Introduction to Linear Algebra
- Gilbert Strang

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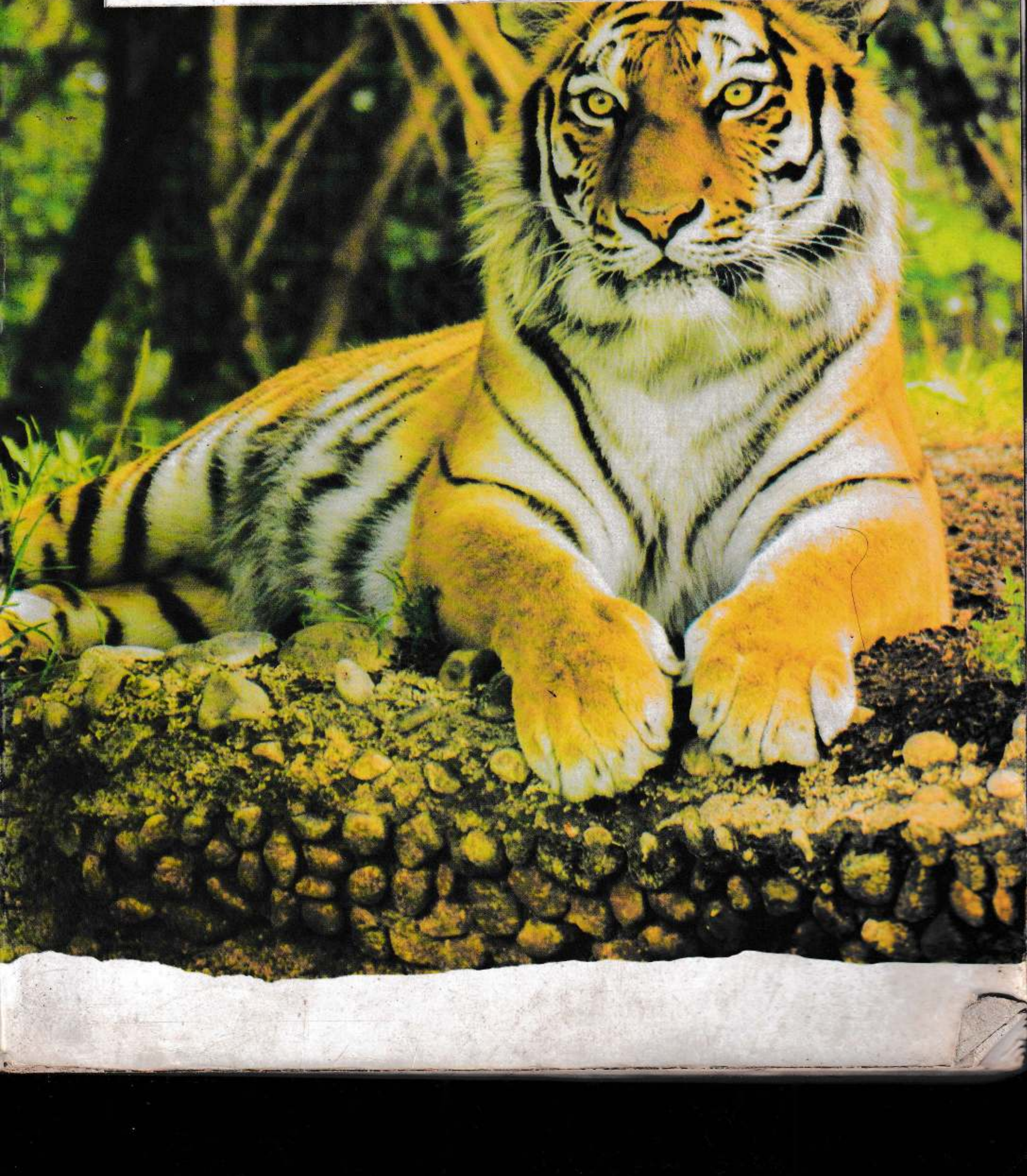
Introduction to Vectors
↓
Solving Linear Equations



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	- Gilbert Strang, MIT			
	(5th Edition)			

$\boxed{1}$ INTRODUCTION TO VECTORS

Linear Algebra
- → add vectors to get $\vec{v}+\vec{w}$
- → multiply vectors by # $c\vec{v}, d\vec{w}$

→ linear combination $c\vec{v}+d\vec{w}$

Represent vector, $\vec{v}$
- → Two numbers
- → Arrow from $(0,0)$
- → Point in the plane

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (a, b, c)$$

* The combinations $c\vec{u}$ fill a line through (0,0,0)

The combinations $c\vec{u} + d\vec{v}$ fill a plane through (0,0,0)

The combinations $c\vec{u} + d\vec{v} + e\vec{w}$ fill 3D space.

$\boxed{1.1(A)}$ The linear combination of $v = (1,1,0)$ and $w = (0,1,1)$ fill a plane in $\mathbb{R}^3$. Describe that plane. Find a vector that is not a combination of $\vec{v}$ and $\vec{w}$ – not on the plane.

Ans:

Combinations

$$c\vec{v} + d\vec{w} = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix}$$ fill a plane

Ex. of vectors in that plane are: $(0,0,0)$, $(2,3,1)$, $(5,7,2)$, $(\pi, 2\pi, \pi)$

$(1,2,3)$ is not in the plane, since $1+3 \neq 2$.

Ex: 2 Put a weight of 4 at the point $x=-1$ and a weight of 2 at the point $x=2$. The x-axis will balance on the centre point (like a see-saw).

The weights balance because

$$4(-1)+2(2)=0$$

* (A) moment is the product of the distance to some point, raised to some power, and some physical quantity such as the force, charge, mass etc at that point.

Ex:-

- The moment of force (torque) is the 1st moment

$$\tau = r\times F$$

- Angular momentum is the 1st moment of momentum, ie., $L = r\times p$

Note: momentum itself is not a moment

- The electric dipole moment is also a 1st moment, $p = qd$ for 2 opposite point charges (or) $\int r\rho(r)d^3r$ for a distributed charge with charge density $\rho(r)$.

- The total mass is the 0^{th} moment of mass.

- The centre of mass is the 1^{st} moment of mass, normalized by total mass:

$R = \frac{\sum_i r_i m_i}{M}$ for a collection of point masses

(or) $\frac{\int r \rho(r) d^3r}{M}$ for an object with mass distribution $\rho(r)$.

- The moment of inertia is the 2^{nd} moment of mass, $I = mr^2$ for a point mass, $\sum_i r_i^2 m_i$ for a collection of point masses, $\int r^2 \rho(r) d^3r$ for an object with mass distribution $\rho(r)$.

Ans: $W = (w_1, w_2) = (4, 2)$; $V = (v_1, v_2) = (-1, 2)$

The eqn. of the see-saw to balance is

$w_1 v_1 + w_2 v_2 = 0 \iff \vec{W} \cdot \vec{V} = 0$

Length of a vector

$$\text{length} = |v| = \sqrt{v \cdot v}$$
$$= \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)^{1/2}$$

$|(1,1,1,1)| = \sqrt{1^2+1^2+1^2+1^2} = 2$, is the diagonal through a unit cube in 4D space

* The length of the diagonal through a unit cube in n-D space is $\sqrt{n}$

$\vec{V} \perp \vec{W}$

Pythagoras: $|\vec{V}|^2 + |\vec{W}|^2 = |\vec{V} - \vec{W}|^2$

$$\cancel{v_1^2} + \cancel{v_2^2} + \cancel{w_1^2} + \cancel{w_2^2} = (v_1 - w_1)^2 + (v_2 - w_2)^2$$

$$= \cancel{v_1^2} + \cancel{w_1^2} - 2v_1w_1 + \cancel{v_2^2} + \cancel{w_2^2} - 2v_2w_2$$

$\cancel{v_1 + v_2}$ v_1w_2

$$v_1w_1 + v_2w_2 = 0 \iff \vec{V} \cdot \vec{W} = 0$$

Check O.M(4) & +1(11)

$|\vec{v}+\vec{w}| \leq |\vec{v}|+|\vec{w}|$: triangle inequality

$|\vec{v}\cdot\vec{w}| \leq |\vec{v}||\vec{w}|$: Cauchy-Schwarz inequality

$|\vec{v}\cdot\vec{w}|^2 + |\vec{v}\times\vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2$: Lagrande's identity

* $|\vec{v}-\vec{w}|^2 = |\vec{v}|^2+|\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta$ — Cosine law

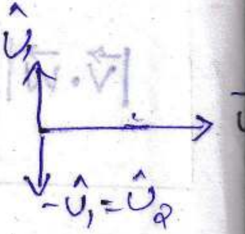
$\boxed{12(B)}$ Find a unit vector $\hat{u}$ in the direction of $u = (3$
Find a unit vectors U that is $\perp$ to $\hat{u}$.

$$Rot^{ccw}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Ans: $|\vec{u}| = 5$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \left(3/5, 4/5\right)$$

$$\hat{U}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$



$$\hat{U}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} = -\hat{U}_1$$

Matrices

3 vectors : $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Their linear combination in 3D space are $x_1 u + x_2 v + x_3 w$.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

$$A\vec{x} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\text{"difference matrix"}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 0 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$$

$$= \begin{bmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (-1,1,0) \cdot (x_1, x_2, x_3) \\ (0,-1,1) \cdot (x_1, x_2, x_3) \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = b$$

$$\Longrightarrow \quad A^{-1} b = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{\text{sum matrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x$$

$Ax = b$ & $x = A^{-1}b$

|||

$\frac{dx}{dt} = b$ & $x = \int_0^t b\,dt$

Backward difference of $x(t) = t^2$: $x(t) - x(t-1) = t^2 - (t-1)^2 = t^2 - (t^2 - 2t + 1$

$= 2t - 1$

The best choice is a centered difference that uses $x(t+1) - x(t-1)$. Divide that Δx by the distance Δt from $t-1$ to $t+1$, which is 2:

Centered difference of $x(t) = t^2$: $\dfrac{(t+1)^2 - (t-1)^2}{2} = 2t$

Cyclic differences

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w}^* = \vec{b}$$

$$C\vec{x} = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\text{cyclic difference matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left.\begin{array}{l} |C| = 1 - 1 = 0 \\ \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \end{array}\right\} \left. \text{rank}(C) = 2 \right\}$$ only 2 linearly independent columns.

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

→ No combination of u, v, w^* will produce the vector $b = (1,3,5)$. The combinations don't fill the whole 3D space.

ie.,

All linear combinations $x_1\vec{u}+x_2\vec{v}+x_3\vec{w}^*$ lie on the plane given by $b_1+b_2+b_3=0$

$\boxed{1.3(B)}$ E is an elimination matrix.

$$\vec{b} = E\vec{x} \Longrightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - l x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ans:- $b_1 = x_1$ & $b_2 = x_2 - l x_1$

$\Rightarrow$ E^{-1} will add b_2 to $l b_1$, because E subtracted.

$$\vec{x} = E^{-1}\vec{b} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ l b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\longrightarrow E^{-1} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}$$

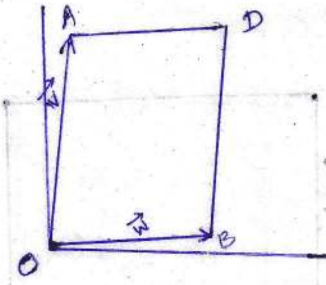
$\boxed{1.3(C)}$ $C\vec{x} = \vec{b} \Longrightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\text{Centered difference matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ 0 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

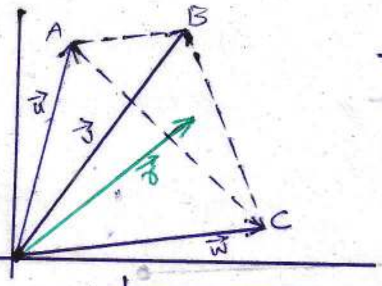
* Row 'i' of Cx is x_{i+1} (right of center) minus x_{i-1} (left of center).

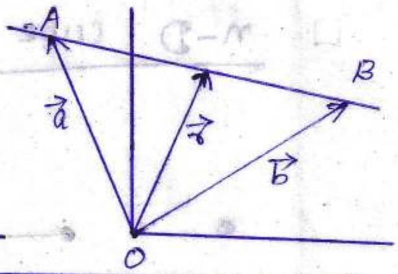
C is not invertible.

$$C^* \vec{x} = \vec{b} \Longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_0 \\ x_3 - x_1 \\ x_4 - x_2 \\ 0 - x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

C^* is invertible.

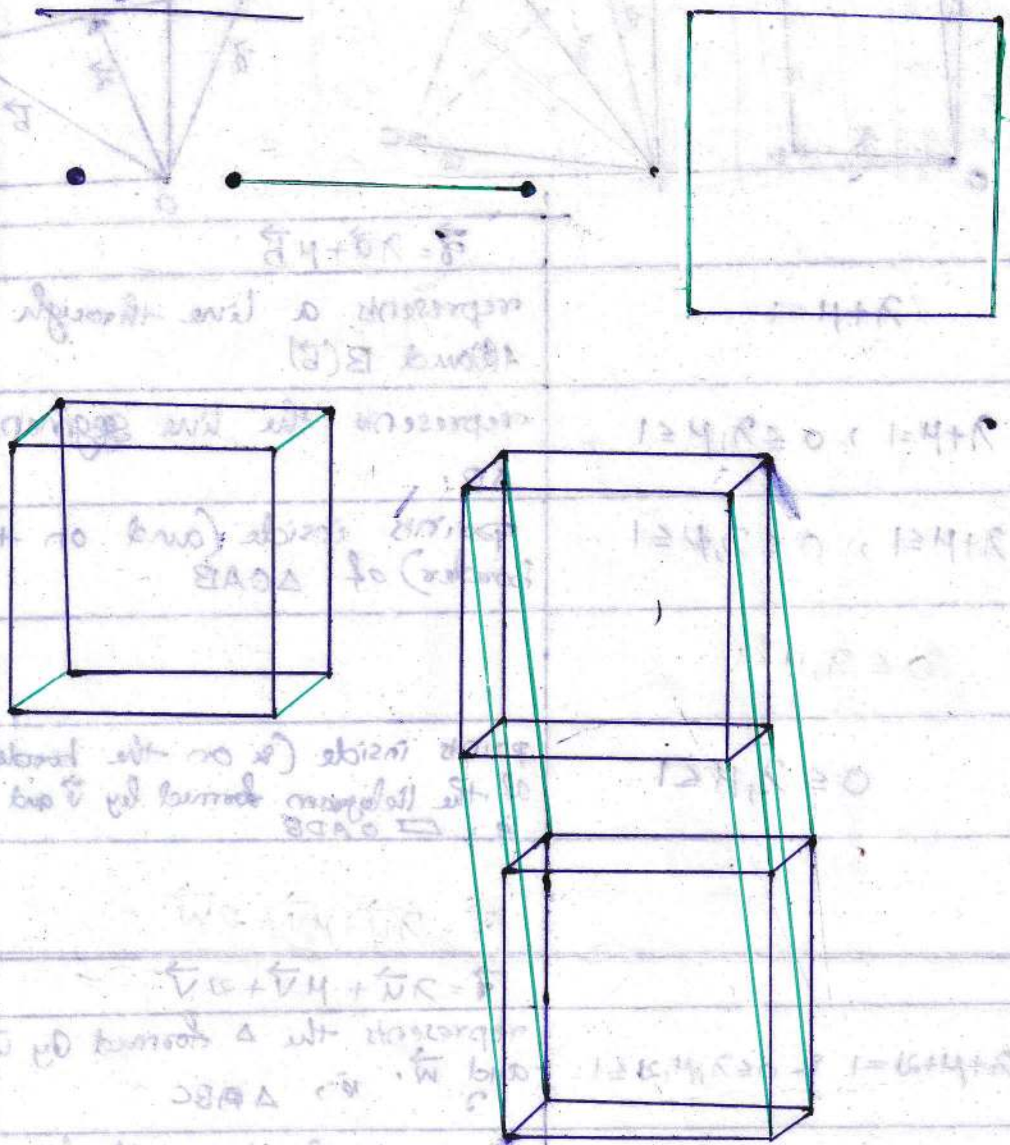






	$\vec{r} = \lambda\vec{a} + \mu\vec{b}$
$\lambda + \mu = 1$	represents a line through $A(\vec{a})$ and $B(\vec{b})$
$\lambda + \mu = 1$, $0 \le \lambda, \mu \le 1$	represents the line segment AB.
$\lambda + \mu \le 1$, $0 \le \lambda, \mu \le 1$	points inside (and on the border) of ΔOAB
$0 \le \lambda, \mu \le 1$	points inside (& on the border) of the ||elogram formed by $\vec{v}$ and $\vec{w}$ ie; ▱ OADB
	$\vec{r} = \lambda\vec{u} + \mu\vec{v} + \nu\vec{w}$
$\lambda + \mu + \nu = 1$ & $0 \le \lambda, \mu, \nu \le 1$	represents the Δ formed by $\vec{u}, \vec{v}$ and $\vec{w}$. ie; ΔABC
$\lambda + \mu + \nu \le 1$ & $0 \le \lambda, \mu, \nu \le 1$	represent the trapezoid formed by $\vec{u}, \vec{v}, \vec{w}$. ie; OABC

n-D cube



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2 1

4 4 1

8 12 6 1

16 32 24 8 1

Recursion formula : $\boxed{\begin{bmatrix} n \\ k \end{bmatrix} = 2\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}$

where, $\begin{bmatrix} n \\ k \end{bmatrix}$: # of k-dimensional faces of the n-D cube.

$$\boxed{\begin{bmatrix} n \\ k \end{bmatrix} = 2^{n-k}\binom{n}{k}}$$

P.S. 1.2(30)

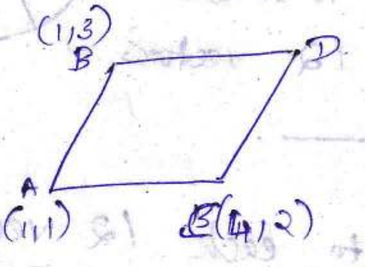
* The largest 'm' for which there exists $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ such that for all $i \neq j$ we have $\vec{v}_i \cdot \vec{v}_j < 0$ is $(m+1)$

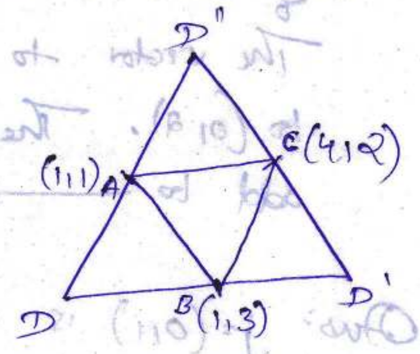
* Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ such that $\vec{v}_i \cdot \vec{v}_j < 0$ for all $i \neq j$, then any 'm' vectors among $(m+1)$ are linearly independent.

P.S. 1.1

9. If 3 corners of a ||elogram are (1,1), (4,2), (1,3). What are all 3 of the possible 4th corners?

Ans:





$$\vec{CD} = \vec{CA} + \vec{CB} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} x-4 \\ y-2 \end{bmatrix} \Longrightarrow D(x,y) = (-2, 2)$$

$$\vec{AD'} = \vec{AC} + \vec{AB} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} x'-1 \\ y'-1 \end{bmatrix}$$

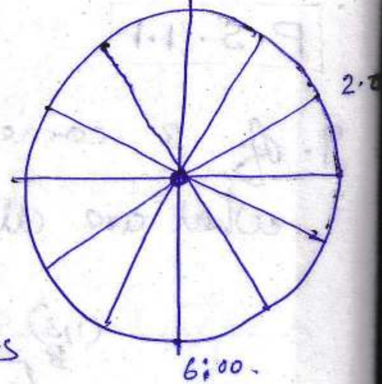
$$\Longrightarrow D'(x', y') = (4, 4)$$

$$\vec{BD''} = \vec{BA} + \vec{BC} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} x''-1 \\ y''-3 \end{bmatrix}$$

$$\Longrightarrow D''(x'', y'') = (4, 0)$$

14.

Suppose, the 12 vectors start from 6:00 at the bottom instead of (0,0) at the ~~centre~~ centre. The vector to 12:00 is doubled to (0, 2). The new 12 vectors add to ______.



Ans: $j = (0,1)$ is added to each 12 vectors.

$$\sum = 12\,(0,1) = 12j = \underline{\underline{(0,12)}}$$

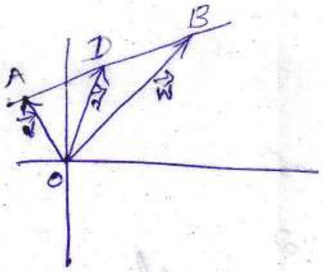
16. Mark the point $-\vec{v}+2\vec{w}$ and any other combination $c\vec{v}+d\vec{w}$ with $c+d=1$. Draw the line of all combinations that have $c+d=1$

Ans: $\vec{r} = c\vec{v}+d\vec{w}$

$c+d=1 \longrightarrow \vec{r} = \vec{v}+\vec{d}\,(\vec{w}-\vec{v}) = \vec{w}+c\,(\vec{v}-\vec{w})$

$\longrightarrow r = c\vec{v}+d\vec{w}$ with $c+d=1$ are on the line that passes through $\vec{v}$ and $\vec{w}$.

$\vec{r} = \vec{v} + d(\vec{w} - \vec{v})$



$d=0, \quad \vec{r}_0 = \vec{v}$

$d=1, \quad \vec{r}_1 = \vec{w}$

$0 \le c,d \le 1$ & $c+d=1$ $\Longrightarrow$ $\vec{r} = c\vec{v} + d\vec{w}$ represents the line segment b/w $\vec{v}$ and $\vec{w}$.

~~when $c+d \le 1$ and $0 \le c,d \le 1$~~

* $c+d \le 1$ & $0 \le c,d \le 1$ $\Longrightarrow$ represents points inside or on the ΔOAB formed by $\vec{v}$ and $\vec{w}$.

Hack
11/10/2020

Proof:1
Let, $e = c+d$ and $f = \frac{c}{e}$

$0 \le e \le 1$ and $0 \le f \le 1$

$c\vec{v} + d\vec{w} = ef\vec{v} + e(1-f)\vec{w} = f(e\vec{v}) + (1-f)(e\vec{w})$

$f+(1-f)=1 \longrightarrow c\vec{v}+d\vec{w}$ lies on the line segment b/w $e\vec{v}$ & $e\vec{w}$

$0 \le e \le 1 \Rightarrow c\vec{v}+d\vec{w}$ lies on or on the border of the ΔOAB.

Proof : 2. Any point on the line segment OD

$$\vec{OP} = \alpha \vec{r} \quad \text{with} \quad \alpha \in [0,1]$$

$$\vec{r} = \vec{v} + \vec{AD}$$

$$\vec{AD} = \beta \vec{AB} = \beta(\vec{w} - \vec{v}) = \beta\vec{w} - \beta\vec{v}, \text{ with } \beta \in [0,1]$$

$\Longrightarrow$

$$\vec{OP} = \alpha\vec{r} = \alpha\left[\vec{v} + \beta\vec{w} - \beta\vec{v}\right]$$

$$= \alpha(1-\beta)\vec{v} + \alpha\beta\vec{w} = c\vec{v} + d\vec{w}$$

$c+d = \alpha \in [0,1]$ & $c, d \in [0,1]$

Proof : 3. Let the position vector of the line segment AB,

$$\vec{r} = c\vec{v} + d\vec{w} \quad \text{with } c+d = 1 \ \& \ 0 \le c, d \le 1$$

Position vectors corresp. points on OD inside ΔOAB be,

$$\vec{p} = \alpha \vec{r}$$

$$= \alpha \left[c\vec{v} + d\vec{w} \right]$$

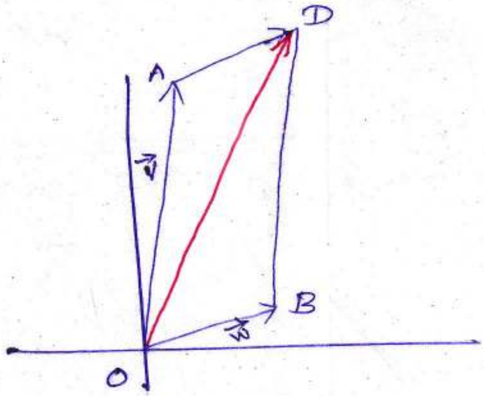
$$= \alpha c \vec{v} + \alpha d \vec{w} = \lambda \vec{v} + \mu \vec{w}$$

with $0 \leq \alpha \leq 1$.

where $\underline{\lambda + \mu = \alpha(c+d) \leq 1}$ and $\underline{\lambda, \mu \in [0,1]}$.

18. Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all combinations $c\vec{v} + d\vec{w}$

Stark
10/10/2020



Ans:

Any point inside ▱ OADB.

i.e;

Points inside $\triangle OAD$ (or) Points inside $\triangle OBD$

$\vec{r} = a\vec{v} + b(\vec{v}+\vec{w})$ (or) $\vec{r} = \alpha(\vec{v}+\vec{w}) + \beta\vec{w}$

with $a+b \le 1$, and $a, b \in [0,1]$

with $\alpha+\beta \le 1$ and $\alpha, \beta \in [0,1]$

$\vec{r} = (a+b)\vec{v} + b\vec{w}$

$= c\vec{v} + d\vec{w}$

(or) $\vec{r} = \alpha\vec{v} + (\alpha+\beta)\vec{w}$

$= c\vec{v} + d\vec{w}$

where, ~~with~~

$c = a+b \in [0,1]$

$d = b \in [0,1]$

where,

$c = \alpha \in [0,1]$

$d = \alpha+\beta \in [0,1]$.

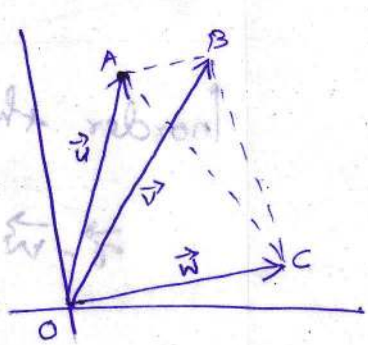
$\Longrightarrow \vec{r} = c\vec{v} + d\vec{w}$ are points inside (or on) the ||elogram OADB.

19. Restricted by $c \geq 0$ and $d \geq 0$, draw the cone of all combinations $c\vec{v} + d\vec{w}$

Ans: With $c \geq 0$, and $d \geq 0$, we get infinite cone or wedge b/w $\vec{v}$ and $\vec{w}$.

20. ~~What cond~~

- Under what restrictions on c, d, e, will the combinations $c\vec{u} + d\vec{v} + e\vec{w}$ fill in the dashed triangle ?



Ans:

With $0 \leq c, d, e \leq 1$ and $c + d + e = 1$,

$\vec{r} = c\vec{u} + d\vec{v} + e\vec{w}$ represents $\triangle ABC$

Proof : 1 $\vec{r} = c\vec{u} + d\vec{v} + e\vec{w}$

doubt at $c, d, e \in [0,1]$

$\vec{v} - \vec{u}$, $\vec{v} - \vec{w}$, $\vec{r} - \vec{u}$ are coplanar.

$\vec{r} - \vec{u} = \lambda(\vec{v} - \vec{u}) + \mu(\vec{v} - \vec{w}) \Rightarrow \vec{r} = (1-\lambda)\vec{u} + (\mu+\lambda)\vec{v} - \mu\vec{w}$

$$\frac{1-\lambda}{c} = \frac{\lambda+\mu}{d} = \frac{-\mu}{e} = 1$$

$c = 1-\lambda, \; d = \lambda+\mu, \; e = -\mu$

$c+d+e = 1$

$\boxed{\text{Proof : 2}}$

Now,
Shift the origin to $C(\vec{w})$,

Inorder that a $(\vec{r}-\vec{w})$ vector to lie in the ΔCA

$$\vec{r}-\vec{w} = \lambda(\vec{v}-\vec{w}) + \mu(\vec{u}-\vec{w}),$$

with $0 \le \lambda, \mu \le 1$ & $\lambda+\mu \le 1$

$$\vec{r} = \mu\vec{u} + \lambda\vec{v} + (1-\lambda-\mu)\vec{w}$$

$$\vec{r} = c\vec{u} + d\vec{v} + e\vec{w}$$

$\mu = c \;;\; \mu, c \ge 0$

$\lambda = d \;;\; \lambda, d \ge 0$

$1-(\lambda+\mu) = e \;;\; \lambda+\mu \le 1 \implies e \ge 0$

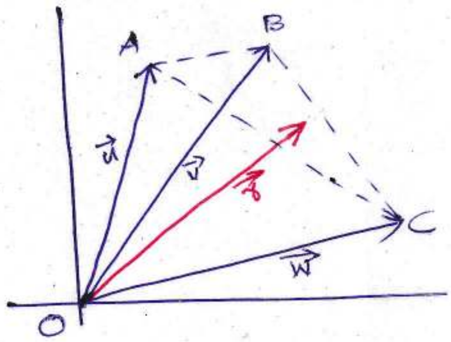
$\cancel{1} - (\lambda + \mu) = \cancel{1} - (c+d) = e$

$\Longrightarrow \quad \underline{c+d+e = 1}$.

22.

With $0 \le c, d, e \le 1$ and $c+d+e \le 1$,

$\vec{r} = c\vec{u} + d\vec{v} + e\vec{w}$ represents the tetrahedron OABC.



Proof

Stack
12/10/2020

For $\vec{r}$ to be any point on the ΔABC,

$\vec{r} = c\vec{u} + d\vec{v} + e\vec{w}$ with $c+d+e = 1$ & $0 \le c, d, e \le 1$

For any point inside the tetrahedron OABC, the position vector

$\vec{p} = \alpha \vec{r}$

$= \alpha c \vec{u} + \alpha d \vec{v} + \alpha e \vec{w} = \psi \vec{u} + \phi \vec{v} + \eta \vec{w}$

with $0 \le \alpha \le 1$.

where,

$$\psi+\phi+\eta = \alpha(c+d+e) \le 1 \quad \& \quad \psi, \phi, \eta \in [0,1]$$

29. Find 2 different combinations of the 3 vectors $\vec{u}=(1,3)$ and $\vec{v}=(2,7)$, and $\vec{w}=(1,5)$ that produce $\vec{b}=(0,1)$

If I take any 3 vectors $\vec{u},\vec{v},\vec{w}$ in the plane, will there be 2 different combinations that produce $\vec{b}=(0,1)$

Ans: ~~For~~ $\cancel{\vec{b}=(0,0)}$.

$\cancel{c\vec{u}+d\vec{v}+e\vec{w}=0} \Longrightarrow \cancel{c(2\vec{u})+d(2\vec{v})+e(2\vec{w})=0}$

For $\vec{b}=(0,0)$.

If $c\vec{u}+d\vec{v}+e\vec{w}=0 \longrightarrow (2c)\vec{u}+(2d)\vec{v}+(2e)\vec{w}=0$

For $\vec{b}=(0,1)$

$\vec{u}-\vec{v}+\vec{w}=(0,1)$

$$\begin{aligned} x+3y&=0\\ x+5y&=0\\ \hline 2y&=0 \Rightarrow y=0\\ x&=0 \end{aligned}$$

$\cancel{2u+v+w}=$

$-2u+v+0w=(0,1)$

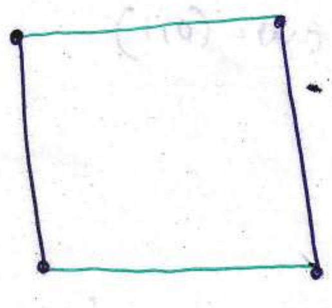
27. How many corners does a cube have in 4D?

~~How~~ many 3D faces?

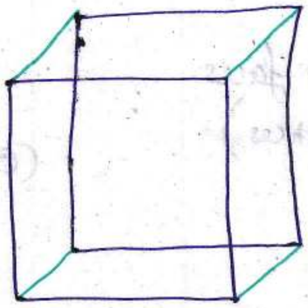
How many edges?

Ans: 1D cube : 2 copies of 0D cube, with a line segment joining the two.

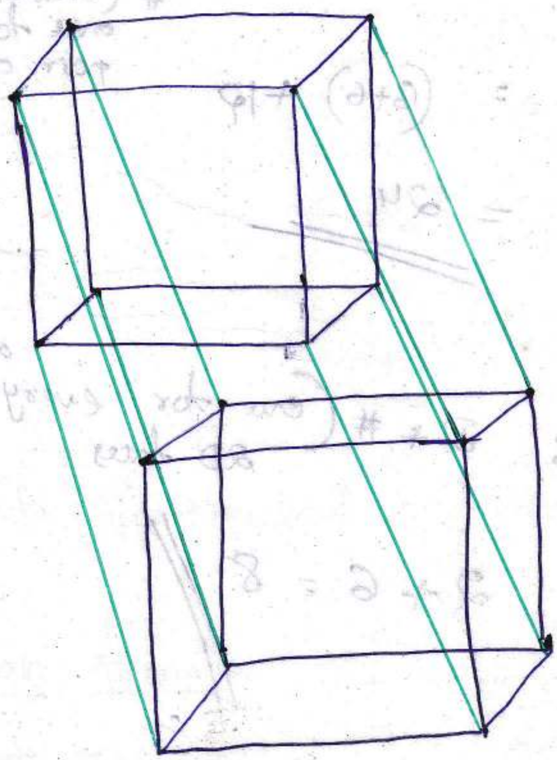
2D cube : 2 copies of 1D cube with line segments joining both pairs of corresp. vertices



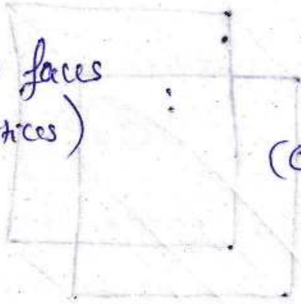
3D cube : 2 copies of the 2D cube (2 squares) with line segments joining all 4 pairs of corresp. vertices.



4-D cube : 2 copies of the 3D cube (2 cubes) with line segments joining all 8 pairs of corresp. vertices



Faces of the 4D cube



0D faces (vertices)

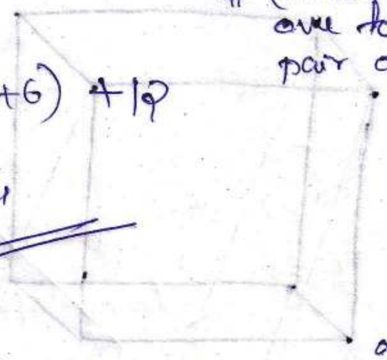
$8+8=16$

(0D) $(\_,\_,\_,\_)$

vertices = $2^4=16$

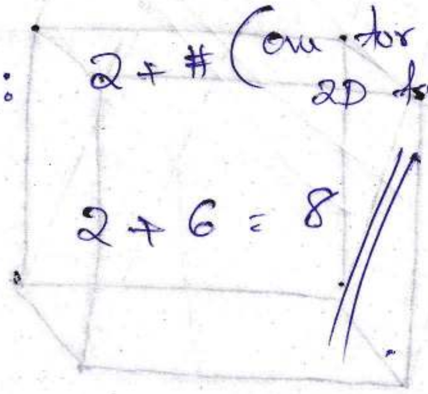
1D faces (edges) : $(12+12)+8=32$

2D faces : # (2D faces of 2 end cubes) + # (created by joins) one for every pair of edges.



$= (6+6)+12$

$= 24$



3D faces : 2 + # (one for every opp. pair of 2D faces)

$2+6=8$

of faces.

Dimension of cube	0D	1D	2D	3D	4D
0 (point)	1				
1 (segment)	2	1			
2 (square)	4	4	1		
3 (cube)	8	12	6	1	
4 (hypercube)	16	32	24	8	1

# of k dimensional faces of the n-D cube $= \begin{bmatrix} n \\ k \end{bmatrix}$

Recursion formula,

$$\boxed{\begin{bmatrix} n \\ k \end{bmatrix} = 2\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}}$$

Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

1
2 1
4 4 1
8 12 6 1
16 32 24 8 1
32 80 80 40 10 1

$$\begin{bmatrix} n \\ k \end{bmatrix} = 2^{n-k} \binom{n}{k}$$

$\#$ of corners (0D face) $= 2^{4-0}\binom{4}{0}$

$= 16$

(OR)

$(-,-,-,-)$

$\#$ of corners $= 2^4 = 16$

$\#$ of edges (1D face) $= 2^{4-1}\binom{4}{1}$

$= 8 \times 4 = 32$

$\#$ of 3D faces $= 2^{4-3}\binom{4}{3} = 2 \times 4 = 8$

P.S.(1.2)

6.
(a) Describe every vector $\vec{w} = (w_1, w_2)$ that is $\perp$ to $\vec{v} = (2, -1)$

Ans: $\vec{w} = c(1, 2) = (c, 2c)$

(b) All vectors $\perp$ to $V = (1,1,1)$ lie on a ________ in 3D

Ans: $\vec{r} \cdot \vec{v} = 0 \longrightarrow$ All vectors $\perp$ to $(1,1,1)$ lie on the plane, $x + y + z = 0$

(c) The vectors $\perp$ to both $(1,1,1)$ & $(1,2,3)$ lie on a ________

Ans: lie on a line in 3D space

8. True/False

(a) If $\vec{u} = (1,1,1)$ $\perp$ to $\vec{v}$ & $\vec{w}$, then $\vec{v} \parallel \vec{w}$

Ans: False

(b) If ~~[illegible]~~ $\vec{u}$ is $\perp$ to $\vec{v}$ & $\vec{w}$, then $\vec{u} \perp \vec{v}+2\vec{w}$

Ans: $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w} = 0.$ True

$\vec{u}\cdot(\vec{v}+2\vec{w}) = \vec{u}\cdot\vec{v} + 2\vec{u}\cdot\vec{w} = 0$

(c) If $\vec{u}$ and $\vec{v}$ are $\perp$ unit vectors, then

$|\vec{u}-\vec{v}| = \sqrt{2}$

Ans: $|\vec{v}| = |\vec{w}| = 1$ &

$(\vec{u}-\vec{v})^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}$

$= 1+1-0 = 2$

$|\vec{u}-\vec{v}| = \sqrt{2}$

16. How long is the vector $\vec{v} = (1,1,\ldots,1)$ in 9D ? Find a unit vector $\vec{u}$ in the same direction as $\vec{v}$ and a unit vector $\vec{w}$ that is $\perp$ $\vec{v}$?

* Hyperplane is a subspace whose dimension is one less than that of its ambient space.

Ans: $|v| = \sqrt{9} = 3$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \vec{v}/3.$

$\vec{w} = \frac{1}{\sqrt{2}}(0,0,\ldots,0,1,-1)$ is a unit vector in the 8D hyperplane $\perp$ to $\vec{v}$.

22. Schwarz inequality,

$$|\vec{v}\cdot\vec{w}| \le |\vec{v}||\vec{w}|$$

Proof $(v_1w_1 + v_2w_2)^2 \le (v_1^2+v_2^2)(w_1^2+w_2^2)$

$$\cancel{v_1^2w_1^2} + \cancel{v_2^2w_2^2} + 2v_1v_2w_1w_2 \le \cancel{v_1^2w_1^2} + \cancel{v_2^2w_2^2} + v_1^2w_2^2 + v_2^2w_1^2$$

$$(v_1w_2 - v_2w_1)^2 \ge 0 \quad \text{true}$$

24. One-line proof of the inequality $|\vec{u}\cdot\vec{U}| \le 1$ for unit vectors (u_1, u_2) and (U_1, U_2):

$$|u\cdot U| = u_1U_1 + u_2U_2 \le |u_1||U_1| + |u_2||U_2|$$

$$\le \left(\frac{u_1^2+U_1^2}{2}\right) + \left(\frac{u_2^2+U_2^2}{2}\right) = 1$$

[AM-GM inequality]

28. If $\vec{v} = (1,2)$, draw all vectors $\vec{w} = (x,y)$ in the xy-plane with $\vec{v}\cdot\vec{w} = x + 2y = 5$.
Why do those $\vec{w}$'s lie along a line ?
Which is the shortest $\vec{w}$?

Ans: $\vec{v}\cdot\vec{w} = x + 2y = 5 \longrightarrow y = \dfrac{5 - x}{2}$

$|\vec{v}\cdot\vec{w}| \le |\vec{v}||\vec{w}| \longrightarrow 5 \le \sqrt{5}\,|\vec{w}| \Longrightarrow |\vec{w}| \ge \sqrt{5}$

29. If $|\vec{v}| = 5$ & $|w| = 3$, what are the smallest & largest possible values of $|\vec{v} - \vec{w}|$? What are the smallest & largest possible values of $\vec{v}\cdot\vec{w}$?

Ans: $|\vec{v}| - |\vec{w}| \le |\vec{v} - \vec{w}| \le |\vec{v}| + |\vec{w}|$

$$2 \le |\vec{v} - \vec{w}| \le 8$$

$$|\vec{v}\cdot\vec{w}| \le |\vec{v}||\vec{w}| = 15$$

$$-15 \le \vec{v}\cdot\vec{w} \le 15$$

30) Can 3 vectors in xy plane have $\vec{u}\cdot\vec{v} < 0$ and $\vec{v}\cdot\vec{w} < 0$ and $\vec{v}\cdot\vec{w} < 0$?

What about n-D space?

Ans:

* The largest 'm' ~~[illegible]~~ for which there exists $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ such that for all $i \neq j$ we have $\vec{v}_i \cdot \vec{v}_j < 0$ is $(n+1)$.

* Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{m+1} \in \mathbb{R}^n$ such that $v_i \cdot v_j <$ for all $i \neq j$; then any m vectors among m+1 are linearly independent.

Proof

Assume that, for some $n \geq 0$ we have $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n+2} \in \mathbb{R}^n$ with $\vec{v}_i \cdot \vec{v}_j < 0$ for $i \neq j$.

$\vec{v}_{n+2} \neq 0$ as $\vec{v}_{n+2} \cdot \vec{v}_1 < 0$

Consider the hyperplane V orthogonal to $\vec{v}_{n+2}$ and project all other vectors into this hyperplane.

The projected vectors are of the form,

$$\vec{u}_i = \vec{v}_i - \frac{\vec{v}_i \cdot \vec{v}_{n+2}}{|\vec{v}_{n+2}|^2} \vec{v}_{n+2} = \vec{v}_i + c\,\vec{v}_{n+2}$$

$$\vec{v}_i \cdot \vec{v}_{n+2} < 0 \implies c_i = \frac{-\vec{v}_i \cdot \vec{v}_{n+2}}{|\vec{v}_{n+2}|^2} > 0$$

$$\vec{u}_i \cdot \vec{u}_j = \vec{u}_i \cdot \vec{v}_j + c_j \left(\cancel{\vec{u}_i \cdot \vec{v}_{n+2}}^{\,=0}\right) = \vec{u}_i \cdot \vec{v}_j$$

$$= \vec{v}_i \cdot \vec{v}_j + c_i \left(\vec{v}_{n+2} \cdot \vec{v}_j\right) < 0$$

$\therefore$ By identifying the hyperplane V with $\mathbb{R}^{n-1}$, we found $(n+1)$ vectors in $\mathbb{R}^{n-1}$ with pairwise -ve dot product.

Repeat the process,

we'll get ~~5~~ 8 vectors with -ve do[t] product in $\mathbb{R}^3$.

~~4~~ vectors ——— in $\mathbb{R}^2$

3 " ——— in $\mathbb{R}^1$

2 vectors ——— in $\mathbb{R}^0$

By induction $\Longrightarrow$ $(n+2)$ such vectors in [$\mathbb{R}^n$] are not possible for any n.

31. Pick any # such that add to $x+y+z=0$. Find the angle b/w your vector $\vec{v}=(x,y,z)$ & the vector $\vec{w}=(z,x,y)$.

Ans:

$$\frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|}=\cos\theta=\frac{\cancel{x}\,z+xy+y\,z}{x^2+y^2+z^2}$$

$$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=0$$

$$\Longrightarrow\quad xy+yz+zx=-\frac{(x^2+y^2+z^2)}{2}$$

$$\cos\theta=\frac{-1}{2}$$

33. Find 4 $\perp$ unit vectors of the form $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)$

Ans: The columns of the 4x4 "Hadamard Matrix" (times $\frac{1}{2}$) are $\perp$ unit vectors:

$$\frac{1}{2}H=\frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$$

34.

Avg. size of the dot products $|\hat{u}\cdot\hat{U}_j|$

Ans:

$$\text{Avg.} = \langle |\hat{u}\cdot\hat{U}_j| \rangle$$

$$= \int_0^{\pi} \frac{|\cos\theta|\, d\theta}{\pi} = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta\, d\theta$$

$$= \frac{2}{\pi}\left[+\sin\theta\right]_0^{\pi/2} = \frac{2}{\pi}\left[1-0\right] = 2/\pi$$

P.S. (1.3)

2. Solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

S : Sum matrix

Sum of the 1st 5 odd #?

Ans: $(y_1, y_2, y_3) = (1, 3, 5)$

$(y_1, y_2, y_3) = (1, 0, 0)$

7. If the columns combine into $Ax = 0$ then each of the rows has $r \cdot x = 0$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By rows : $\begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The 3 rows also lie in a plane. Why is that plane $\perp$ to x?

Ans: $Ax = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} x = \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \end{bmatrix}$

$r_1 \cdot x = r_2 \cdot x = r_3 \cdot x = 0$

$\Longrightarrow$ all 3 rows are $\perp$ to the solution x

$\therefore$ the whole plane of rows is $\perp$ to x.

9. What is the cyclic 4×4 matrix?

Find all solutions to $Cx = 0$.

Ans: $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

when $x = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$, any const vector.

* 4 columns of C lie in a "3D hyperplane" inside 4D space.

10. A forward difference matrix Δ is upper triangular :

$$\Delta Z = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

$\Delta^{-1} =$?

Ans:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -b_1 - b_2 - b_3 \\ -b_2 - b_3 \\ -b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}}_{\Delta^{-1}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

11. Show that the forward diff. $(t+1)^n - t^n$ will begin with the derivative of t^n, which is ________.

Ans: $(t+1)^n - t^n = \left[t^n + nt^{n-1} + \cdots\right] - t^n$

$= \underline{\underline{nt^{n-1}}} + \cdots$

12. 4×4 centered difference matrix & its inverse –

Ans: $$C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - 0 \\ x_3 - x_1 \\ x_4 - x_2 \\ 0 - x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(14.) If (a,b) is a multiple of (c,d) with $abcd \neq 0$, show that (a,c) is a multiple of (b,d).

Ans: $(a,b) = k(c,d) \Longrightarrow \frac{a}{c} = \frac{b}{d}$

$\boxed{ad = bc}$ $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$

$\therefore$ If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows, then it also has dependent columns.

2 SOLVING LINEAR EQUATIONS

Column picture of $A\vec{x}=\vec{b}$: a combination of 'n' columns of 'A' produces vector $\vec{b}$.

$$A\vec{x} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

Row picture of $A\vec{x}=\vec{b}$: 'm' equation from 'm' rows give 'm' ~~planes~~ hyperplanes meeting at x.

$$A\vec{x} = \begin{bmatrix}\vec{r}_1\\ \vec{r}_2\\ \vdots\\ \vec{r}_m\end{bmatrix}\cdot\vec{x} = \begin{bmatrix}\vec{r}_1\cdot\vec{x}\\ \vec{r}_2\cdot\vec{x}\\ \vdots\\ \vec{r}_m\cdot\vec{x}\end{bmatrix} = \begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix} = \vec{b}$$

⊙ dot product gives the eq^n of each plane:

$\vec{r}_1\cdot\vec{x} = b_1$; $\vec{r}_2\cdot\vec{x} = b_2$; $\cdots$; $\vec{r}_m\cdot\vec{x} = b_m$

* Think,

It is a lot easier to see a combination of 4 vectors in 4D space, than to visualize how 4 hyperplanes might possibly meet at a point. (Even one hyperplane is hard enough).

$\boxed{Q.1 (B)}$

$$\left.\begin{aligned} x+3y+5z&=4 \\ x+2y-3z&=5 \\ 2x+5y+2z&=8 \end{aligned}\right\} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & -3 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} = b$$

① Are any of the 3 ||el planes ||el to $x+3y+5z=4$?
Eqn. of planes ||el to $x+3y+5z=4$?

Ans: No, planes are ||el

Plane ||el to $x+3y+5z=4$ is of the form

$x+3y+5z=k$, $k\in\mathbb{R}$

② Take the dot product of each column of A (and also b) with $y=(1,1,-1)$. How do these dot products show that no combination of columns equals b?

Ans: $\vec{a}_1\cdot\vec{y} = \vec{a}_2\cdot\vec{y} = \vec{a}_3\cdot\vec{y} = 0$ & $\vec{b}\cdot\vec{y} = 1 \neq 0$.

$\vec{a}_1 x + \vec{a}_2 y + \vec{a}_3 z = \vec{b}$

$(\vec{a}_1\cdot\vec{y})x + (\vec{a}_2\cdot\vec{y})y + (\vec{a}_3\cdot\vec{y})z = \vec{b}\cdot\vec{y}$

$0 = 1 \implies$ <u>No solution</u>

3. Find 3 different right side vectors b^{*}, b^{**} and b^{***} that do allow solutions.

Ans: Solution when $\vec{b}$ is a linear combination of the columns.

For, $\vec{x}^{*} = (1,0,0)$, $x^{**} = (0,0,0)$, $x^{***} = (0,1,0)$

$$b^{*} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad b^{**} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad b^{***} = \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}$$

The idea of elimination

- The 1^{st} two equations are $a_{11}x_1 + a_{12}x_2 + \cdots = b_1$ and $a_{21}x_1 + a_{22}x_2 + \cdots = b_2$.
- Multiply the 1^{st} equation by $\frac{a_{21}}{a_{12}}$ and subtract from the 2^{nd}. : x_1 is eliminated
- The corner entry a_{11} is the 1^{st} "pivot" and the ratio $\frac{a_{21}}{a_{12}}$ is the 1^{st} "multiplier".
- Eliminate x_1 from every remaining equation i by subtracting $\frac{a_{i1}}{a_{11}}$ times the 1^{st} equation.
- Now, the last $(n-1)$ equations contain $(n-1)$ unknowns $x_2, x_3, \ldots, x_n$. Repeat to eliminate x_2.
- Elimination breaks down if zero appears in the pivot. Exchanging 2 equations may save it.

Breakdown of Elimination

At some point, the method might ask us to divide by zero. We can't do it!
The elimination process has to stop.

$\boxed{\text{Ex:1}}$ Permanent failure with no solution

$x - 2y = 1$
$3x - 6y = 11$

Ans: $eq^n.2 \to (eq^n.2) - 3(eq^n.1)$: $\begin{array}{l} x - 2y = 1 \\ 0y = 8 \end{array}$

$\Longrightarrow$ No solution to $0y = 8$.

$\boxed{\text{Ex:2}}$ Failure with infinitely many solutions, $\vec{b} = (1, 3)$

$\left.\begin{array}{l} x - 2y = 1 \\ 3x - 6y = 3 \end{array}\right\}$ $eq^n.2 \to eq^n 2 - 3 eq^n 1$: $\begin{array}{l} x - 2y = 1 \\ 0y = 0 \end{array}$

$\Longrightarrow$ Every 'y' satisfies $0y = 0$. The unknown 'y' is "free"

$\boxed{\text{Ex: 3}}$ Temporary failure (zero in pivot) :

$$\left.\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned}\right\}$$

Exchange the two equations

$$\begin{aligned} 3x - 2y &= 5 \\ 2y &= 4 \end{aligned}$$

- A linear system $A\vec{x} = \vec{b}$ becomes upper triangular $(U\vec{x} = \vec{c})$ after elimination.

- We subtract l_{ij} times equation j from eq^n i to make (i,j) entry zero.

- The multiplier is $l_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$

Pivots can not be zero.

- When zero is in the pivot position, exchange rows if there is a non-zero below it.

- The upper triangular ~~[illegible]~~ $U\vec{x} = \vec{c}$ is solved by back substitution (starting at the bottom).

- When breakdown is permanent, $A\vec{x} = \vec{b}$ has no solution or infinitely many.

Q.2(A) 1st & 2nd pivot? $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

multiplier in the 1st step?

Why $l_{31} = 0$?

If you exchange $a_{33} = 2$ to $a_{33} = 1$,
why does elimination fail ?

Ans:
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

1st pivot : 1 , 2nd pivot : 1

$l_{21} = 1$

$l_{31} = 0$ Because $a_{31} = 0$.

'A' is a band matrix, everything stays zero outside the band.

If $a_{33} = 1$,
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

No 3rd pivot $\Longrightarrow$ elimination fails.

$\boxed{Q2.(B)}$ If 'A' is a triangular matrix (upper/lower) Where do you see its pivots? When does $Ax=b$ have exactly one solution for every b?

Ans: Pivots are along the main diagonal.

Elimination succeeds when all those numbers are non-zero.

A is upper triangular : back substitution

A is lower triangular : forward substitution.

$\boxed{Q.2(C)}$ Use elimination to reach upper triangular matrices U. Solve by Back substitution (or) explain why this is impossible.

$$x+y+z=7$$
$$x+y-z=5$$
$$x-y+z=3$$

&

$$x+y+z=7$$
$$x+y-z=5$$
$$-x-y+z=3$$

Ans :

$$x+y+z=7$$
$$0y-2z=-2$$
$$-2y+0z=-4$$

$$x+y+z=7$$
$$-2y+0z=-4$$
$$-2z=-2$$

Back substitution.

$z=1,\ y=2,\ x=4$

$$\left.\begin{aligned} x+y+z &= 7 \\ 0y-2z &= -2 \\ 0y+2z &= 10 \end{aligned}\right\} \quad 0z = 8 \longleftarrow \text{No 3rd pivot}$$

No pivot in column 1.

2. No solution!

Elimination using matrices

(Comparing to the identity matrix)

* The elementary matrix (or) elimination matrix E_{ij} has the extra non-zero entry $-l$ in the i, j position. Then E_{ij} subtracts a multiple 'l' of row j from row i.

while, multiplying from the right subtracts a multiple of 'l' of column i from column j

- $|E_{ij}| = 1$

Ex:-

$R_2 \to R_2 - 2R_1$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{(21)}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - 2a_{11} & a_{22} - 2a_{12} & a_{23} - 2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$C_1 \to C_1 - 2C_2$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|cc} a_{11} - 2a_{12} & a_{12} & a_{13} \\ a_{21} - 2a_{22} & a_{22} & a_{23} \\ a_{31} - 2a_{32} & a_{32} & a_{33} \end{array}\right]$$

Row Exchange matrix : P_{ij} is the identity matrix
* with rows i and j reversed. When this permutation matrix P_{ij} multiplies (from the left) a matrix, it exchanges rows i and j.

while, multiplying ~~[illegible]~~ on the right swaps columns i and j.

- $|P_{ij}| = -1$

Ex:-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad R_2 \longleftrightarrow R_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{bmatrix} \quad C_2 \longleftrightarrow C_3$$

The Augmented Matrix

Elimination does the same row operations to 'A' and to $\vec{b}$.

$\therefore$ We can ~~enlarge~~ include 'b' as an extra column and follow it through elimination.

The matrix 'A' is enlarged (or) augmented by the extra column 'b':

Augmented matrix : $\begin{bmatrix} A & \vec{b} \end{bmatrix}$

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \cdots \vec{a}_n & \vec{b} \end{bmatrix}$$

Rules for Matrix Operations

The Different ways – Matrix multiplication

A Take the dot product of each row of A with each column of B.

$$(AB)_{ij} = A_i \cdot b_j$$

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} (A_1 \cdot b_1) & (A_1 \cdot b_2) & \cdots & (A_1 \cdot b_p) \\ (A_2 \cdot b_1) & (A_2 \cdot b_2) & \cdots & (A_2 \cdot b_p) \\ \vdots & \vdots & & \vdots \\ (A_n \cdot b_1) & (A_n \cdot b_2) & \cdots & (A_n \cdot b_p) \end{bmatrix}$$

[B] Matrix 'A' times every column of B.

$$AB = A\left[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_p\right] = \left[A\vec{b}_1\ A\vec{b}_2 \cdots A\vec{b}_p\right]$$

$$A\vec{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix} = v_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$A\vec{v}$ is a linear combination of columns of A $\Big\{ = v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n$

[C] Every row of A times B.

$$AB = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} B = \begin{bmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_n B \end{bmatrix}$$

$$vB = \left[v_1\ v_2 \cdots v_n\right]\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} = \left[v_1 b_{11} + v_2 b_{21} + \cdots + v_n b_{n1},\ v_1 b_{12} + v_2 b_{22} + \cdots + v_n b_{n2}, \cdots, v_1 b_{1p} + v_2 b_{2p} + \cdots + v_n b_{np}\right]$$

vB is a linear combination of rows of B.

$$= v_1(b_{11}, b_{12}, \cdots, b_{1p}) + v_2(b_{21}, b_{22}, \cdots, b_{2p}) + \cdots + v_n(b_{n1}, b_{n2}, \cdots, b_{np})$$
$$= v_1\vec{B}_1 + v_2\vec{B}_2 + \cdots + v_n\vec{B}_n$$

→ Every row of AB is a linear combination of the rows of B

⇒ Every vector which is a linear combination of rows of AB is also a linear combination of rows of B.

i.e., the rowspace of AB is contained in the rowspace of B.

D

$$AB = \left[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n\right]\begin{bmatrix}\vec{B}_1\\ \vec{B}_2\\ \vdots\\ \vec{B}_n\end{bmatrix} = \vec{a}_1\vec{B}_1 + \vec{a}_2\vec{B}_2 + \cdots + \vec{a}_n\vec{B}_n$$

$$= \vec{a}_1 \otimes \vec{B}_1^T + \vec{a}_2 \otimes \vec{B}_2^T + \cdots + \vec{a}_n \otimes \vec{B}_n^T$$

$$= \begin{bmatrix}a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1p}\end{bmatrix} + \begin{bmatrix}a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{bmatrix}\begin{bmatrix}b_{21} & b_{22} & \cdots & b_{2p}\end{bmatrix} + \cdots\cdots + \begin{bmatrix}a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{bmatrix}\begin{bmatrix}b_{n1} & b_{n2} & \cdots & b_{np}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1p}\\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1p}\end{bmatrix} + \begin{bmatrix}a_{12}b_{21} & a_{12}b_{22} & \cdots & a_{12}b_{2p}\\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2p}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m2}b_{21} & a_{m2}b_{22} & \cdots & a_{m2}b_{2p}\end{bmatrix} + \cdots + \begin{bmatrix}a_{1n}b_{n1} & a_{1n}b_{n2} & \cdots & a_{1n}b_{np}\\ a_{2n}b_{n1} & a_{2n}b_{n2} & \cdots & a_{2n}b_{np}\\ \vdots & \vdots & \ddots & \vdots\\ a_{mn}b_{n1} & a_{mn}b_{n2} & \cdots & a_{mn}b_{np}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1} & a_{11}b_{12}+a_{12}b_{22}+\cdots a_{1n}b_{n2} & \cdots & a_{11}b_{1p}+a_{12}b_{2p}+\cdots a_{1n}b_{np}\\ a_{21}b_{11}+a_{22}b_{22}+\cdots a_{2n}b_{n1} & a_{21}b_{12}+a_{22}b_{22}+\cdots a_{2n}b_{n2} & \cdots & a_{21}b_{21}+a_{22}b_{2p}+\cdots a_{2n}b_{np}\\ \vdots & \vdots & & \vdots\\ a_{m1}b_{11}+a_{m2}b_{21}+\cdots a_{mn}b_{n1} & a_{m1}b_{12}+a_{m2}b_{22}+\cdots a_{mn}b_{n2} & \cdots & a_{m1}b_{1p}+a_{m2}b_{2p}+\cdots a_{mn}b_{np}\end{bmatrix}$$

* Multiplication by a matrix A with a vector $\vec{v}$ is a linear transformation.

* A transformation is linear if it meets these requirements

$$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$$

$$T(c\vec{v}) = c\,T(\vec{v}) \quad \forall c$$

$$\Updownarrow$$

T is a linear transformation

$\Rightarrow$ Linear Algebra

Laws of Matrix Operations

Addition law

$A+B=B+A$ (commutative law)

$c(A+B)=cA+cB$ (distributive law)

$A+(B+C)=(A+B)+C$ (associative law)

Multiplication law

$AB \neq BA$

$A(B+C)=AB+AC$

$(A+B)C=AC+BC$

[distributive laws]

$A(BC)=(AB)C$ (associative law)

Block Matrices & Block Multiplication

check #2(4)

Ex:-

$$E\,[A\,|\,b] = [EA\,|\,Eb]$$

$[A|b]$ has 2 blocks of different sizes.
No problem to multiply blocks times blocks, when their shapes permit.

If blocks of A can multiply blocks of B, then block multiplication of AB is allowed.

* $$\boxed{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]}$$

Proof: $[A\,|\,B]^T = \left[\begin{array}{c} A^T \\ \hline B^T \end{array}\right]$ & $\left[\begin{array}{c} A \\ \hline B \end{array}\right]^T = [A^T\,|\,B^T]$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^T = \begin{bmatrix} \left[\begin{array}{c} A \\ \hline C \end{array}\right]^T \\ \left[\begin{array}{c} B \\ \hline D \end{array}\right]^T \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$$

* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a block symmetric matrix if $A^T = A$, $B^T = C$, & $D^T = D$

Ex: 3. Let blocks of 'A' be its 'n' columns. Let the blocks of 'B' be its 'n' rows. Then block multiplication 'AB' adds up columns times rows:

$$\left[\vec{a}_1 \middle| \vec{a}_2 \middle| \cdots \middle| \vec{a}_n\right]\begin{bmatrix} \vec{B}_1 \\ \hline \vec{B}_2 \\ \hline \vdots \\ \hline B_n \end{bmatrix} = \left[\vec{a}_1\vec{b} + \vec{a}_2\vec{b}_2 + \cdots + \vec{a}_n\vec{b}_n\right]$$

$$\left[\begin{array}{c|c} 1 & 4 \\ 1 & 5 \end{array}\right]\left[\begin{array}{cc} 3 & 2 \\ \hline 1 & 0 \end{array}\right] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Ex: 4 Elimination by blocks.

$$A = \left[\begin{array}{c|c|c} 1 & x & x \\ 3 & x & x \\ 4 & x & x \end{array}\right]; \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$$

Using inverse matrices, a block matrix E can do elimination on a whole (block) column.
Suppose a matrix has 4 blocks A, B, C, D

Block Elimination : $\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right]$

A : pivot block

$S = D - CA^{-1}B$: Schur complement

Applying column operations,

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\underbrace{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}_{M}\left[\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right] = \underbrace{\begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}}_{D_M}$$

- $\text{rank}(M) = \text{rank}(D_M) = \text{rank}(A) + \text{rank}(S)$

rank of a matrix is unchanged under elementary transformations.

Check OM(23)

$\text{rank}[E_1 M E_2] = \text{rank}(D_M) = \text{rank}(M)$

- If M is $n \times n$,

  $$\det(M) = \det(A) \cdot \det S$$

- If M is $n \times n$ and non-singular, then S is also non-singular

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & S \end{array}\right]$$

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & 0 \\ \hline 0 & S \end{array}\right]\left[\begin{array}{c|c} I & A^{-1}B \\ \hline 0 & I \end{array}\right]$$

$$M^{-1} = \left[\begin{array}{c|c} I & -A^{-1}B \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & S^{-1} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]$$

$$= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1}$$

$$M^{-1} = \left[\begin{array}{c|c} A^{-1} & -A^{-1}BS^{-1} \\ \hline 0 & S^{-1} \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]$$

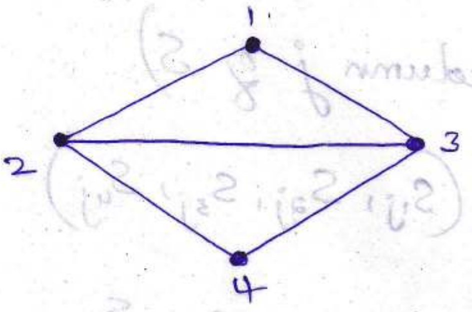
$$\boxed{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ \hline -S^{-1}CA^{-1} & S^{-1} \end{array}\right]}$$

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\cdot\det(S)$$
$$= \det(A)\cdot\det(D - CA^{-1}B)$$

$$\det\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det(A)\cdot\det(D)$$

2.4(A)

A graph or a network has 'n' nodes. Its adjacency matrix S is $n\times n$. This is a 0-1 matrix with $S_{ij}=1$ when nodes i & j are connected by an edge.



$$S=\begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{bmatrix}$$

$(S^2)_{ij}$ counts the walks of length 2 b/w node i and node j.

b/w node 2 & 3 : 2-1-3 & 2-4-3.

\# of walks = 2.

From node 1 to node 1 : 1-2-1 & 1-3-1

$$S^2=\begin{bmatrix}2&1&1&2\\1&3&2&1\\1&2&3&1\\2&1&1&2\end{bmatrix};\quad S^3=\begin{bmatrix}2&5&5&2\\5&4&5&5\\5&5&4&5\\2&5&5&2\end{bmatrix}$$

$(S^3)_{ij}$: counts the walks of length 3 b/w node i and node j.

→ S^N counts all the N-step paths b/w pairs of nodes.

Why?

$$(S^2)_{ij} = (\text{row } i \text{ of } S) \cdot (\text{column } j \text{ of } S)$$

$$= (S_{i1}, S_{i2}, S_{i3}, S_{i4}) \cdot (S_{1j}, S_{2j}, S_{3j}, S_{4j})$$

$$= S_{i1} S_{1j} + S_{i2} S_{2j} + S_{i3} S_{3j} + S_{i4} S_{4j}$$

If there is a 2-step path $i \to 1 \to j$, the 1st multiplication gives $S_{i1} S_{1j} = (1)(1) = 1$.

If $i \to 1 \to j$ is not a path, then either $i \to 1$ or $1 \to j$ is missing. So the multiplication gives $S_{i1} S_{1j} = 0$ in that case.

i.e., $(S^2)_{ij}$ is adding up 1's for all the 2-step paths $i \to k \to j$. So it counts those paths.

* Matrix multiplication is suited to counting paths on a graph - channels of communication b/w employees in a company.

Inverse Matrices

Inverse of an elimination matrix

E_{ij} : subtracts a multiple 'l' of row j from row i.

E_{ij}^{-1} : adds a multiple 'l' of row j to row i

Ex:-

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-5a_{11} & a_{22}-5a_{12} & a_{23}-5a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A^{-1} : Gauss - Jordan Elimination

$$AA^{-1} = A\left[\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n\right] = \left[\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n\right] = I$$

To invert an $n \times n$ matrix A, we have to solve 'n' systems of equations.

$A\vec{x}_1 = \vec{e}_1 = (1,0,0,\cdots,0)$,

$A\vec{x}_2 = \vec{e}_2 = (0,1,0,\cdots 0)$

$\vdots$

$A\vec{x}_n = \vec{e}_n = (0,0,0,\cdots,1)$

The Gauss-Jordan method computes A^{-1} by solving all 'n' equations together.

Ex: $\rightarrow$

$$\begin{bmatrix} K & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} K \mid I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

* Gauss would finish by back substitution. The contribution of Jordan is to continue with the elimination. He goes all the way to the reduced echelon form.

0 above 3rd pivot:
$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

0 above 2nd pivot
$$\rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

divide by 2
" by 3/2
" by 4/3
$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] = \begin{bmatrix} I & x_1 & x_2 & x_3 \end{bmatrix}.$$
$$= \begin{bmatrix} I \mid K^{-1} \end{bmatrix}$$

$$\boxed{\text{Gauss-Jordan: Multiply } [A|I] \text{ by } A^{-1} \text{ to get } [I|A^{-1}]}$$

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow K^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

- K is symmetric across its main diagonal. Then K^{-1} is also symmetric
- K is tridiagonal. But K^{-1} is a dense matrix with no zeros.

  That is one of the reasons we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.

- The product of pivots is $2 \times \frac{3}{2} \times \frac{4}{3} = 4$ is the determinant of K.

* 'A^{-1}' exists exactly when 'A' has a full set of 'n' pivots.

Proof.

- If A does not have 'n' pivots, elimination will lead to a zero row.
- These elimination steps are taken by an invertible matrix M. So a row of MA is zero.
- If $AC = I$ had been possible, then ~~[illegible]~~

$M(AC) = (MA)C = M$

The zero row of MA, times C, gives a zero row of M itself.

- An invertible matrix M can't have a zero row!

$\Longrightarrow$ 'A' must have 'n' pivots if $AC = I$.

* If A is invertible & upper triangular, so is A^{-1}.

* A triangular matrix is invertible iff no diagonal entries are zero.

(strictly)

* Diagonally dominant matrices are invertible.

$$\boxed{|a_{ii}| > \sum_{j\neq i} |a_{ij}| \text{ for all } i \in \{1,2,\dots n\} \implies |A| \neq 0}$$

Ex:- $\begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}$; diagonally dominant.

$\begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$ is not diagonally dominant

$\begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$ is strictly diagonally dominant.

Proof [Elementary]

Assume, ~~let~~ 'A' is a (strictly) diagonally dominant matrix and there exists a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$;

Let x_i is the largest entry of $\vec{x}$ by absolute value. ie; $x_{max} = x_i$

$$\sum_j a_{ij} x_j = a_{ii} x_i + \sum_{j \neq i} a_{ij} x_j = 0$$

$$\Longrightarrow \quad a_{ii} x_i = -\sum_{j \neq i} a_{ij} x_j$$

$$a_{ii} = -\sum_{j \neq i} a_{ij} \frac{x_j}{x_i}$$

$$|a_{ii}| = \left| \sum_{j \neq i} a_{ij} \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} \left| a_{ij} \frac{x_j}{x_i} \right|$$

Since, $\left| \frac{x_j}{x_i} \right| < 1$,

$$|a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \left| \frac{x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{ij}|$$

which is a contradiction

$\Rightarrow A\vec{x} = \vec{0}$ only when $\vec{x} = \vec{0}$.

$\therefore$ A is invertible

Proof [Gershgorin Circle theorem]

Every eigenvalue of A lies within atleast one of the Gershgorin discs $D(a_{ii}, R_i)$

ie.,

Every eigenvalue of matrix A satisfies

$$|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| = R_i \quad , \text{for } i \in \{1, 2, \ldots, n\}$$

Begin by dividing the eigenvector $\vec{x}$ associated with the eigenvalue λ by its largest magnitude entry, so there exists an index i such that $x_i = 1$ & all other entries $|v_j| \leq 1$

The i^{th} row of the eigenvalue equation $A\vec{x} = \lambda\vec{x}$ is :

$$\sum_j a_{ij} x_j = \lambda x_i = \lambda = \sum_{j \neq i} a_{ij} x_j + a_{ii}$$

Applying triangle inequality,

$$|\lambda - a_{ii}| = \left|\sum_{j\neq i} a_{ij} x_j\right| \leq \sum_{j\neq i} |a_{ij}||x_j|$$

$$|\lambda - a_{ii}| \leq \sum_{j\neq i} |a_{ij}| = R_i$$

If $\lambda = 0$ is an eigenvalue of A,
then
there exists an i such that

$$|a_{ii}| \leq \sum_{j\neq i} |a_{ij}|$$

such a matrix is not strictly diagonally dominant.

$\Longrightarrow$ Contradiction

$\Rightarrow$ A strictly diagonally dominant matrix can not have a zero eigenvalue.

$\Rightarrow |A| = \prod \lambda_i \neq 0$.

$\Rightarrow$ A is non-singular.

Gershgorin Circle theorem

Let 'A' be a complex $n \times n$ matrix, with entries a_{ij}. For $i \in \{1, 2, \dots, n\}$, let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the i^{th} row. Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at a_{ii} with radius R_i. Such a disc is called a Gershgorin disc.

* Every eigenvalue of A lies within atleast one of the Gershgorin discs $D(a_{ii}, R_i)$.

Proof

Let 'λ' be an eigenvalue of 'A'. Choose a corresp. eigenvector $\vec{x}$ such that one component x_i is equal to 1 & others are of abs. value less than or equal to 1. ie., $x_i = 1$ & $|x_j| \leq 1$ for $j \neq i$.

There is always such an $\vec{x}$, which can be obtained simply by dividing any eigenvector by its component with largest modulus

$A\vec{x} = \lambda\vec{x}$

$$\sum_j a_{ij} x_j = \lambda x_i = \lambda \cdot 1 = \lambda = \sum_{j \neq i} a_{ij} x_j + a_{ii}$$

$$|\lambda - a_{ii}| = \left|\sum_{j \neq i} a_{ij} x_j\right| \leq \sum_{j \neq i} |a_{ij}||x_j|$$

$$\boxed{|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| = R_i}$$

* The eigenvalues of 'A' must also lies within the Gershgorin discs corresp. to the columns of 'A'.

Proof

Apply the theorem to A^T

$\boxed{Q.5(C)}$ Apply Gauss-Jordan method to invert this triangular Pascal matrix, L. — adding each entry to the entry on its left gives the entry below. The entries of L are "binomial coefficients".

Triangular Pascal matrix, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

Ans: $$[L|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right] = [I|L^{-1}]$$

Elimination = Factorization : A=LU

Elimination without row exchanges

Many key ideas of linear algebra, when you look at them closely, are really factorizations of a matrix.

The factorization that comes from elimination is $A=LU$. The factors L and U are triangular matrices.

Ex:-

Forward from A to U : $E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$

Back from U to A : $E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \boxed{A=LU}$

If 'A' is a 3×3 matrix,

$$E_{32} E_{31} E_{21} A = U \Longrightarrow A = \left(E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}\right) U$$

$$A = LU$$

Elimination without row exchanges

$A = LU$: The upper triangular U has the pivots on its diagonal.

The lower triangular L has all 1's on its diagonal.

- Each pivot is a ratio of upper left determinants.

Ex:1. Elimination subtracts $\frac{1}{2}$ times row 1 from row 2. Last step subtracts $\frac{2}{3}$ times row 2 from row 3.

Ans:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU$$

$$E_{31} = 0, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~A=~~

Ex: 2.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= LU$$

* When a row of 'A' starts with zeros, so does that row of L
* When a column of A start with zeros, so does that column of U.

If a row starts with zero, we don't need an elimination step. 'L' has a zero, which saves computer time.

Similarly, a zero at the start of a column survive into U.

Zeros in the middle of a matrix are likely to be filled in.

A = LU is unsymmetric because U has the pivots on its diagonal where L has 1's.

$\Longrightarrow$ Divide U by a diagonal matrix D that contains the pivots.

$$U = \begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots \\ & d_2 & u_{23} & \cdots \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots \\ & 1 & u_{23}/d_2 & \cdots \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$= DU$

The triangular factorization can be written as:

$$\boxed{A = LU \quad \text{or} \quad A = LDU}$$

One square system = Two triangular systems

The matrix L contains our memory of Gaussian elimination. It holds the numbers that multiplied the pivot rows, before subtracting them from lower rows.

When do we need this & how do we use it in solving $Ax = b$?

We need $\underline{L}$ as soon as there is a right side b. The factors L & U were completely decided by the left side (the matrix A). On the right side of $Ax = b$, we use L^{-1} and U^{-1}. That <u>solve</u> step deals with 2 triangular matrices.

$$L^{-1}A = U \implies A = LU$$

$$U^{-1}L^{-1}A = I$$

$$\therefore \; L^{-1}b = C \;\; \& \;\; U^{-1}L^{-1}b = U^{-1}C = x$$

How does solve work on b ?

The original system $Ax = b$ is factored into 2 triangular systems :

Forward & Backwards : Solve $Lc = b$ and then solve $Ux = c$

To see that x is correct, multiply $Ux = c$ by L. Then $LUx = Lc$ is just $Ax = b$.

Ex:3

$$Ax = b : \quad \begin{aligned} u + 2v &= 5 \\ 4u + 9v &= 21 \end{aligned} \implies \begin{aligned} u + 2v &= 5 \\ v &= 1 \end{aligned} \qquad Ux = c$$

Ans: $L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$$Lc = b : \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} c = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Longrightarrow c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$Ux = c : \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \longrightarrow x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

L & U can go into the n^2 storage locations that originally held 'A' (now forgettable)

The Cost of Elimination

1^{st} stage of elimination produces zeros below the 1^{st} pivot in column 1.
To find each entry below the pivot row requires 1 multiplication & 1 subtraction.

1^{st} stage $\Longrightarrow$ n^2 multiplications & n^2 subtractions.

It is actually less, $n^2 - n$, because row 1 does not change.

The next stage clears out the 2^{nd} column below the 2^{nd} pivot. The working matrix is now of size $(n-1)$.

2^{nd} stage $\Longrightarrow$ $(n-1)^2$ multiplications & subtractions.

Rough count to reach U:

$$n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{3} n \left(n + \frac{1}{2}\right)(n+1)$$

When n is large,

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1) \longrightarrow \frac{n^3}{3}$$

$$\int_0^n x^2\,dx = \left[\frac{x^3}{3}\right]_0^n = \frac{n^3}{3}$$

A to U

Elimination on 'A' requires about $\frac{1}{3}n^3$ multiplications & $\frac{1}{3}n^3$ subtractions.

What about the right side b ?

We subtract multiples of b_1 from the lower components $b_2, \dots, b_n$.

1st stage : $(n-1)$ steps

2nd stage : $(n-2)$ steps.

Back substitution:

Computing x_n uses one step (divide by the last pivot). The next unknown uses 2 steps. When we reach x_1, it will require n steps ($(n-1)$ substitutions of the other unknowns, then division by the 1st pivot).

The total count on the right side, from b to c to x — forward to the bottom & back to the top — is exactly n^2.

$$[(n-1)+(n-2)+\cdots+1]+[1+2+\cdots+n]$$

$$= n+2[1+2+\cdots+(n-1)]$$

$$= n+2\frac{(n-1)(n)}{2} = n^2$$

Solve: Each right side needs n^2 multiplications and n^2 subtractions.

A band matrix 'B' has only 'w' non-zero diagonals.

The zero entries outside the band stay zero in elimination (they are zero in L & U).

Clearing out the 1^{st} column needs w^2 multiplications & subtractions (w zeros to be produced below the pivot, each one using a pivot row of length w).

Then clearing out all 'n' columns to reach U, needs no more than 'nw^2'.

Band Matrix

A to U : $\frac{1}{3}n^3$ reduces to nw^2

Solve : n^2 reduces to $2nw$

2·6(A) The symmetric Pascal matrix P is a product of triangular Pascal matrices L and U.

Ans:
$$P=\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&2&5&9\\0&3&9&19\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&3&10\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}=U$$

$$P=\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}$$

$$=LU$$

$\Rightarrow$ For Pascal, U is the transpose of L

$\boxed{2.6(B)}$ Solve $Px = b = (1,0,0,0)$

Two triangular systems : $Lc = b$ & $Ux = c$

$\underline{\underline{Lc = b}}$

$$\begin{aligned} c_1 \quad\quad\quad\quad &= 1 \\ c_1 + c_2 \quad\quad &= 0 \\ c_1 + 2c_2 + c_3 \quad &= 0 \\ c_1 + 3c_2 + 3c_3 + c_4 &= 0 \end{aligned} \Longrightarrow c = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$\underline{\underline{Ux = c}}$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_2 + 2x_3 + 3x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ x_4 &= -1 \end{aligned} \Longrightarrow x = \begin{bmatrix} 4 \\ -6 \\ 4 \\ -1 \end{bmatrix}$$

Transposes & Permutations

$$(A^T)_{ij} = A_{ji}$$

We defined A^T by flipping the matrix across its main diagonal. That's not mathematics!

A^T is the matrix that makes these 2 inner products equal for every x and y:

$$\boxed{\begin{aligned}&(A\vec{x})\cdot\vec{y} = \vec{x}\cdot(A^T\vec{y}) \iff (A\vec{x})^T\vec{y} = \vec{x}^T(A^T\vec{y})\\ &\langle A\vec{x},\vec{y}\rangle = \langle \vec{x}, A^T\vec{y}\rangle\end{aligned}}$$

* $(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$(A^{-1})^T = (A^T)^{-1}$

* transpose of a lower triangular matrix is upper triangular.

Inverse of a lower triangular matrix is lower triangular.

* $A = LDU \Longrightarrow A^T = U^T D^T L^T$

* A^T is invertible exactly when A is invertible.

$$A^{-1}A = I \Longrightarrow A^T(A^{-1})^T = I$$

$$\underline{A^T(A^T)^{-1} = I}$$

□ Symmetric matrices in elimination

$S^T = S$

$S = LDU \longrightarrow S^T = U^T D^T L^T = U^T D L^T = LDU = S$

$\Longrightarrow S = LDL^T$, where $U = L^T$

> If $S = S^T$ is factored into LDU with no row exchanges, then $U = L^T$
>
> The symmetric factorization of a symmetric matrix is, $S = LDL^T$

$S^T = S$ makes elimination faster, because we can work with half the matrix (plus the diagonal).

The work of elimination is cut in half, from $\frac{n^3}{3}$ multiplications to $\frac{n^3}{6}$.

The storage is also cut in half. We only keep L and D, not U which is just L^T

$PA = LU$ Factorization with Row Exchanges

Permutation matrices

* A permutation matrix P has the rows of the identity I in any order.

* There are $n!$ permutation matrices of order n.

There are $3! = 6$ 3 by 3 permutation matrices

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{32} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{21} P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

PA = LU

$$A = (E_{ij}^{-1} \, E^{-1} \cdots) U = LU$$

Every elimination step was carried out by an E_{ij} & it was inverted by E_{ij}^{-1}. Those inverses were compressed into one matrix L.

$\Longrightarrow \quad A = LU$

But, it doesn't work always!

Sometimes, row exchanges are needed to produce pivots.

$$A = (E^{-1} \cdots P^{-1} \cdots E^{-1} \cdots P^{-1} \cdots) U$$

We now compress those row exchanges into a single permutation matrix P.

There are 2 ways to collect the P_{ij}'s :

1. Row exchanges done in advance.

- Their product P puts the rows of A in the right order, so that no exchanges are needed for PA :

$$\Longrightarrow \boxed{PA = LU}$$

2. If we hold row exchanges until after elimination.

- the pivot rows are in a strange order. P_1 puts them in the correct triangular order in U_1.

$$\Longrightarrow \boxed{A = L_1 P_1 U_1}$$

Ex:-

$$\underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \longrightarrow \underset{PA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \longrightarrow \underset{l_{31} = 2}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}}$$

$$\longrightarrow \underset{l_{32} = 3.}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU$$

* If 'A' is invertible, a permutation ~~P~~ will put its rows in the right order to factor $PA = LU$.

* There must be a full set of pivots after row exchanges for 'A' to be invertible.

$$\operatorname{sgn}(P) = \begin{cases} +1, & \text{\# of row exchanges is even} \\ -1, & \text{\# of row exchanges is odd} \end{cases}$$