pi}^{\pi} f(x) \cos mx\,dx$$

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = a_m \pi \Longrightarrow \boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx}$$

for $n = 1, 2, 3, \ldots$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m \int_{-\pi}^{\pi} \cos(mx)\cos(mx)\,dx = a_m \pi$$

$$a_m = \frac{\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx}{\int_{-\pi}^{\pi} (\cos mx)^2\,dx} = \frac{(f(x), \cos(mx))}{(\cos(mx), \cos(mx))}$$

$$\boxed{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx}$$

$$\boxed{\begin{aligned} a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \end{aligned}}$$

for $n = 1, 2, \ldots$

If f be a piecewise continuous function on $[-\pi, \pi]$. Then the Fourier series of f is the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

where the coefficients a_n and b_n in this series are defined by

$$\frac{a_0}{2} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)\,dx}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

and are called Fourier coefficients of f.

The Fourier series of a periodic function $f(x)$ of period T is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}\right)$$

for some set of Fourier coefficients a_k & b_k defined by the integrals:

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n x}{L}\right)dx \Leftrightarrow \frac{V_k^T b}{V_k^T V}$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi n x}{L}\right)dx$$

The length of a typical $f(x)$:

$$(f, f) = \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots)^2 \, dx$$

$$= \int_{-\pi}^{\pi} (a_0^2 + a_1^2 \cos^2 x + b_1^2 \sin^2 x + a_2^2 \cos^2 2x + \cdots) \, dx$$

[orthogonality

$$\boxed{\|f\|^2 = (f, f) = 2\pi a_0^2 + \pi (a_1^2 + b_1^2 + a_2^2 + \cdots)}$$

$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots$ is an orthonormal basis for our function space.

These are unit vectors.

We could combine them with coefficients $A_0, A_1, A_2, \ldots\ldots$ to yield a function $F(x)$.

Function length = Vector length

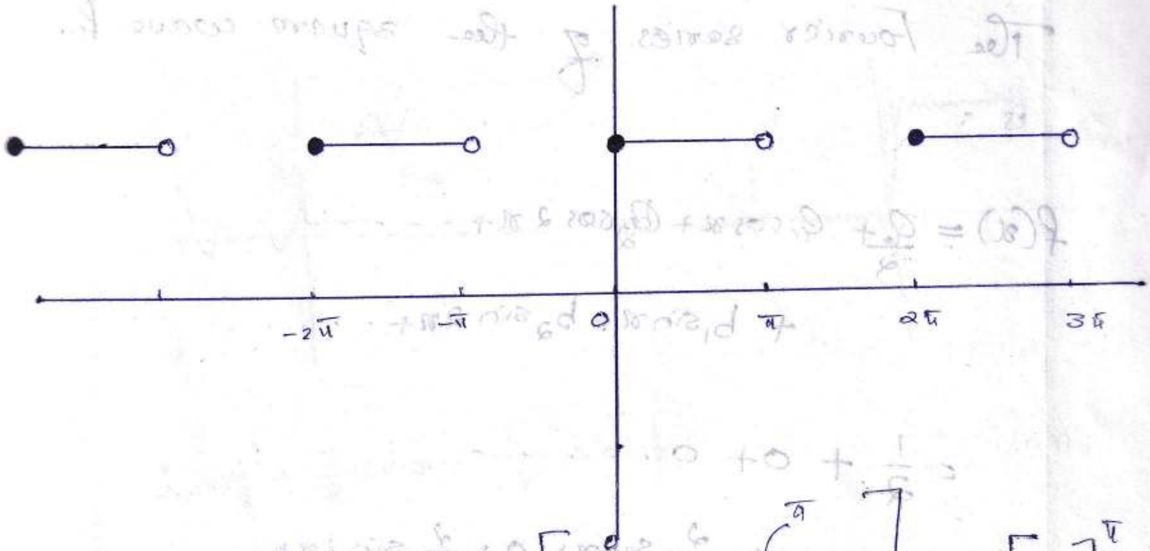
$$\|F\|^2 = (F, F) = A_0^2 + A_1^2 + B_1^2 + A_2^2 + \ldots\ldots$$

The function has finite length exactly when the vector of coefficients has finite length. Fourier series gives us a perfect match between the Hilbert spaces for functions and for vectors. The function is in L^2 and, its Fourier coeff. are in ℓ^2.

* The function space contains $f(x)$ exactly when the Hilbert space contains the vector $v = (a_0, a_1, b_1, a_2, \ldots)$ of Fourier coefficients of $f(x)$. Both must have finite length.

Ex:- Find the Fourier coeff. and Fourier series of the square-wave function f defined by,

$$f(x) = \begin{cases} 0, & \text{if } -\pi \le x < 0 \\ 1, & \text{if } 0 \le x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} dx\right] = 0 + \frac{1}{2\pi}\Big[x\Big]_0^{\pi}$$

$$= \frac{1}{2\pi} \times \pi = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} = \frac{1}{n\pi}\left(\quad\right) = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx) = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{-1}{n\pi}\Big[\cos(n\pi) - \cos(0)\Big]$$

$$b_n = \frac{-1}{n\pi}(\cos n\pi - 1) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

The Fourier series of the square wave function is :

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots$$

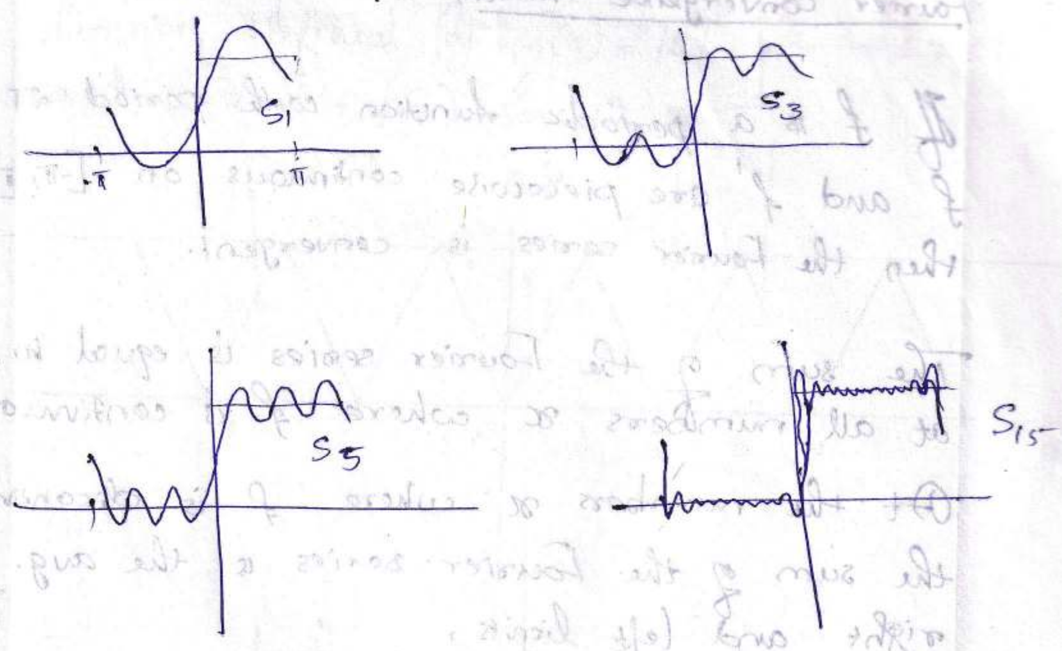
$$+ b_1 \sin x + b_2 \sin 2x + \cdots$$

$$= \frac{1}{2} + 0 + 0 \cdots$$

$$+ \frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin[(2k-1)x], \text{ where } k \in \mathbb{Z}$$



$$S_n = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \cdots + \frac{2}{n\pi}\sin(nx)$$

As n increases, $S_n(x)$ becomes a better approximation to the square-wave function. It appears that the graph of $S_n(x)$ is approaching the graph of $f(x)$, except where $x=0$ or x is an integer multiple of π. i.e., it looks as if f is equal to the sum of its Fourier series except at the points where f is discontinuous.

Fourier convergence theorem

If f is a periodic function with period 2π and f and f' are piecewise continuous on $[-\pi, \pi]$, then the Fourier series is convergent.

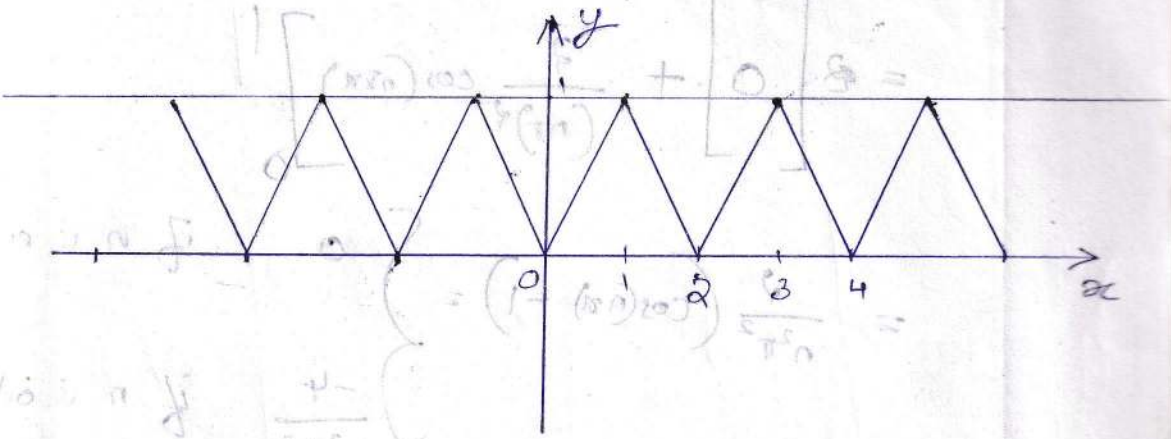
The sum of the Fourier series is equal to $f(x)$ at all numbers x where f is continuous.

At the numbers x where f is discontinuous the sum of the Fourier series is the avg of the right and left limits,

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

- Gibbs phenomenon

Ex:- Find the Fourier series of the triangular wave function defined by $f(x) = |x|$ for $-1 \le x \le 1$ and $f(x+2) = f(x)$ for all x



$L = 1$.

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos(n\pi x)\, dx$$

$\cos\left(\frac{2\pi n x}{T}\right) = \cos\left(\frac{2\pi n x}{2}\right) = \cos(\pi n x)$

$$= \frac{2}{2} \int_{-1}^{1} |x| \cos(n\pi x)\, dx$$

$$= \int_{-1}^{0} -x \cos(n\pi x)\, dx + \int_{0}^{1} x \cos(n\pi x)\, dx$$

$$\int_{1}^{0} t \cos(n\pi t)\, dt + \int_{0}^{1} x \cos(n\pi x)\, dx \qquad t = -x \quad t: 1 \to 0$$

$$= 2 \int_{0}^{1} x \cos(n\pi x)\, dx \qquad \left[\begin{array}{l} y = |x| \cos(n\pi x) \\ \text{is an even function} \end{array} \right.$$

$$= 2 \left[x \frac{\sin(n\pi x)}{n\pi} - \int \frac{\sin(n\pi x)}{n\pi}\, dx \right]$$

$$a_n = 2\left[\frac{x\sin(n\pi x)}{n\pi} + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= 2\left[0\right] + \left.\frac{2}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left(\cos(n\pi) - 1\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_{-1}^{1} |x| \sin(n\pi x) = 0$$

The Fourier series is :

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x) - \frac{4}{9\pi^2}\cos(3\pi x) - \frac{4}{25\pi^2}\cos(5\pi x) - \cdots$$

$$|x| = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}\cos\left[(2k-1)\pi x\right] \quad \text{for all } x.$$

Ex:- The double angle formula in trigonometry is

$\cos 2x = 2\cos^2 x - 1 \implies \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x.$

Complex form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - i b_n}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{a_n + i b_n}{2} e^{-inx}$$

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{inx}$$

where,

$$C_0 = \frac{a_0}{2}, \quad C_n = \frac{a_n - i b_n}{2}, \quad C_{-n} = \frac{a_n + i b_n}{2}$$

C_n: complex Fourier coefficients.

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) e^{-inx} \, dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\int_{-\pi}^{\pi} f(x)\, e^{imx}\, dx = \int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} C_n e^{imx}\right) e^{inx}$$

$$\int_{-\pi}^{\pi} C_n e^{2inx}\, dx = \frac{C_n}{2in}\left[e^{2inx}\right]_{-\pi}^{\pi}$$

$$= \frac{C_n}{2in}\left[e^{2\pi i n} - e^{-2\pi i n}\right] = \frac{C_n}{n}\sin(2\pi n)$$

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx = \int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} C_n e^{imx}\right) e^{-imx}\, dx$$

$$= \int_{-\pi}^{\pi} C_n e^{0}\, dx = C_n\,(2\pi)$$

$$\Longrightarrow \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

The complex form of the Fourier series of a periodic function $f(x)$ of period T is :

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{i2\pi n x}{L}}$$

where, the Complex Fourier coefficient C_n is

$$C_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i\frac{2\pi n x}{L}} dx, \qquad n = 0, \pm 1, \pm 2, \cdots$$

Ex: Fourier series of

$$f(x) = \operatorname{sgn}(x) = \begin{cases} -1, & -\pi \le x \le 0 \\ 1, & 0 < x \le \pi \end{cases}$$

Ans:

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{0} (-1)\, e^{-inx} dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left[\left[\frac{+e^{-inx}}{+in}\right]_{-\pi}^{0} - \left[\frac{e^{-inx}}{in}\right]_{0}^{\pi}\right]$$

$$= \frac{-i}{2\pi n}\left[1 - e^{in\pi} - e^{-in\pi} + 1\right] = \frac{-i}{2\pi n}\left[2 - \left(e^{in\pi} + e^{-in\pi}\right)\right]$$

$$= \frac{-i}{2\pi n}\left[2 - 2\cos n\pi\right] = \frac{i}{\pi n}\left(\cos n\pi - 1\right)$$

$$= \frac{i}{\pi n}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{if } n = 2k \\ \dfrac{-2i}{(2k-1)\pi}, & n = 2k-1 \end{cases}$$

$\therefore$ The Fourier series of the function in complex form is:

$$f(x) = \operatorname{sgn}(x) = \frac{-2i}{\pi}\sum_{k=-\infty}^{+\infty} \frac{e^{i(2k-1)x}}{2k-1}$$

$n = 2k-1$

$$f(x) = \operatorname{sgn}(x) = \frac{-2i}{\pi} \sum_{k=-\infty}^{\infty} \frac{e^{i(2k-1)x}}{2k-1} = \frac{-2i}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n}$$

$$= \frac{-2i}{\pi} \sum_{n=1}^{\infty} \left[\frac{e^{-inx}}{-n} + \frac{e^{inx}}{n} \right]$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\frac{e^{inx} - e^{-inx}}{2in} \right]$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)}{2k-1}$$

$$f(\pi/2) = 1 = \frac{4}{\pi}\left[\sin(\pi/2) + \frac{\sin\left(\frac{3\pi}{2}\right)}{3} + \frac{\sin\left(\frac{5\pi}{2}\right)}{5} + \dots \right]$$

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$$

$$\boxed{\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]}$$

Ex:- Using the complex form find the Fourier series of the function $f(x) = x^2$, defined on $[-1,1]$.

Ans:

$$\cancel{C_n = \frac{1}{2}\int_{-1}^{1} x^2 e^{-inx}\,dx = \frac{1}{2}\left[x^2 \frac{e^{-inx}}{-in}\Big|_{-1}^{1} - \int_{-1}^{1} x \frac{e^{-inx}}{-in}\,dx \right]}$$

$$C_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-i\frac{2\pi n x}{L}} = \frac{1}{2}\int_{-1}^{1} x^2 e^{-i\pi n x}$$

$$= \frac{1}{2}\left[x^2 \frac{e^{-i\pi n x}}{-i\pi n} - \int_{-1}^{1} 2x \frac{e^{-i\pi n x}}{-i\pi n}\,dx \right.$$

$$= \frac{1}{2}\left[x^2 \frac{e^{-i\pi n x}}{-i\pi n}\right]_{-1}^{1} + \frac{1}{2(i\pi n)}\left[2x \frac{e^{-i\pi n x}}{(-i\pi n)} - \int_{-1}^{1} \frac{2e^{-i\pi n x}}{-i\pi n}\right]$$

$$= \frac{1}{2} x^2 \frac{e^{-i\pi n x}}{-i\pi n} + \frac{\cancel{2} x e^{-i\pi n x}}{\cancel{2}\pi^2 n^2} - \frac{\cancel{2}}{\cancel{2}\pi^2 n^2} \frac{e^{-i\pi n x}}{-i\pi n}$$

$$= \left[\frac{x^2 e^{-i\pi n x}}{-2i\pi n} + \frac{2x e^{-i\pi n x}}{2\pi^2 n^2} + \frac{2e^{-i\pi n x}}{2i\pi^3 n^3}\right]_{-1}^{1}$$

$$= \frac{e^{-i\pi n}}{-2i\pi n} + \frac{2e^{-i\pi n}}{2\pi^2 n^2} + \frac{2e^{-i\pi n}}{2i\pi^3 n^3} + \frac{e^{i\pi n}}{2i\pi n} + \frac{2e^{i\pi n}}{2\pi^2 n^2} - \frac{2e^{i\pi n}}{2i\pi^3 n^3}$$

$$= \frac{1}{\pi n} \frac{e^{i\pi n} - e^{-i\pi n}}{2i} + \frac{2}{\pi^2 n^2} \frac{e^{i\pi n} + e^{-i\pi n}}{2} - \frac{2}{\pi^3 n^3} \frac{e^{i\pi n} - e^{-i\pi n}}{2i}$$

$$= \frac{1}{n\pi} \cancel{\sin(n\pi)} + \frac{2}{n^2\pi^2} \cos(n\pi) - \frac{2}{n^3\pi^3} \cancel{\sin(n\pi)}$$

$$c_n = \frac{2}{n^2\pi^2}(-1)^n$$

$$C_0 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{6}\left[1+1\right] = 1/3.$$

The Fourier extension is complex form

is :

$$f(x) = x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}(-1)^n e^{in\pi x}$$

$$+ \sum_{n=1}^{\infty} \frac{2}{(-n)^2\pi^2}(-1)^{-n} e^{-in\pi x}$$

$$(-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n$$

$$f(x) = x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,\frac{e^{in\pi x}+e^{-in\pi x}}{2}$$

$$= \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$$

Ex:

Ans: f(

Ex: 3 $\quad f(x) = \dfrac{a\sin x}{1 - 2a\cos x + a^2}$, $|a| < 1$.

Ans: $$f(x) = \dfrac{a\,\dfrac{e^{ix} - e^{-ix}}{2i}}{1 - \cancel{2}a\,\dfrac{e^{ix}+e^{-ix}}{\cancel{2}} + a^2}$$

$$= \frac{1}{2i}\,\frac{a\left(e^{ix} - e^{-ix}\right)}{1 - a\left(e^{ix} + e^{-ix}\right) + a^2}$$

$$= \frac{1}{2i}\,\frac{a\left(e^{ix} - e^{-ix}\right)}{1 - ae^{ix} + ae^{-ix} + a^2 e^{ix}\cdot e^{-ix}}$$

$$= \frac{1}{2i}\,\frac{a\left(e^{ix} - e^{-ix}\right)}{1 - ae^{ix} - ae^{-ix}\left(1 - ae^{ix}\right)}$$

$$= \frac{1}{2i} \times \frac{a\left(e^{ix} - e^{-ix}\right)}{\left(1 - ae^{ix}\right)\left(1 - ae^{-ix}\right)}$$

$$= \frac{1}{2i}\left[\frac{A}{1 - ae^{ix}} + \frac{B}{1 - ae^{-ix}}\right]$$

$$a\left(e^{ix} - e^{-ix}\right) = A\left(1 - ae^{-ix}\right) + B\left(1 - ae^{ix}\right)$$

$$= A - Aae^{-ix} + B - Bae^{ix}$$

$x = 0$:

$A + B = (A+B)\,a \Rightarrow (A+B)(1-a) = 0$

$\underline{B = -A}$

$$a(e^{ix}-e^{-ix}) = aA(e^{ix}-e^{-ix})$$

$$\Longrightarrow A=1,\ B=-1$$

$$f(x) = \frac{1}{2i}\left(\frac{1}{1-ae^{ix}} - \frac{1}{1-ae^{-ix}}\right)$$

$$|ae^{ix}| = |a||e^{ix}| = |a|<1$$

$$|ae^{-ix}| < 1$$

$$\frac{1}{1-ae^{ix}} = 1+ae^{ix}+a^2e^{2ix}+\cdots = \sum_{n=0}^{\infty} a^n e^{inx}$$

$$\frac{1}{1-ae^{-ix}} = 1+ae^{-ix}+a^2e^{-2ix}+\cdots = \sum_{n=0}^{\infty} a^n e^{-inx}$$

$$f(x) = \frac{1}{2i}\sum_{n=0}^{\infty} a^n\left(e^{inx}-e^{-inx}\right)$$

$$= \sum_{n=0}^{\infty} a^n \sin(nx)$$

$$= \sum_{n=1}^{\infty} a^n \sin(nx).$$

Ex: $f(t) = \cosh(t-1)$ for $0 \le t \le 1$

even function $f(t)$ is periodic with period, $T = 2$.

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{-i\frac{2\pi n t}{T}}\, dt = \frac{1}{2}\int_0^2 e^{-i\frac{2\pi n t}{2}} \times \cosh(t-1)\, dt$$

$$\int \cosh t \, dt = \sinh t + c$$

$$\int \sinh t \, dt = \cosh t + c$$

$$I = \frac{1}{2}\left[e^{-in\pi t}\sinh(t-1) - \int (-i\pi n) e^{-i\pi n t} \sinh(t-1)\right.$$

$$= \frac{1}{2}\left[e^{-in\pi t}\sinh(t-1) + i\pi n \left[e^{-i\pi n t}\cosh(t-1) - \int (-i\pi n) e^{-i\pi n t}\cosh(t-1)\right.\right.$$

$$2I = \frac{2}{2} e^{-in\pi t}\sinh(t-1) + i\pi n e^{-i\pi n t}\cosh(t-1) - \pi^2 n^2 \times 2I$$

$$I(2 + 2\pi^2 n^2) = \frac{2}{2} e^{-i\pi n t}\sinh(t-1) + i\pi n e^{-i\pi n t}\cosh(t-1)$$

$$I = \frac{1}{1+\pi^2 n^2} \times \frac{1}{2}\left[e^{-i\pi n t}\left(\sinh(t-1) + i\pi n \cosh(t-1)\right)\right]_0^2$$

$$= \frac{1}{2(1+\pi^2 n^2)}\left[\sinh(1) + \cancel{i\pi n \cosh(1)} - \sinh(-1) - \cancel{i\pi n \cosh(-1)}\right]$$

$$= \frac{\sinh(1)}{1+\pi^2 n^2}$$

The complex Fourier series is:

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{i\frac{2\pi n t}{T}}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{\sinh(1)}{1+\pi^2 n^2} e^{i\pi n t}$$

Fourier Transforms

Fourier transforms as a limit of Fourier series

Fourier transforms (FTs) are an extension of Fourier series that can be used to describe non periodic functions on an infinite interval. The key idea is to see that a non-periodic function can be viewed as a periodic one, but taking the limit of $L \to \infty$.

Fourier series only includes modes with wavenumbers $k_n = \frac{2n\pi}{L}$ with adjacent modes separated by $\delta k = \frac{2\pi}{L}$.

What happens to our Fourier series if we let $L \longrightarrow \infty$?

Consider the complex Fourier series for $f(x)$:

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{ik_n x}$$

where,

$$C_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-ik_n x}\, dx$$

and the allowed wavenumbers are $k_n = \frac{2n\pi}{L}$.

The separation of adjacent wavenumbers (ie., for $n \rightarrow n+1$) is $\delta k = \frac{2\pi}{L}$

$\underline{\delta k}$

$\delta k = \frac{2\pi}{L}$: As $L \to \infty$, the modes become more and more finely separated in k.

In the limit, we are then interested in the variation of C as a function of the continuous variable k.

- The factor $\frac{1}{L}$ outside the integral looks problematic for taking the limit $L \to \infty$, but this can be evaded by defining a new quantity :

$$\tilde{f}(k) = L \times C(k) = \lim_{L \to \infty} \int_{-L/2}^{L/2} f(x)\, e^{-in\delta k x}\, dx$$

$$L \to \infty,\ \delta k \to 0,\ \lim_{\substack{L \to \infty \\ \delta k \to 0}} n\,\delta k = k$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

The function $\tilde{f}(k)$ ['f tilde', more commonly 'f twiddle' ; f_k is another common notation] is the Fourier transform of the non-periodic function f.

$$\boxed{\tilde{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx}$$

We need the inverse Fourier transform, this gives us back the function $f(x)$ if we know $\tilde{f}$.

We just need to rewrite the Fourier series,

Mode spacing $\delta k = \frac{2\pi}{L}$

$$\cancel{f(x) = \sum_{k_n} c(k)}$$

$$f(x) = \lim_{L \to \infty} \sum_{n=-\infty}^{+\infty} c(\delta k) e^{in\delta k x}$$

$$= \lim_{L \to \infty} \sum_{n=-\infty}^{+\infty} \frac{L}{2\pi} c(\delta k)\, e^{in\delta k x} \cdot \delta k$$

$$\lim_{\delta k \to 0} \sum g(k)\, \delta k = \int g(k)\, dk$$

$$k = n\delta k : -\infty \to +\infty \Longleftarrow n : -\infty \to +\infty$$

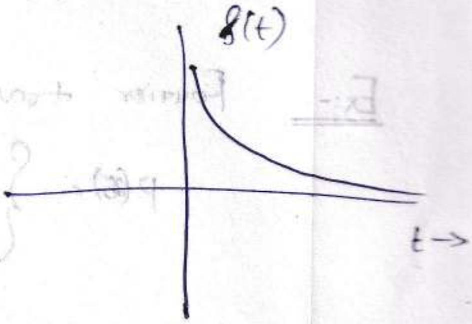
$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(k)\, e^{ikx}\, dk}$$

is the inverse Fourier transform.

Ex:- Find the Fourier transform of the one-sided exponential function

$$f(t) = \begin{cases} 0 & , \ x < 0 \\ e^{-\alpha x} & , \ x > 0 \end{cases}$$

where α : the constant



Ans: $\tilde{f}(k) = \int_0^\infty e^{-\alpha x} e^{-ikx} dx$

$$= \int_0^\infty e^{-(\alpha+ik)x} dx = \left[\frac{e^{-(\alpha+ik)x}}{-(\alpha+ik)} \right]_0^\infty$$

$$= \frac{1}{\alpha+ik}$$

This real function has a complex Fourier transform

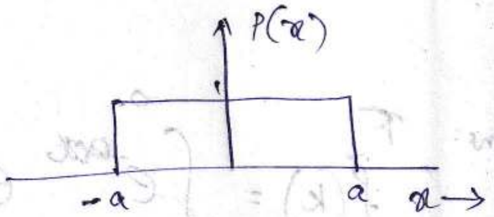
$$\tilde{f}\left(e^{-\alpha x} u(x)\right) = \frac{1}{\alpha+ik} \ , \ \alpha > 0$$

where, $u(t)$ is the Heaviside unit step function:

$$u(t) = \begin{cases} 0 & , \ t < 0 \\ 1 & , \ t > 0 \end{cases}$$

Ex:- Fourier transform of the rectangular pulse

$$p(x) = \begin{cases} 1 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$



Ans:

$$\tilde{f}(k) = \int_{-a}^{a} (1)\, e^{-ikx}\, dx = \left[\frac{e^{-ikx}}{-ik} \right]_{-a}^{a}$$

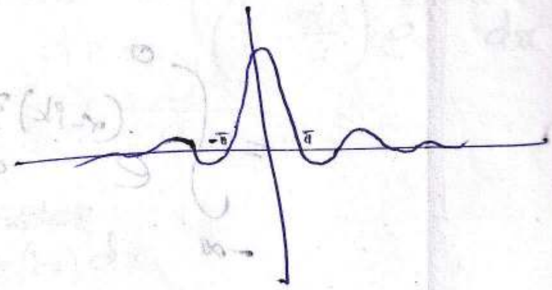
$$= \frac{e^{-ika} - e^{ika}}{-ik} = \frac{2i\sin(ka)}{ik}$$

$$= \frac{2\sin(ka)}{k} = 2a \cdot \frac{\sin(ka)}{(ka)}$$

$\frac{\sin x}{x}$: sinc function.

For $a=1$,

$f_1(k) = \frac{2\sin k}{k}$: even fuction.



Ex:- Fourior transform of the 2 sided exponential function

$$f(x) = \begin{cases} e^{\alpha x} , & t < 0 \\ e^{-\alpha x} , & t > 0. \end{cases}$$

where $\alpha > 0$.

Ans: $$\tilde{f}(k) = \int_{-\infty}^{0} e^{\alpha x} e^{-ikx} dx + \int_{0}^{\infty} e^{-\alpha x} e^{-ikx} dx$$

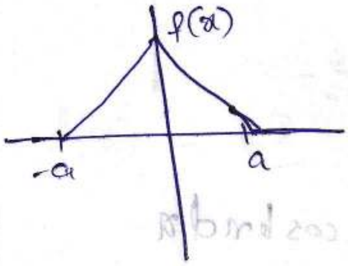
$$= \int_{-\infty}^{0} e^{(\alpha - ik)x} dx + \int_{0}^{\infty} e^{-(\alpha + ik)x} dx$$

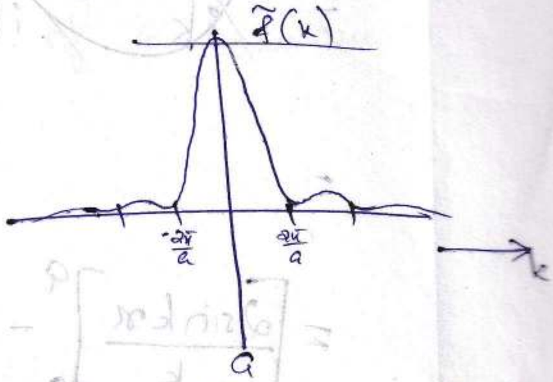
$$= \left[\frac{e^{(\alpha - ik)x}}{\alpha - ik} \right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha + ik)}}{-(\alpha + ik)} \right]_{0}^{\infty}$$

$$= \frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} = \frac{2\alpha}{\alpha^2 + k^2}$$

Ex: Fourier transform of the unit "triangle" function of width $2a$:

$$f(x) = \begin{cases} 1 + \dfrac{x}{a}, & -a \le x < 0 \\ 1 - \dfrac{x}{a}, & 0 \le x \le a \end{cases}$$





Ans: $$\tilde{f}(k) = \int_{-a}^{0} \left(1 + \frac{x}{a}\right) e^{-ikx}\, dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{-ikx}\, dx$$

$$= 2 \int_{0}^{a} \left(1 - \frac{x}{a}\right) \cos(kx)\, dx$$

$$= 2\left[\frac{e^{-ikx}}{-ik}\right]_0^a - \frac{2}{a} \int_0^a x\, e^{-ikx}\, dx$$

$$= \frac{2}{-ik}\left[e^{-ika} - 1\right] - \frac{2}{a}\left\{ x \frac{e^{-ikx}}{-ik} - \int \frac{e^{-ikx}}{-ik}\, dx \right\}$$

$$= \frac{2\left(e^{-ika} - 1\right)}{-ik} - \frac{2}{a}\left[\frac{e^{-ikx}\, x}{-ik} - \frac{e^{-ikx}}{(-ik)^2} \right]_0^a$$

$$= \frac{2\left(e^{-ika}-1\right)}{-ik} - \frac{2}{a}\left[\frac{a e^{-ika}}{-ik} + \frac{e^{-ika}}{k^2} - \frac{1}{k^2}\right]$$

$$= \cancel{\frac{-2e^{-ika}}{ik}} + \cancel{\frac{2e^{-ika}}{ik}} \mp \frac{2}{ik} - \frac{2e^{-ika}}{ak^2} + \frac{2}{ak^2}$$

$$= \left[\frac{2\sin kx}{k}\right]_0^a - \frac{2}{a}\int x\cos kx\,dx$$

$$= \frac{2\sin(ka)}{k} - \frac{2}{a}\left[x\frac{\sin(kx)}{k} - \int \frac{\sin kx}{k}\,dx\right]$$

$$= \frac{2\sin(ka)}{k} - \frac{2}{a}\left[\frac{x\sin(kx)}{k} + \frac{\cos(kx)}{k^2}\right]_0^a$$

$$= \cancel{\frac{2\sin(ka)}{k}} - \frac{2}{a}\left[\cancel{\frac{a\sin(ka)}{k}} + \frac{\cos(ka)}{k^2} - \frac{1}{k^2}\right]$$

$$= \frac{2}{ak^2}\left[1 - \cos ka\right] = 2\times\frac{2\sin^2\left(\frac{ka}{2}\right)}{ak^2}$$

$$= \cancel{a}\times\frac{4\sin^2\left(ka/2\right)}{a^2k^2} = a\,\frac{\sin^2\left(ka/2\right)}{\left(ka/2\right)^2}$$

Ex:-

$f($

Ans: T

on (22) Appendix

$\tilde{f}($

Ex:- Fourier transform of a Gaussian

$$f(x) = e^{-\frac{x^2}{2a^2}}$$

Ans: $$I = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2a^2}} dx = \sqrt{2\pi a^2} = \sqrt{2\pi}\, a$$

on (22) Appendix

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} e^{-x^2/2a^2} \cdot e^{-ikx} dx$$

$$= \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2a^2} + ikx\right)} dx$$

$$\frac{x^2}{2a^2} + ikx = \frac{1}{2a^2}\left(x^2 + 2ika^2 x\right) + \cancel{ikx} - \cancel{ikx}$$

$$= \frac{1}{2a^2}\left(x + ika^2\right)^2 + \frac{1}{2a^2}\left(k^2 a^4\right) + \cancel{ikx} - \cancel{ikx}$$

$$= \frac{1}{2a^2}\left(x + ika^2\right)^2 + \frac{k^2 a^2}{2}$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} e^{-\frac{(x+ika^2)^2}{2a^2}} \cdot e^{-\frac{k^2a^2}{2}}\, dx$$

$$= e^{-\frac{k^2a^2}{2}} \underbrace{\int_{-\infty}^{+\infty} e^{-u^2/2a^2}\, du}_{\sqrt{2\pi}\, a}$$

Set

$u = x + ika^2$

$du = dx$

$$= \sqrt{2\pi}\, a\, e^{-\frac{k^2a^2}{2}}$$

Ex:- The delta function

The Dirac delta function

$$\delta(x) = \begin{cases} +\infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases} \quad \text{--- (1)}$$

not compatible

No such function satisfies both the equations.

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \quad \text{--- (2)}$$

Dirac suggested a way to circumvent this problem is to interpret the above integral as:

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \delta_\epsilon(x)\, dx$$

where, $\delta_\epsilon(x)$ is a generic function of both x & ϵ such that

$$\lim_{\epsilon \to 0^+} \delta_\epsilon(x) = \begin{cases} +\infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases} \quad \text{--- (3)}$$

$$\int_{-\infty}^{+\infty} \delta_\epsilon(x)\, dx = 1 \quad \text{--- (4)}$$

$\Rightarrow$ The Dirac delta function $\delta(x)$ is a "generalized function" (but, strictly speaking, not a function) which satisfy (1) & (2) with the condition that the integral in (2) must be interpreted according to eq^n. (3) where the functions $\delta_\epsilon(x)$ satisfy eq^n. (3) & (4)

There are infinite functions $\delta_\epsilon(x)$ which satisfy eq's (3) & (4).

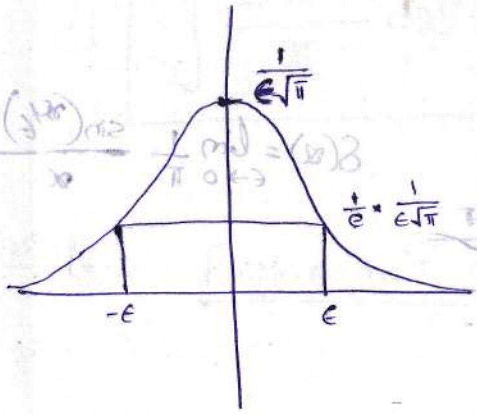
Ex:-

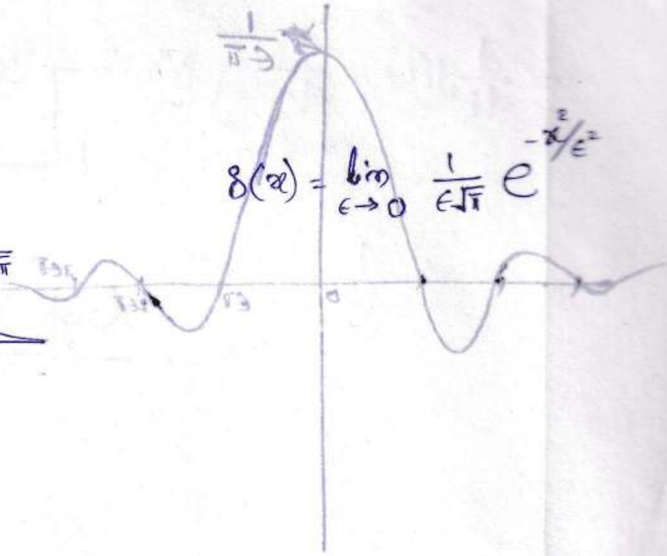
* $\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}$ Gaussian

* $\delta_\epsilon(x) = \begin{cases} 1/\epsilon & \text{for } |x| \le \epsilon/2 \\ 0 & \text{for } |x| > \epsilon/2 \end{cases}$

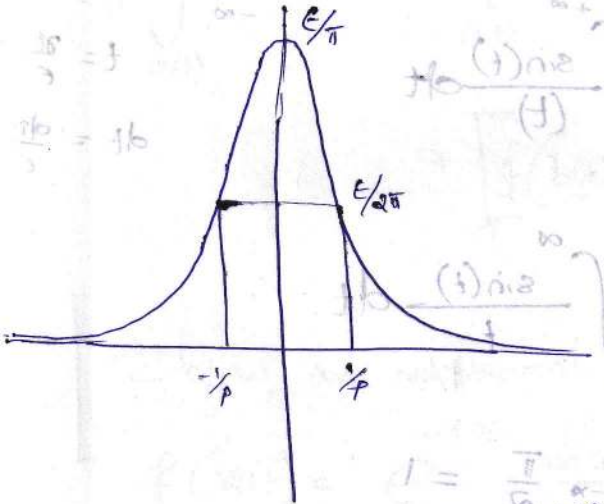
* $\delta_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$ Lorentzian

* $\delta_\epsilon(x) = \frac{1}{\pi} \frac{\sin(x/\epsilon)}{x}$ Dirichlet





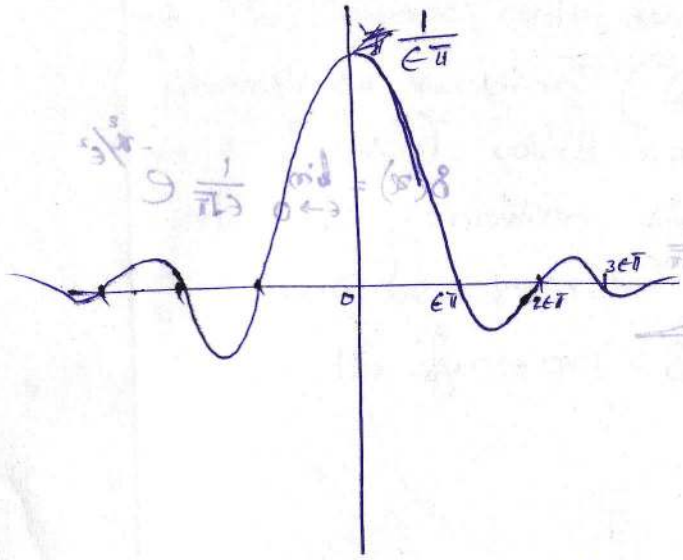
$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon\sqrt{\pi}}\, e^{-x^2/\epsilon^2}$$



$$\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\pi}\,\frac{\epsilon}{x^2+\epsilon^2}$$

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\epsilon\,dx}{x^2+\epsilon^2} = \lim_{\epsilon\to 0}\frac{1}{\pi}\left[\tan^{-1}\left(\frac{x}{\epsilon}\right)\right]_{-\infty}^{+\infty}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \lim_{\epsilon\to 0} 1 = 1$$



$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sin(x/\epsilon)}{x}$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(x/\epsilon)}{x}\, dx = \lim_{\epsilon \to 0} \frac{1}{\epsilon\pi} \int_{-\infty}^{+\infty} \frac{\sin(x/\epsilon)}{x/\epsilon}\, dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(t)}{(t)}\, dt \qquad t = \frac{x}{\epsilon} \qquad dt = \frac{dx}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin(t)}{t}\, dt$$

$$= \lim_{\epsilon \to 0} \frac{2}{\pi} \times \frac{\pi}{2} = 1$$

Method:1

Leibn

$\frac{d}{dt} \int_{a(t)}$

Defi

$f($

and

$$\boxed{\int \frac{\sin(x)}{x}\, dx} \leftarrow ?$$

Method:1 Feynman's trick.

Leibniz rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\, dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\, dx$$

$$+$$

$$\left[f(b(t),t)\frac{d}{dt}b(t) - f(a(t),t)\frac{d}{dt}a(t) \right]$$

Define an additional variable t and let

$$f(x,t) = e^{-tx}\frac{\sin x}{x}$$

and $g(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}\, dx$

$$\frac{d}{dt}g(t) = \frac{d}{dt}\int_0^\infty f(x,t)\,dx$$

$$= \frac{d}{dt}\int_0^\infty e^{-tx}\,\frac{\sin x}{x}\,dx$$

$$= \int_0^\infty \frac{\partial}{\partial t} e^{-tx}\cdot\frac{\sin x}{x}\,dx$$

$$= -\int_0^\infty e^{-tx}\sin x\,dx$$

$$= -\int_0^\infty e^{-tx}\,\frac{e^{ix}-e^{-ix}}{2i}\,dx$$

$$= \frac{-1}{2i}\int_0^\infty\left[e^{-x(t-i)} - e^{-x(t+i)}\right]dx$$

requires proof

interchange of operations: limits / integrals

$$= \frac{-1}{2i}\left[\frac{-1}{t-i}e^{-x(t-i)} + \frac{1}{t+i}e^{-x(t+i)}\right]_0^\infty$$

$$= \frac{-1}{2i}\left[0-\left(\frac{-1}{t-i}+\frac{1}{t+i}\right)\right]$$

$$= \frac{-1}{2i}\left[\frac{1}{t-i} - \frac{1}{t+i}\right]$$

$$\cancel{\frac{dg}{dt}} = \frac{-1}{2i}\left[\frac{\cancel{t}+i-\cancel{t}+i}{t^2+1}\right] = \frac{-1}{t^2+1}$$

$$\frac{dg(t)}{dt} = \frac{d}{dt}\left[\int_0^\infty \cancel{f(x)}\, e^{-tx} \cdot \frac{\sin x}{x}\, dx\right] = \frac{-1}{t^2+1}$$

$$g(t) = \int \frac{-dt}{t^2+1} = -\tan^{-1}(t) + C$$

requires proof

Interchange of operations i.e. limits of integrals

$$\lim_{t\to\infty} g(t) = 0 \Longrightarrow C = \lim_{t\to\infty} \tan^{-1}(t) = \pi/2.$$

$$g(t) = -\tan^{-1}(t) + \pi/2$$

$$g(0) = \frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}\, dx$$

Method: 2

$$\int_0^\infty e^{-xt}\,dt = \frac{1}{x}$$

$$\int_0^{+\infty} \frac{\sin x}{x}\,dx = \int_0^\infty \int_0^\infty e^{-xt} \sin x \, dx\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-xt}\, \frac{e^{ix} - e^{-ix}}{2i}\, dx\, dt$$

$$= \frac{1}{2i} \int_0^\infty \int_0^\infty \left[e^{-x(t-i)} - e^{-x(t+i)} \right] dx\, dt$$

$$= \frac{1}{2i} \int_0^\infty \left[\frac{-1}{t-i} e^{-x(t-i)} + \frac{1}{t+i} e^{-x(t+i)} \right]_0^\infty dt$$

$$= \frac{1}{2i} \int_0^\infty \left[0 - \left(\frac{-1}{t-i} + \frac{1}{t+i} \right) \right] dt$$

$$= \frac{1}{2i} \int_0^\infty \frac{2i}{t^2+1}\, dt$$

$$= \frac{2}{2}\int_0^\infty \frac{dt}{t^2+1} = \tan^{-1} t\Big]_0^\infty$$

$$= \pi/2$$

Dirac delta continue.

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = \lim_{\epsilon\to 0}\int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon}\,dx = \frac{1}{\epsilon}\,\epsilon \;\lim_{\epsilon\to 0} 1 = 1$$

Shifting property of Dirac delta function.

For any function $f(x)$ continuous at x_0,

$$\boxed{\int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)}$$

This gives a sense of measure to the Dirac delta function. - it measures the value of $f(x)$ at the point x_0.

Proof

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x)$$

where,
$$\delta_\epsilon(x) = \begin{cases} 1/\epsilon & \text{for } |x| \le \epsilon/2 \\ 0 & \text{for } |x| > \epsilon/2 \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = \lim_{\epsilon \to 0} \int_{x_0-\epsilon/2}^{x_0+\epsilon/2} f(x)\,\delta_\epsilon(x-x_0)\,dx$$

$$= \lim_{\epsilon \to 0} \int_{x_0-\epsilon/2}^{x_0+\epsilon/2} f(x) \cdot \frac{1}{\epsilon}\,dx$$

Over this very small range of x, the function $f(x)$ can be thought to be constant. and can be taken out of the integral.

$$= f(x_0) \lim_{\epsilon \to 0} \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} \frac{1}{\epsilon}\, dx$$

$$= f(x_0) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \times \epsilon$$

$$= f(x_0) \lim_{\epsilon \to 0} 1 = f(x_0).$$

⟹ Any function $\delta(x - x_0)$ that satisfies the shifting property is the Dirac delta function.

$$\boxed{\int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx = f(0)}$$

Scaling property.

$$\boxed{\delta(ax) = \frac{\delta(x)}{|a|}} \Longrightarrow |a|\,\delta(ax) = \delta(x)$$

with, $a \neq 0$

Proof.

(i) $$|a|\,\delta(ax) = \begin{cases} |a|(+\infty) & \text{for } ax = 0 \\ |a|\,(0) & \text{for } ax \neq 0. \end{cases}$$

$$\Longrightarrow |a|\,\delta(ax) = \begin{cases} +\infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$$

(iii) For $a>0$,

$$\int_{-\infty}^{+\infty} |a|\delta(ax)\,dx = \int_{-\infty}^{+\infty} |a|\,\delta(t)\,\frac{dt}{a}$$

put $t = ax$

$dt = a\,dx$

$x: -\infty \to +\infty$

$t = ax: -\infty \to +\infty$

$$= \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

For $a<0$,

$x: -\infty \to +\infty$

$t = ax: +\infty \to -\infty$

$$\int_{-\infty}^{+\infty} |a|\delta(ax)\,dx = \int_{+\infty}^{-\infty} |a|\;\delta(t)\,\frac{dt}{a}$$

$$= \int_{+\infty}^{-\infty} -a\cdot\delta(t)\cdot\frac{dt}{a}$$

$$= \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

$$\Longrightarrow \quad |a|\,\delta(ax) = \delta(x)$$

$$\boxed{\delta(-x) = \delta(x)}$$

$$\boxed{\delta(x^2-a^2) = \frac{1}{2|a|}\Big(\delta(x-a) + \delta(x+a)\Big) \quad \Updownarrow \quad \int_{-\infty}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx = \frac{1}{2|a|}\Big(f(a) - f(-a)\Big)}$$

Proof

Set $x=\sqrt{y} \Longrightarrow dx = \frac{dy}{2\sqrt{y}}$

$\cancel{x^2 = y \Longrightarrow 2x\,dx = dy}$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx = \int_{-\infty}^{0} f(x)\,\delta(x^2-a^2)\,dx + \int_{0}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx$$

$$= \int_{0}^{\infty} f(-x)\,\delta(x^2-a^2)\,dx + \int_{0}^{\infty} f(x)\,\delta(x^2-a^2)\,dx$$

$$= \int_{0}^{\infty} \cancel{f(-\sqrt{y})\,\delta(y-a^2)\frac{dy}{2\sqrt{y}}} + \int_{0}^{\infty} f(\sqrt{y})\,\delta(y-a^2)\frac{dy}{2\sqrt{y}}$$

$$\boxed{\begin{gathered}\delta(x^2-a^2)=\frac{1}{2|a|}\Big(\delta(x-a)+\delta(x+a)\Big)\\ \Updownarrow \\ \int_{-\infty}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx=\frac{1}{2|a|}\Big(f(a)-f(-a)\Big)\end{gathered}}$$

Proof

$$x=\sqrt{y} \implies dx=\frac{dy}{2\sqrt{y}}$$

Set ~~$x^2=y \implies 2x\,dx=dy$~~

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx=\int_{-\infty}^{0} f(x)\,\delta(x^2-a^2)\,dx+\int_{0}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx$$

$$=\int_{0}^{\infty} f(-x)\,\delta(x^2-a^2)\,dx+\int_{0}^{\infty} f(x)\,\delta(x^2-a^2)\,dx$$

$$=\int_{0}^{\infty} f(-\sqrt{y})\,\delta(y-a^2)\,\frac{dy}{2\sqrt{y}}+\int_{0}^{\infty} f(\sqrt{y})\,\delta(y-a^2)\,\frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2}\int_{0^+}^{\infty} \left[\frac{f(-\sqrt{x}) + f(\sqrt{x})}{\sqrt{x}}\right] \delta(x-a^2)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x^2-a^2)\,dx = \frac{1}{2}\left[\frac{f(-a)+f(a)}{|a|}\right] = \frac{f(a)+f(-a)}{2|a|}$$

$$\Rightarrow \quad \delta(x^2-a^2) = A\,\delta(x-a) + B\,\delta(x+a)$$

$$\int_{-\infty}^{+\infty} f(x)\left[A\,\delta(x-a) + B\,\delta(x+a)\right] dx = A\,f(a) + B\,f(-a)$$

$$\frac{f(a)+f(-a)}{2|a|} = A\,f(a) + B\,f(-a)$$

$$A = B = \frac{1}{2|a|}$$

$$\delta(x^2-a^2) = \frac{1}{2|a|}\Big(\delta(x-a) + \delta(x+a)\Big)$$

In general,

$$\boxed{\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|} \Longleftrightarrow \int_{-\infty}^{+\infty} f(x)\,\delta(g(x))\,dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}}$$

Composition rule

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} \delta(x - x_0)\, e^{-ikx}\, dx = e^{-ikx_0}$$

when $x_0 = 0$, $\tilde{f}(k) = 1$

Taking the inverse FT

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx_0}\, e^{ikx}\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(x - x_0)}\, dk$$

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk}$$

Discrete Fourier Transform (DFT)

The DFT is the equivalent of the continuous Fourier transform for functions that are sampled discretely at N equally spaced points x_n separated by $\Delta x = L/N$ such that $x_n = n \Delta x = n \left(L/N\right)$.

The Fourier transform of the original signal $f(x)$ would be

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} f(x) \, e^{-ikx} \, dx$$

Let N samples be

$$f[x_n] = f[0], f[1], \ldots\ldots, f[N-1]$$

$$L = N\Delta x \longrightarrow \Delta x = L/N$$

$$x_n = n\Delta x = n(L/N)$$

$k_m = m\,\delta k$, where $\delta k = \frac{2\pi}{L}$ & $k_m = \frac{2m\pi}{L}$

Given function $f(x)$ is zero outside the region $x \in [0, N]$, we can express its Fourier transform as :

$$\tilde{f}(k) = \int_0^N f(x)\, e^{-ikx}\, dx$$

The integral can be approximated by summing over the N-sampled values:

$$\tilde{f}(k_m) \approx \sum_{n=0}^{N-1} f(x_n)\, e^{-ik_m x_n}\, \Delta x$$

$$= \sum_{n=0}^{N-1} f(x_n)\, e^{-ik_m x_n} \cdot \left(\frac{L}{N}\right)$$

$$= \frac{L}{N} \sum_{n=0}^{N-1} f(x_n)\, e^{-ik_m x_n}$$

The discrete Fourier transform (DFT) of the sequence x_n is given by

$$\boxed{f[k_m] = \sum_{n=0}^{N-1} f[x_n]\, e^{-ik_m x_n}}$$

The value of DFT is (up to the constant of proportionality L/N) an approximation of the value of the Fourier transform at the frequency n/L.

$$e^{-ik_m x_n} = e^{-i\frac{2m\pi}{L}\cdot n\Delta x} = e^{-i\cdot\frac{2m\pi}{\cancel{L}}\cdot n\cdot\frac{\cancel{L}}{N}}$$

$$= e^{-i\frac{2m\pi}{N}n}$$

$$\boxed{\begin{aligned} f[k_m] &= \sum_{n=0}^{N-1} f[x_n]\, e^{-ik_m x_n} \\ &= \sum_{n=0}^{N-1} f[x_n]\, e^{-i\frac{2m\pi}{L}x_n} \\ &= \sum_{n=0}^{N-1} f[x_n]\, e^{-i\frac{2m\pi}{N}n} \end{aligned}}$$

Similarly,

the inverse Fourier transform can be expressed as:

$$f(x) = \frac{1}{2\pi}\int_0^N \tilde{f}(k)\, e^{ikx}\, dk$$

the integral can be approximated by summing over the N-sampled values:

$$f(x_n) \approx \frac{1}{2\pi}\sum_{m=0}^{N-1} f(k_m)\, e^{ik_m x_n}\, \delta k$$

$$= \frac{1}{2\pi}\sum_{m=0}^{N-1} f(k_m)\, e^{ik_m x_n} \cdot \frac{2\pi}{L}$$

$$\boxed{= \frac{1}{L}\sum_{m=0}^{N-1} f(k_m)\, e^{ik_m x_n}}$$

Substitute, $f(k_m) = \frac{L}{N} f[k_m]$

$$\cancel{f[x_n] = \sum_{m=0}^{N-1} f[k_m]}$$

$$f[x_n] = \frac{1}{\cancel{L}} \sum_{m=0}^{N-1} \frac{\cancel{L}}{N} f[k_m]\, e^{i k_m x_n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f[k_m]\, e^{i k_m x_n}$$

The inverse DFT is:

$$\boxed{f[x_n] = \frac{1}{N} \sum_{m=0}^{N-1} f[k_m]\, e^{i k_m x_n}}$$

$$\boxed{\begin{aligned} f[x_n] &= \frac{1}{N}\sum_{m=0}^{N-1} f[k_m]\, e^{i k_m x_n} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} f[k_m]\, e^{i \frac{2m\pi}{L} x_n} \\ &= \frac{1}{N}\sum_{m=0}^{N-1} f[k_m]\, e^{i \frac{2m\pi}{N} n} \end{aligned}}$$

Change of Basis

DFT of x_n'

$$f[k_m] = \sum_{n=0}^{N-1} f[x_n]\, e^{-i\frac{2\pi m}{N}n}$$

The IDFT is :

$$f[x_n] = \frac{1}{N}\sum_{k_m=0}^{N-1} f[k_m]\, e^{i\frac{2\pi m}{N}n}$$

A discrete function of finite duration consisting of N-samples can be thought of as a vector in $\mathbb{C}^N$.

$$f[x_n] = f[0]e_0 + f[1]e_1 + \cdots\cdots + f[N-1]e_{N-1}$$

$$= \sum_{k=0}^{N-1} f[k]\,\delta[x_n - k]$$

$$\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{bmatrix} = \begin{bmatrix} f[0] \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f[1] \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f[2] \\ \vdots \\ 0 \end{bmatrix} + \cdots\cdots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f[N-1] \end{bmatrix}$$

Real space basis functions: $e_{x_k} = \delta(x - x_k)$

$$C_k = \left(e_{x_k}, f(x)\right) = \int_{-\infty}^{+\infty} f(x)\, \delta(x - x_k)\, dx = f(x_k)$$

$$\left(e_{x_0}, e_{x_1}\right) = \int_{-\infty}^{+\infty} \delta^*(x - x_0)\, \delta(x - x_1)\, dx$$

$$= \int_{-\infty}^{+\infty} \delta(x - x_0)\, \delta(x - x_1)\, dx = \delta(x_0 - x_1)$$

$\Longrightarrow$ $e_{x_k} = \delta(x - x_k)$ are orthonormal.

The DFT is used to represent ~~this~~ the vector in $\mathbb{C}^N$, corresponds to a discrete function of finite length consisting of N samples, as a linear combination of vectors $v_k \in \mathbb{C}^N$ of the form:

$$v_k = \left(1, e^{i\frac{2\pi k}{N}}, e^{i\frac{2\pi(2k)}{N}}, \dots, e^{i\frac{2\pi(N-1)k}{N}}\right)$$

$$k = 0, 1, \dots, N-1$$

$$\langle v_l, v_k\rangle = \sum_{n=0}^{N-1} e^{i\frac{2\pi l n}{N}} \overline{e^{i\frac{2\pi k n}{N}}} = \sum_{n=0}^{N-1} e^{i\frac{2\pi(l-k)n}{N}}$$

$$= \frac{1 - e^{i2\pi(l-k)}}{1 - e^{i\frac{2\pi(l-k)}{N}}} = \begin{cases} 0 & \text{for } l \neq k \\ N & \text{for } l = k \end{cases}$$

$$f[k_m] = \sum_{n=0}^{N-1} f[x_n]\, e^{-ik_m x_n} = \sum_{n=0}^{N-1} f[x_n]\, e^{-i\frac{2m\pi}{L}x_n}$$

$$= \sum_{n=0}^{N-1} f[x_n]\, e^{-i\frac{2m\pi}{N}n} \quad \Big| \; n = 0, 1, \dots, N-1$$

$$f[x_n] = \frac{1}{N}\sum_{m=0}^{N-1} f[k_m]\, e^{ik_m x_n} = \frac{1}{N}\sum_{m=0}^{N-1} f[k_m]\, e^{i\frac{2m\pi}{N}n}$$

$$\mathrm{DFT}(f(x_n)) = \tilde{f}(k) = \tilde{C}_k = \sum_{n=0}^{N-1} f(x_n) e^{-i\frac{2\pi k}{N}n}$$

$$k = 0, 1, \ldots, N-1$$

$$\tilde{f}(k) = \langle x, v_k \rangle$$

The vectors $\{v_k\}_{k=0}^{N-1}$ form an orthogonal basis for $\mathbb{C}^N$, and the DFT can be understood as computing the coefficients for representing a vector in this basis.

Let $\tilde{C}_k$ be the coefficients of x in the basis $\{v_k\}_{k=0}^{N-1}$ so that

$$x = \tilde{c}_0 v_0 + \tilde{C}_1 v_1 + \cdots + C_{N-1} v_{N-1}$$

$$\langle x, v_k \rangle = C_k \langle v_k, v_k \rangle$$

$\langle v_k, v_k \rangle = N$ and $\langle x, v_k \rangle = \tilde{f}(k)$

$$\Longrightarrow \quad \tilde{c}_k = \frac{\tilde{f}(k)}{N}$$

$$\therefore \quad x = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{f}(k)\, v_k$$

Discrete Inverse Fourier transform (IDFT).

The N-point discrete Fourier transform (N-pt DFT) is a linear map from $\mathbb{C}^N$ to $\mathbb{C}^N$.

$$f[x_n] = \frac{1}{N}\sum_{m=0}^{N-1} \tilde{f}[k_m]\, e^{i\frac{2\pi m}{N}n} = \sum_{m=0}^{N-1} \tilde{c}_k\, e^{i\frac{2\pi m}{N}n}$$

$$\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{bmatrix} = \tilde{c}_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \tilde{c}_1 \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{N}} \\ e^{i\frac{4\pi}{N}} \\ \vdots \\ e^{i\frac{2\pi(N-1)}{N}} \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} 1 \\ e^{i\frac{4\pi}{N}} \\ e^{i\frac{8\pi}{N}} \\ \vdots \\ e^{i\frac{4\pi(N-1)}{N}} \end{bmatrix} + \cdots\cdots$$

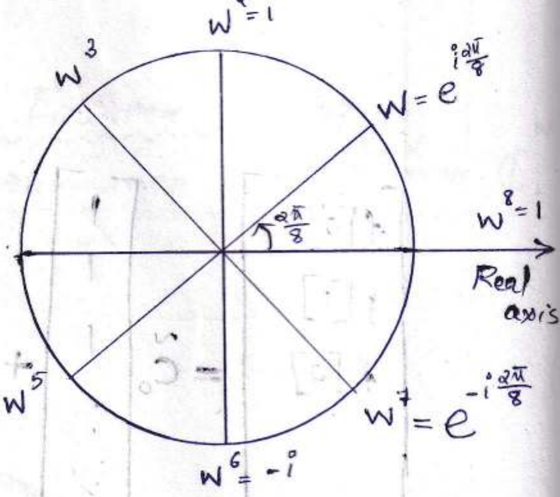
$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_{N-1} \end{bmatrix} = FC$$

where, $W = e^{i\frac{2\pi}{N}}$, is the N^{th} root of unity

and $W^N = W^{2N} = \cdots = 1$

Ex:-

There are 8 solutions to $z^8 = 1$. Their spacing is $45^\circ = \frac{2\pi}{8}$ rad.



$w^N = 1 \Longrightarrow$ ~~$w^{N-1} = 0$~~ $1 - w^N = 0$

~~$w^N - 1 = (w-1)$~~ $1 - w^N = (1-w)\left(1 + w + w^2 + \cdots + w^{N-1}\right) = 0$

$\Longrightarrow 1 + w + w^2 + \cdots + w^{N-1} = 0$ & $w^N = 1$

$\Rightarrow$ The columns of F are orthogonal.

$$
Fc = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_{N-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} \tilde{c}_0 + \tilde{c}_1 + \tilde{c}_2 + \cdots + \tilde{c}_{N-1} \\ \tilde{c}_0 + \tilde{c}_1 W + \tilde{c}_2 W^2 + \cdots + \tilde{c}_{N-1} W^{N-1} \\ \tilde{c}_0 + \tilde{c}_1 W^2 + \tilde{c}_2 W^4 + \cdots + \tilde{c}_{N-1} W^{2(N-1)} \\ \vdots \\ \tilde{c}_0 + \tilde{c}_1 W^{N-1} + \tilde{c}_2 W^{2(N-1)} + \cdots + \tilde{c}_{N-1} W^{(N-1)^2} \end{bmatrix}
$$

* F is symmetric $(F^T = F)$, but not Hermitian.

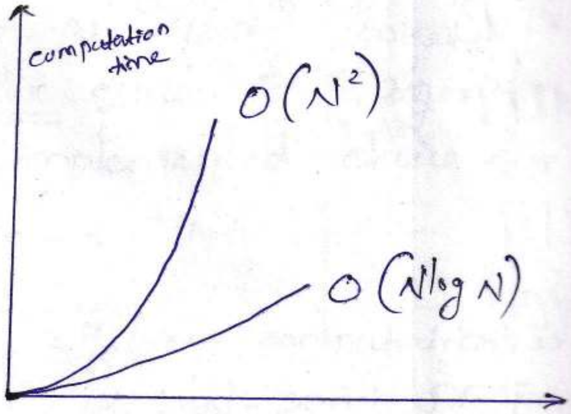
$\frac{1}{2}F$ is a unitary matrix, i.e., $\left(\frac{1}{2}F^H\right)\left(\frac{1}{2}F\right) = I$

$$F^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W^{-1} & W^{-2} & \cdots & W^{-(N-1)} \\ 1 & W^{-2} & W^{-4} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{-(N-1)} & W^{-2(N-1)} & \cdots & W^{-(N-1)^2} \end{bmatrix} = \frac{\overline{F}}{N}$$

Fast Fourier Transform (FFT)

The Fast Fourier Transform (FFT) is an efficient $O(N \log N)$ algorithm for calculating DFTs.

- originally discovered by Gauss in the early 1800's.
- rediscovered by Cooley and Tukey at IBM in the 1960's.



N	10	100	1000	10^6	10^9
N^2	100	10^4	10^6	10^{12}	10^{18}
$N \log N = \log N^2$	1	200	3000	6×10^6	9×10^9

FFT → Decimation-In-Time (DIT) algorithm

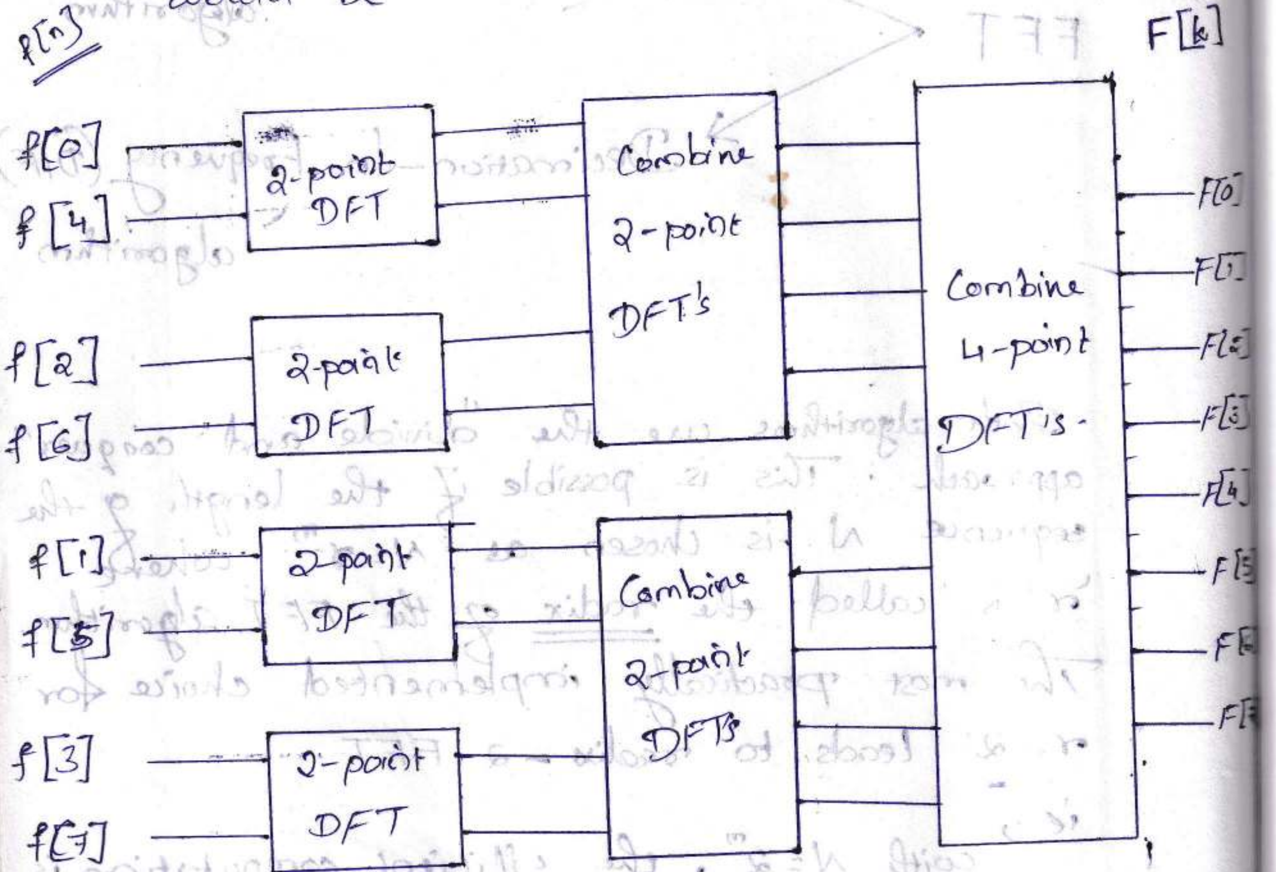
FFT → Decimation-In-Frequency (DIF) algorithm

Both algorithms use the "divide and conquer" approach. This is possible if the length of the sequence N is chosen as $N = r^m$, where 'r' is called the radix of the FFT algorithm. The most practically implemented choice for $r = 2$ leads to radix-2 FFT.

ie.,

with $N = 2^m$, the efficient computation is achieved by breaking the N-point DFT into two $\frac{N}{2}$-point DFTs, then breaking each $\frac{N}{2}$-point DFT into two $\frac{N}{4}$-point DFTs and continuing this process until 2-point DFTs are obtained.

For $N = 2$, the D.I.T algorithm decomposition would be



A Decimation-In-Time (DIT) algorithm

The time-domain sequence $f[n]$ is decimated into two $\frac{N}{2}$-point sequences, one composed of even indexed values of $f[n]$, and the other composed of odd indexed values of $f[n]$. ie.,

$$g[n] = f[2n]$$
$$h[n] = f[2n+1]$$

The N-point DFT of $f[n]$ is given by

$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-i\frac{2m\pi}{N}n}$$

$$= \sum_{n=0}^{N-1} f[n] W_N^{nm} \longrightarrow \text{Twiddle factor}$$

$$F[k_m] = \sum_{n=0}^{N-1} f[n]\, W_N^{nm}$$

$$= \sum_{n,\,\text{even}} f[n]\, W_N^{nm} + \sum_{n,\,\text{odd}} f[n]\, W_N^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n]\, W_N^{2nm} + \sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\, \underbrace{W_N^{(2n+1)m}}_{W_N^{2nm} \times W_N^{m}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n]\, W_N^{2nm} + \sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\, W_N^{2nm} \cdot W_N^{m}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n]\, W_N^{2nm} + W_N^{m} \sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\, W_N^{2nm}$$

$$W_N^{2nm} = e^{-i\frac{2m\pi}{N}(2n)} = e^{-i\frac{2m\pi}{N/2}n} = W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n]\, W_{N/2}^{nm} + W_N^{m} \sum_{n=0}^{\frac{N}{2}-1} f[2n+1]\, W_{N/2}^{nm}$$

$$F[k_m] = \sum_{n=0}^{\frac{N}{2}-1} g[n]\, W_{N/2}^{nm} + W_N^{m} \sum_{n=0}^{\frac{N}{2}-1} h[n]\, W_{N/2}^{nm}$$

$$= G[k_m] + W_N^{m}\, H[k_m]$$

where,

$G[k_m]$ and $H[k_m]$ are the $\frac{N}{2}$-point DFTs of $g[n]$ and $h[n]$ respectively.

$$W_N^{m+\frac{N}{2}} = e^{-i\frac{2\left(m+\frac{N}{2}\right)\pi}{N}} = e^{-i\frac{2m\pi}{N}} * e^{-i\pi} = -e^{-i\frac{2m\pi}{N}} = -W_N^{m}$$

$$W_{N/2}^{n\left(m+\frac{N}{2}\right)} = e^{-i\frac{2\left(m+\frac{N}{2}\right)\pi}{N/2}n} = e^{-i\frac{2m\pi}{N/2}n} * e^{-i2\pi n} = e^{-i\frac{2m\pi}{N/2}n} * 1$$

$$= W_{N/2}^{nm}$$

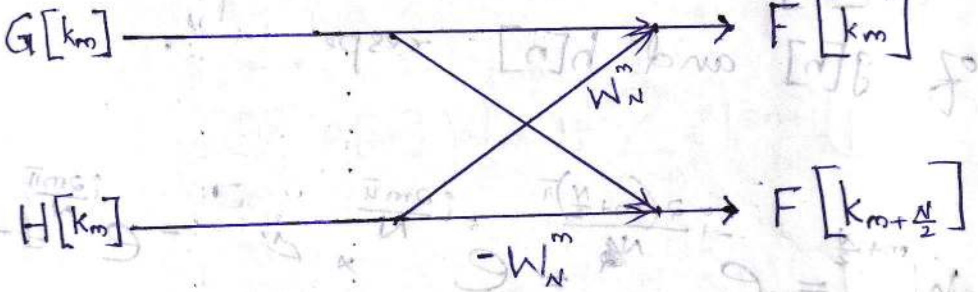
$$F\left[k_{m+\frac{N}{2}}\right] = \sum_{n=0}^{\frac{N}{2}-1} g[n]\, W_{N/2}^{n\left(m+\frac{N}{2}\right)} + W_N^{m+\frac{N}{2}} \sum_{n=0}^{\frac{N}{2}-1} h[n]\, W_{N/2}^{n\left(m+\frac{N}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g[n]\, W_{N/2}^{nm} - W_N^{m} \sum_{n=0}^{\frac{N}{2}-1} h[n]\, W_{N/2}^{nm}$$

$$= G[k_m] - W_N^{m}\, H[k_m]$$

$$F[k_m] = G[k_m] + W_N^m \, H[k_m]$$

$$F[k_{m+\frac{N}{2}}] = G[k_m] - W_N^m \, H[k_m]$$

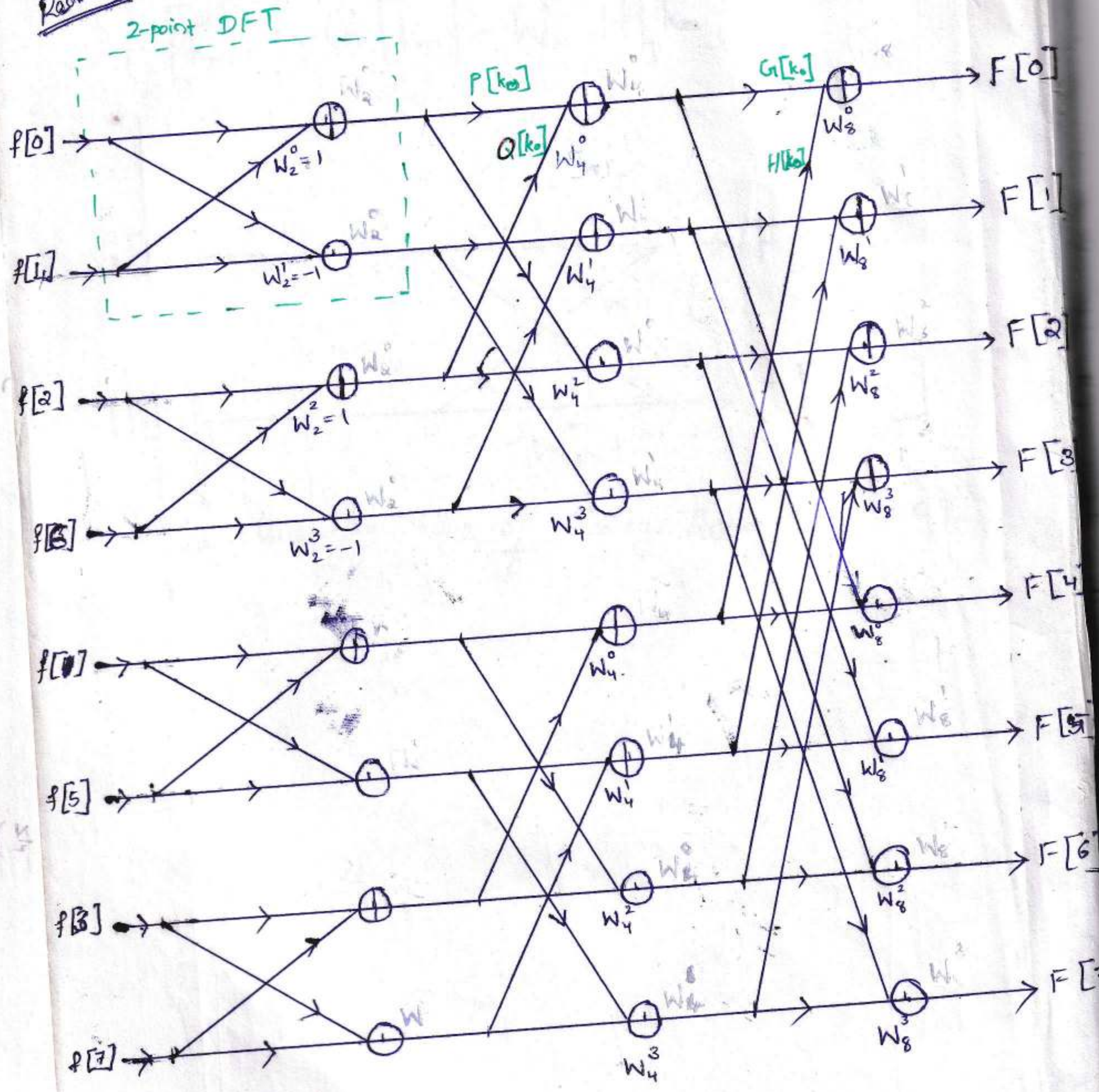


* The Butterfly stage of Cooley-Tukey DIT FFT

The process is repeated for calculating the $\frac{N}{2}$-point DFTs of $g[n]$ and $h[n]$, and this is continued till we get 2-point DFTs.

* The overall butterfly diagram for DIT FFT algorithm for $N=8$ is:

Radix-2



$$F[k_m] = G[k_m] + W_N^m H[k_m]$$

$\frac{N}{2}$-pt DFT of {0,2,4,6}

$\frac{N}{2}$-pt DFT of {1,3,5,7}

N-pt DFT of {0,1,2,3,4,5,6,7}.

$$\boxed{W_N^m = e^{-2\pi i m/N}}$$

$$G[k_m] = P[k_m] + W_{N/2}^m Q[k_m]$$

$\frac{N}{4}$-pt DFT of {0,4}

$\frac{N}{4}$-pt DFT of {2,6}.

$\frac{N}{2}$-pt DFT of {0,2,4,6}

+

$$W_N^m H[k_m] = W_N^m \left(R[k_m] + W_{N/2}^m S[k_m]\right)$$

$\frac{N}{4}$-pt DFT of {1,3}

$\frac{N}{4}$-pt DFT of {5,7}

$\frac{N}{2}$-pt DFT of {1,3,5,7}

F

$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-2\pi i m n/N} = \sum_{n=0}^{N-1} f[n] W_N^{nm}$$

$$= \sum_{\substack{n=0 \\ (\text{even } n)}}^{\frac{N}{2}-1} f[2n] W_N^{2nm} + \sum_{\substack{n=0 \\ (\text{odd } n)}}^{\frac{N}{2}-1} f[2n+1] W_N^{(2n+1)m}$$

$$W_N^{2nm} = e^{-2\pi i m(2n)/N} = e^{-2\pi i m n/\frac{N}{2}} = W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_{N/2}^{nm} + W_N^m \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_{N/2}^{nm}$$

⟹ Any DFT can be constructed from the sum of 2 smaller DFTs. This implies that we can attack the problem using the "divide & conquer" approach.

F

$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-2\pi i m n/N} = \sum_{n=0}^{N-1} f[n] W_N^{nm}$$

$$= \underbrace{\sum_{n=0}^{\frac{N}{2}-1} f[2n] W_N^{2nm}}_{(\text{even } n)} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_N^{(2n+1)m}}_{(\text{odd } n)}$$

$$W_N^{2nm} = e^{-2\pi i m(2n)/N} = e^{-2\pi i m n/\frac{N}{2}} = W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_{N/2}^{nm} + W_N^{m} \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_{N/2}^{nm}$$

⇒ Any DFT can be constructed from the sum of 2 smaller DFTs. This implies that we can attack the problem using the "divide & conquer" approach.

Ex:- <u>N=8 point DFT</u>

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ F(4) \\ F(5) \\ F(6) \\ F(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\ 1 & W^2 & W^4 & W^6 & W^8 & W^{10} & W^{12} & W^{14} \\ 1 & W^3 & W^6 & W^9 & W^{12} & W^{15} & W^{18} & W^{21} \\ 1 & W^4 & W^8 & W^{12} & W^{16} & W^{20} & W^{24} & W^{28} \\ 1 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} & W^{30} & W^{35} \\ 1 & W^6 & W^{12} & W^{18} & W^{24} & W^{30} & W^{36} & W^{42} \\ 1 & W^7 & W^{14} & W^{21} & W^{28} & W^{35} & W^{42} & W^{49} \end{bmatrix} \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \\ f[4] \\ f[5] \\ f[6] \\ f[7] \end{bmatrix}$$

Reordering the matrix to the DIT split in the equation, separating the even & odd index samples:

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \\ F[4] \\ F[5] \\ F[6] \\ F[7] \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^2 & W^4 & W^6 & W & W^3 & W^5 & W^7 \\ 1 & W^4 & W^8 & W^{12} & W^2 & W^6 & W^{10} & W^{14} \\ 1 & W^6 & W^{12} & W^{18} & W^3 & W^9 & W^{15} & W^{21} \\ \hline 1 & W^8 & W^{16} & W^{24} & W^4 & W^{12} & W^{20} & W^{28} \\ 1 & W^{10} & W^{20} & W^{30} & W^5 & W^{15} & W^{25} & W^{35} \\ 1 & W^{12} & W^{24} & W^{36} & W^6 & W^{18} & W^{30} & W^{42} \\ 1 & W^{14} & W^{28} & W^{42} & W^7 & W^{21} & W^{35} & W^{49} \end{array}\right] \left[\begin{array}{c} f[0] \\ f[2] \\ f[4] \\ f[6] \\ \hline f[1] \\ f[3] \\ f[5] \\ f[7] \end{array}\right]$$

Applying the same reordering to each 4×4 blocks :

$$
\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \\ F[4] \\ F[5] \\ F[6] \\ F[7] \end{bmatrix}
=
\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W^{4} & W^{2} & W^{6} & W & W^{5} & W^{3} & W^{7} \\
\hline
1 & W^{8} & W^{4} & W^{12} & W^{2} & W^{10} & W^{6} & W^{14} \\
1 & W^{12} & W^{6} & W^{18} & W^{3} & W^{15} & W^{9} & W^{21} \\
\hline
1 & W^{16} & W^{8} & W^{24} & W^{4} & W^{20} & W^{12} & W^{28} \\
1 & W^{20} & W^{10} & W^{30} & W^{5} & W^{25} & W^{15} & W^{35} \\
\hline
1 & W^{24} & W^{12} & W^{36} & W^{6} & W^{30} & W^{18} & W^{42} \\
1 & W^{28} & W^{14} & W^{42} & W^{7} & W^{35} & W^{21} & W^{49}
\end{array}\right]
\left[\begin{array}{c} f[0] \\ f[4] \\ \hline f[2] \\ f[6] \\ \hline f[1] \\ f[5] \\ \hline f[3] \\ f[7] \end{array}\right]
$$

$$W^{n} = W^{n+Nk}$$

$$W^{n} = -W^{n+N/2}$$

$$W^{nk} = 1$$

$$W^{N} = 1$$

$$W^{N/2} = e^{-i\pi} = -1$$

$$
\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \\ F[4] \\ F[5] \\ F[6] \\ F[7] \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & \omega^2 & -\omega^2 & \omega & -\omega & \omega^3 & -\omega^3 \\
1 & 1 & -1 & -1 & \omega^2 & \omega^2 & -\omega^2 & -\omega^2 \\
1 & -1 & -\omega^2 & \omega^2 & \omega^3 & -\omega^3 & \omega & -\omega \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & \omega^2 & -\omega^2 & -\omega & \omega & -\omega^3 & \omega^3 \\
1 & 1 & -1 & -1 & -\omega^2 & -\omega^2 & \omega^2 & \omega^2 \\
1 & -1 & -\omega^2 & \omega^2 & -\omega^3 & \omega^3 & -\omega & \omega
\end{bmatrix}
\begin{bmatrix} f[0] \\ f[4] \\ f[2] \\ f[6] \\ f[1] \\ f[5] \\ f[3] \\ f[7] \end{bmatrix}
$$

If

f[x

cor

of

Index

0

1

2

3

4

5

6

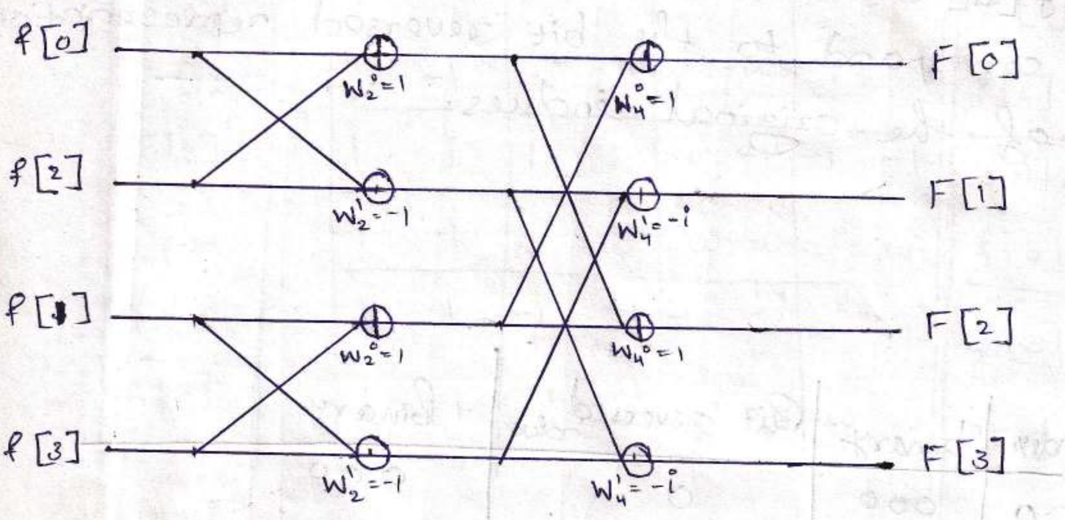
7

If we represent the indices of the $f[x_n]$ vector as Binary numbers, they correspond to the bit reversed representation of the original indices.

Index	binary	Bit reversed index	Binary
0	000	0	000
1	001	4	100
2	010	2	010
3	011	6	110
4	100	1	001
5	101	5	101
6	110	3	011
7	111	7	111

Ex:-

DIT FFT algorithm for $N=4$



$$\boxed{W_N^m = e^{-2\pi i m/N}}$$

$$F[k_m] = G[k_m] + W_N^m H[k_m]$$

$$F[k_{m+\frac{N}{2}}] = G[k_m] - W_N^m H[k_m]$$

We want to multiply F times c as quickly as possible. Normally a matrix times a vector takes n^2 separate multiplications — the matrix has n^2 entries.

If the matrix has zero entries then multiplication can be skipped. But the Fourier matrix has no zeros.

By
entries
that

$F_4 =$

Factors for FFT

$F_4 =$

$=$

Combines th
half sized outp
a way that
the same full

$y = F_4 c$

By using the special pattern W^{jk} for its entries, F can be factored in a way that produces many zeros. This is the FFT.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \& \quad \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}$$

Factors for FFT

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}.$$

Combines the two half sized outputs in a way that produces the same full size output $y = F_4 c$

Performs half sized transforms F_2 & F_2 on the even $f[2n]$ & odd $f[2n]$'s.

When $N = 1024 = 2^{10} \implies \frac{N}{2} = 512$

$$W = e^{-2\pi i/1024}$$

$$F_{1024} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$$

D_{512} : diagonal matrix with entries $(1, W, W^2, \dots, W^{511})$

The Full FFT by Recursion

We reduced F_N to $F_{N/2}$. Keep going to $F_{N/4}$

- Every F_{512} lead to F_{256}. Then 256 leads to 128. That is recursion.

$$\begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} = \begin{bmatrix} I & D & & \\ I & -D & & \\ & & I & D \\ & & I & -D \end{bmatrix} \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \begin{bmatrix} \text{pick } 0, 4, 8, \ldots \\ \text{pick } 2, 6, 10, \ldots \\ \text{pick } 1, 5, 9, \ldots \\ \text{pick } 3, 7, 11, \ldots \end{bmatrix}$$

* The final count for size $N = 2^l$ is reduced from N^2 (for DFT) to $\frac{N}{2}\log_2 N = \frac{1}{2}Nl$ (for FFT)

Reasoning

There are $l = \log_2 N$ levels.

Each level has $\frac{N}{2}$ multiplications from the diagonal D's

$\Longrightarrow$ Final count $= \frac{N}{2} \times l = \frac{1}{2}Nl = \frac{1}{2}N\log_2 N$

$\boxed{B}$ Decimation-In-Frequency (DIF) algorithm

Here,
we decimate the DFT sequence $F[k_m]$ into smaller and smaller subsequences (instead of the time-domain sequence $f[n]$).

$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-i\frac{2\pi m}{N} n}$$

$$= \sum_{n=0}^{N-1} f[n] W_N^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[n] W_N^{nm} + \sum_{n=\frac{N}{2}}^{N-1} f[n] W_N^{nm}$$

$$= \sum_{n=0}^{N/2-1} f[n] W_N^{nm} + \sum_{n=0}^{\frac{N}{2}-1} f[n+\tfrac{N}{2}] W^{(n+\frac{N}{2})m}$$

$$W^{(n+\frac{N}{2})m} = e^{-i\frac{2\pi m}{N}(n+\frac{N}{2})} = e^{-i\frac{2\pi m}{N}n} * e^{-i m\pi}$$

$$= (-1)^m W_N^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left\{ f[n] + (-1)^m f\left[n+\frac{N}{2}\right] \right\} W_N^{nm}$$

$m = 0, 1, \cdots, N-1$

$$W_N^{n(2m)} = e^{-i\frac{2(2m)\pi}{N}n} = e^{-i\frac{2m\pi}{N/2}n} = W_{N/2}^{nm}$$

$$W_N^{n(2m+1)} = e^{-i\frac{2(2m)\pi}{N}n} \times e^{-i\frac{2\pi}{N}n} = W_{N/2}^{nm}\, W_N^{n}$$

Decimating $F[k_m]$ into even and odd indexed samples,

$$F[k_{2m}] = \sum_{n=0}^{\frac{N}{2}-1} \left\{ f[n] + f\left[n+\frac{N}{2}\right] \right\} W_{N/2}^{nm} \qquad k = 0, 1, \ldots, \frac{N}{2}-1$$

$$F[k_{2m+1}] = \sum_{n=0}^{\frac{N}{2}-1} \left\{ f[n] - f\left[n+\frac{N}{2}\right] \right\} W_N^{n}\, W_{N/2}^{nm} \qquad k = 0, 1, \ldots, \frac{N}{2}-1$$

Let,

$$g[n] = f[n] + f\left[n+\frac{N}{2}\right]; \quad 0 \le n \le \frac{N}{2}-1$$

$$h[n] = \left\{ f[n] - f\left[n+\frac{N}{2}\right] \right\} W_N^{n}; \quad 0 \le n \le \frac{N}{2}-1$$

Substituting,

$$G[k_m] = F[k_{2m}] , \quad 0 \le k \le \frac{N}{2}$$

and $$H[k_m] = F[k_{2m+1}] ; \quad 0 \le k \le \frac{N}{2}$$

For $N = 8$,

$$g[0] = f[0] + f[4]$$

$$g[1] = f[1] + f[5]$$

$$g[2] = f[2] + f[6]$$

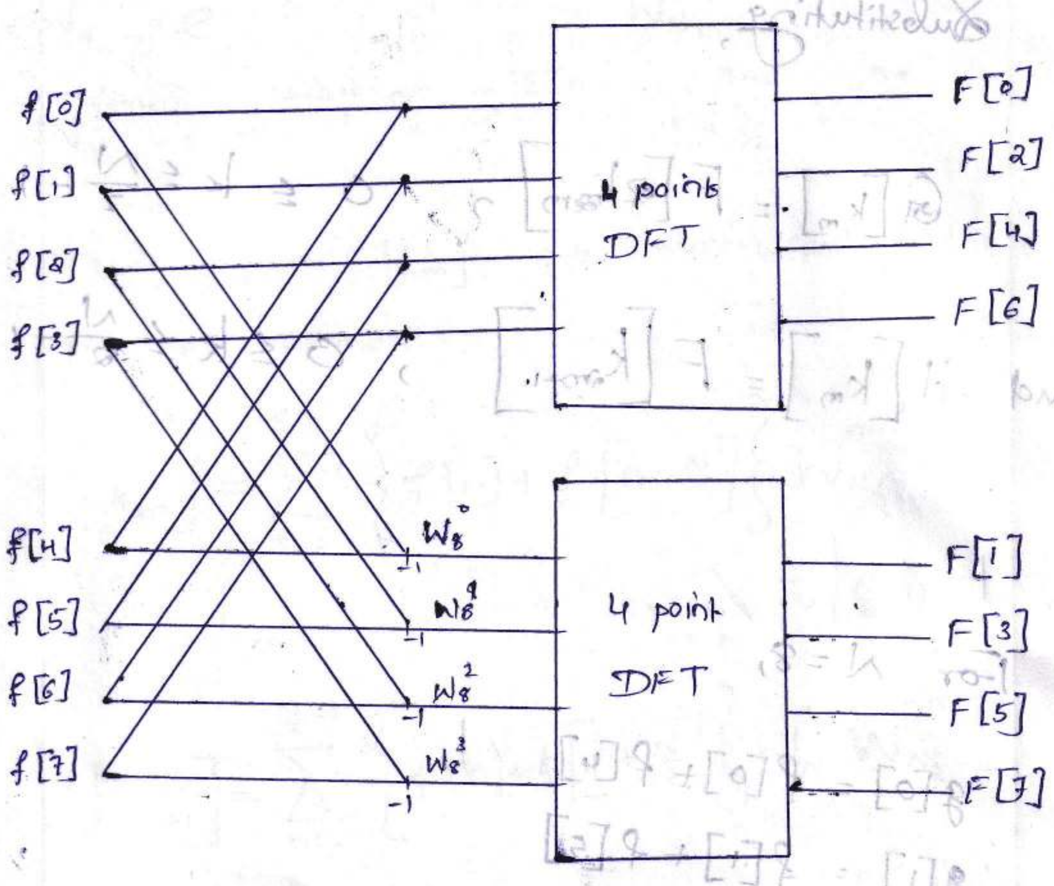
$$g[3] = f[3] + f[7]$$

$$h[0] = \{f[0] - f[4]\} W_8^0$$

$$h[1] = \{f[1] - f[5]\} W_8^1$$

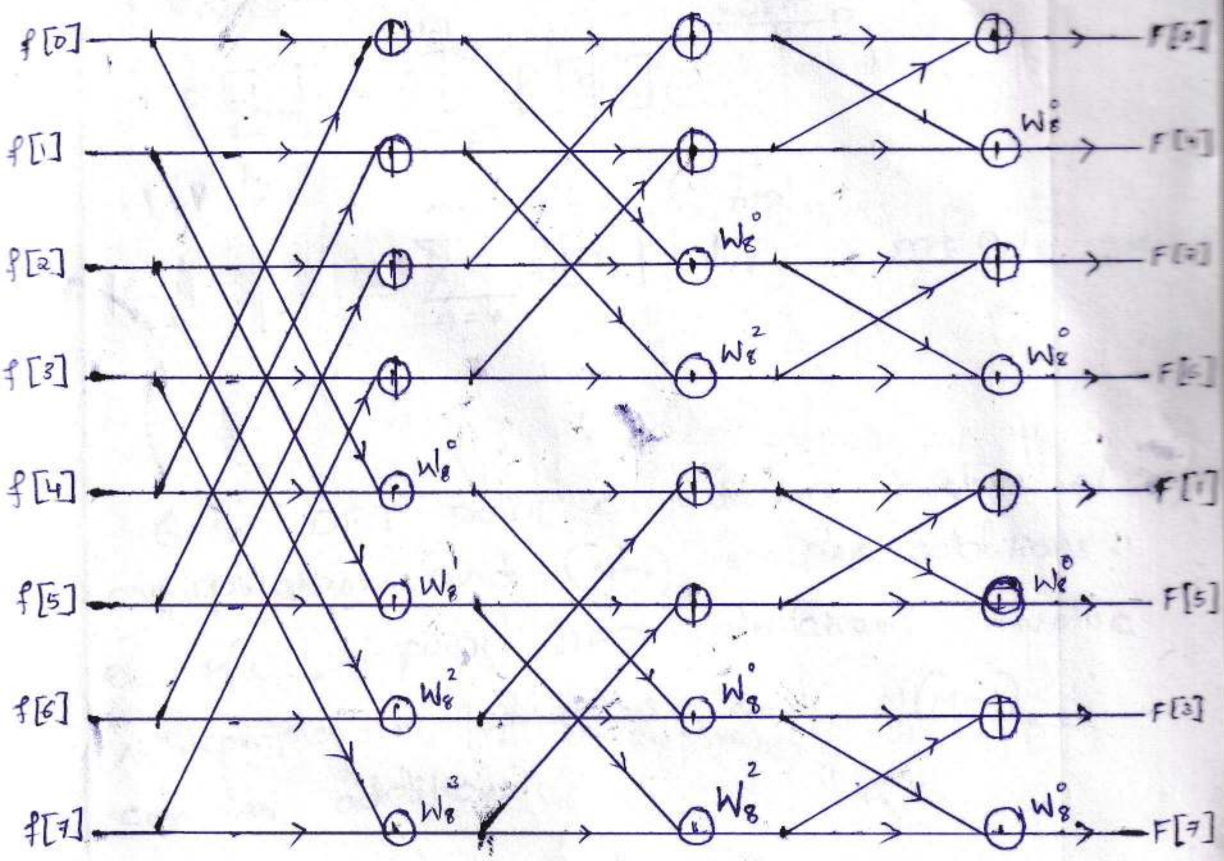
$$h[2] = \{f[2] - f[6]\} W_8^2$$

$$h[3] = \{f[3] - f[7]\} W_8^3$$



The above process of decimation is repeated for $G[k_m]$ and $H[k_m]$, until we reach a 2-point sequence.

* The final butterfly diagram for Decimation-In-Frequency (DIF) FFT algorithm for N=8 is :-



Efficiency of FFT algorithm

An N-point DFT is given by,

$$F[k_m] = \sum_{n=0}^{N-1} f[x_n] e^{-i\frac{2\pi m}{N} n}$$

$$= \sum_{n=0}^{N-1} f[x_n] W_N^{nm} \quad ; m = 0, 1, \ldots$$

Each DFT point computation involves N complex multiplications and (N-1) complex additions. So, the N-point DFT calculations involve N^2 complex multiplications and $N(N-1)$ complex additions.

Ex:- 1024 point sequence

≈ 10^6 complex multiplication & additions

FFT

\# of stages in a butterfly diagram $= \log_2 N$

\# of butterflies in each stage $= \frac{N}{2}$

\# of complex multiplications in each stage butterfly $= 1$

\# of complex additions in each butterfly $= 2$

$\therefore$ Total # of complex multiplications $= \frac{N}{2} \log_2 N$

Total # of complex additions $= N \log_2 N$

Ex:- 1024 point sequence

Approx. 5120 complex multiplications and 10240 complex additions.

⟹ approx. 100 times less addition and 200 times less multiplications than direct computation of DFT.

* As the # of i/p samples increases, the savings in the # of computations also increase.

P.S. 9.3

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutations} \end{bmatrix}$$

2. Invert the 3 factors in eqn. to find a fast factorization of F^{-1}.

Ans: $$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix} = \frac{F^H}{4}$$

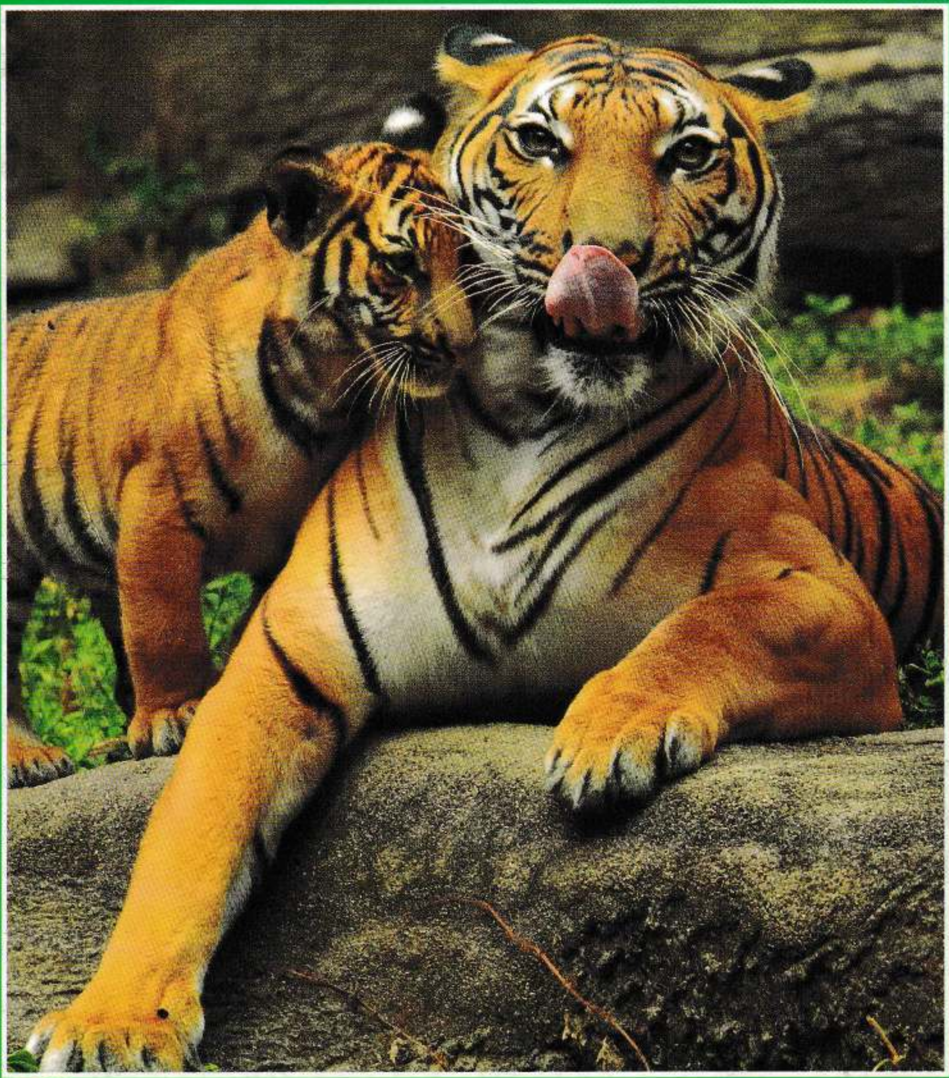
3. F is symmetric. So transpose the equation to find a new FFT

Ans:

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix}$$

Permutation last

4. All entries in the factorization of F_6 involve powers of $w_6 = 6^{th}$ root of 1



YESSAR

Pages Incl.of Cover	192
M.R.P.₹ Incl.of GST	45.00