# $A_n$ is simple for $n \geq 5$

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# Background

## DEFINITION

**Simple groups**: A group is simple when it is nontrivial and there are no normal subgroups besides the trivial group and the group itself.

WHY $n \geq 5$ ?

## HOW

**Lemmas:**

- 3-cycles
- Conjugates
- $A_5$ and $A_6$

**Theorem:** $A_n$ is simple for $n \geq 5$

**Application**

## WHO

# Lemma 1: $A_n$ is generated by 3-cycles

*Proof:*

Identity element: $e = (1) = (1\ 2\ 3)(1\ 3\ 2)$

Nonidentity elements: $\sigma = \tau_1 \tau_2 ... \tau_r$ where $\sigma$ is a product of transpositions.

Case 1: $\tau_i$ and $\tau_{i+1}$ are equal.

We see that $\tau_i\tau_{i+1} = (1) = (123)(132)$. Therefore, $\tau_i\tau_{i+1}$ is the product of two 3-cycles.

Case 2: $\tau_i$ and $\tau_{i+1}$ have exactly one element in common.

Let the common element be a, so let $\tau_i = (ab)$ and $\tau_{i+1} = (ac)$ where $b \neq c$. From this we have $\tau_i\tau_{i+1} = (ab)(ac) = (acb) = (abc)(abc)$. Therefore, $\tau_i\tau_{i+1}$ is the product of two 3-cycles.

Case 3: $\tau_i$ and $\tau_{i+1}$ are disjoint.

Let $\tau_i = (ab)$ and $\tau_{i+1} = (cd)$. Then $\tau_i\tau_{i+1} = (ab)(cd) = (ab)(bc)(bc)(cd) = (bca)(cdb) = (abc)(bcd)$. Therefore, $\tau_i\tau_{i+1}$ is the product of two 3-cycles. $\square$

**Lemma 1.3: For n ≥ 5, the conjugate of all 3-cycles in $A_n$ are 3-cycles.**

## Lemma 1.2: For n ≥ 5, all 3-cycles in $A_n$ are conjugate in $A_n$.

- Consider a 3-cycle (abc). We show that it is conjugate to (123). Conjugacy is an equivalence relation, so it follows that all 3-cycles are conjugate.

$$(123) = [(1a)(2b)(3c)](abc)[(1a)(2b)(3c)]^{-1}.$$

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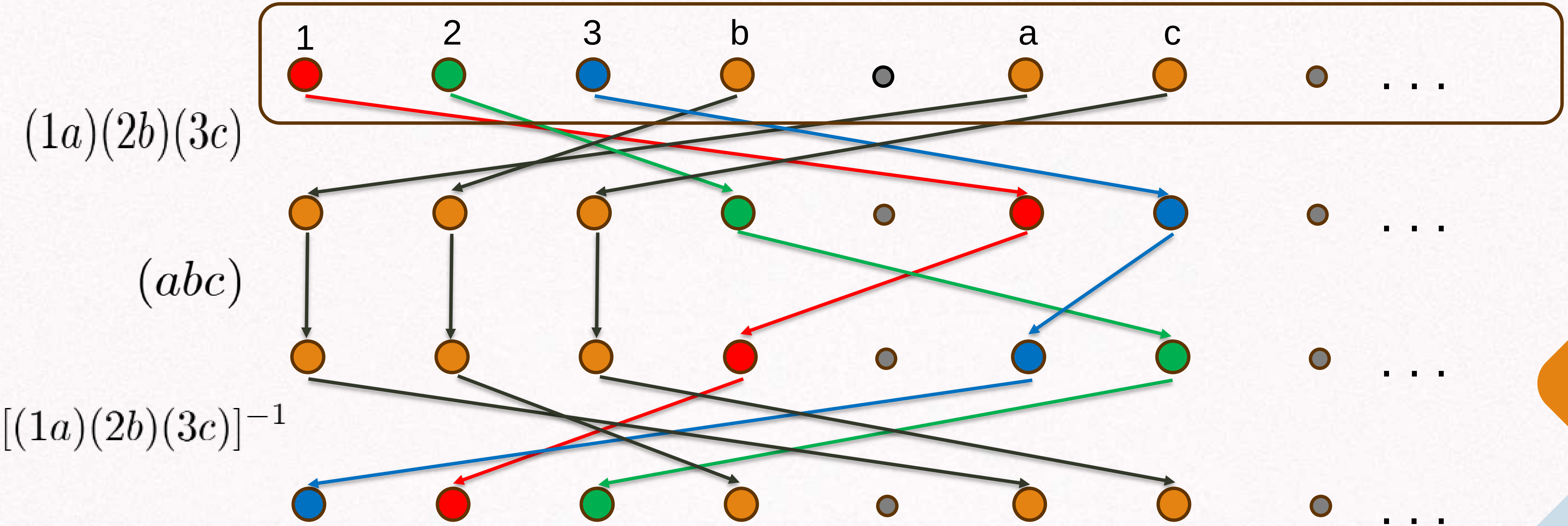

# Lemma 1.2: For $n \geq 5$, all 3-cycles in $A_n$ are conjugate in $A_n$.

- Case 1: $(1a)(2b)(3c) \in A_n$
  - (abc) and (123) are conjugate in $A_n$, as desired
- Case 2: $(1a)(2b)(3c) \notin A_n$
  - We append (45) to the permutation to make it even; the claim follows

$$\begin{aligned}(123) &= (45)(123)(45) \\ &= (45)[(1a)(2b)(3c)](abc)[(1a)(2b)(3c)]^{-1}(45) \\ &= [(45)(1a)(2b)(3c)](abc)[(45)(1a)(2b)(3c)]^{-1}\end{aligned}$$

# $A_5$ is Simple

- Recall:
  - Conjugacy is an equivalence relation
  - Lagrange's Theorem: The order of a subgroup divides the order of the group
- The order of all conjugacy classes in $A_5$ are given in the following table:

Table 1: $A_5$ Conjugacy Classes

| Representative | e | (12345) | (21345) | (12)(34) | (123) |
|---|---|---|---|---|---|
| Order | 1 | 12 | 12 | 15 | 20 |

- Any normal subgroups of $A_5$ must be the disjoint union of conjugacy classes in the table including {e}. However, none of the orders of each subgroup add to a proper divisor (except 1) of $|A_5| = 5! = 120$, so $A_5$ is simple.
- $A_6$ is simple by a similar argument

# $A_n$ is Simple for $n > 6$

Each $N \trianglelefteq A_n$ s.t. $N \neq \{e\}$ contains a 3-cycle

Lemma 1.2

$N$ contains all 3-cycles

Lemma 1

$N = A_n$

# Each $N \trianglelefteq A_n$ s.t. $N \neq \{e\}$ contains a 3-cycle

$H \cong A_6$ s.t. $H \subseteq N$

⬇

$A_6$ contains even permutations

⬇

$A_6$ contains 3-cycles

⬇

$H$ contains 3-cycles

⬇

$N$ contains 3-cycles

# H$\cong A_6$ s.t. $H \subseteq N$

Let

$\sigma \in N \trianglelefteq A_n$ a non-identity element, i.e., $\sigma(l) \neq l$ for some $l \in \{1, 2, \cdots, n\}$

$\tau = (i\ j\ k) \in A_n$ where $i, j, k \neq l$ and $\sigma(l) \in \{i, j, k\}$

$$\sigma, \tau\sigma\tau^{-1} \in N \implies \mu = (\tau\sigma\tau^{-1})\sigma^{-1} \in N$$

$\tau^{-1}$ is a 3-cycle

Lemma 1.3

$$\tau\sigma\tau^{-1}(l) = \tau(\sigma(l)) \neq \sigma(l)$$

$$\therefore \tau\sigma\tau^{-1} \neq \sigma$$

$$\mu = \tau\sigma\tau^{-1}\sigma^{-1} \neq (1)$$

$\sigma\tau^{-1}\sigma^{-1}$ is also a 3-cycle

$\mu = \tau(\sigma\tau^{-1}\sigma^{-1}) \in N$ is a product of two 3-cycles

## H$\cong A_6$ s.t. $H \subseteq N$

$\mu = \tau(\sigma\tau^{-1}\sigma^{-1}) \in N$ is a product of two 3-cycles

⬇

$\mu$ permutes at most 6 numbers in $\{1, \cdots, n\}$

⬇

$H$ : Even permutation of these numbers (augmented)

⬇

$H \cong A_6 \quad \& \quad \mu \in H$

**Is $H$ a subset of $N \trianglelefteq A_n$ ?**

## H$\cong A_6$ s.t. $H \subseteq N$

$h \in A_n$ *and* $n \in N \cap H \implies n \in N$ *and* $N \trianglelefteq A_n$

$\therefore \quad hnh^{-1} \in N$

$h \in H \implies h^{-1} \in H$ *and* $n \in N \cap H \implies n \in H$

$\therefore \quad hnh^{-1} \in H$

For any $h \in H \leq A_n$ & $n \in N \cap H$

$hnh^{-1} \in N \cap H$

$N \cap H \trianglelefteq H$

$H \cong A_6$ Simple

$N \cap H \in \{(1), H\}$

$\mu \in H$

$\mu \in N$

$\mu \neq (1)$

$\mu \in N \cap H$

$N \cap H$ is non-trivial

$N \cap H = H$

$H \subseteq N$

## Each $N \trianglelefteq A_n$ s.t. $N \neq \{e\}$ contains a 3-cycle

$H \cong A_6$ s.t. $H \subseteq N$

↓

$A_6$ contains even permutations

↓

$A_6$ contains 3-cycles

↓

$H$ contains 3-cycles

↓

$N$ contains 3-cycles

# $A_n$ is Simple for $n > 6$

Each $N \trianglelefteq A_n$ s.t. $N \neq \{e\}$ contains a 3-cycle

Lemma 1.2

$N$ contains all 3-cycles

Lemma 1

$N = A_n$

# Application
# s

# Solvability of Quintics

A polynomial is solvable by radical formula ⟺ The Galois group is solvable

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A polynomial is solvable by radical formula ⟺ The Galois group is solvable

$A_5$ is simple ⟹ $S_5$ is not solvable

# Solvability of Quintics

A polynomial is solvable by radical formula ⟺ The Galois group is solvable

$A_5$ is simple ⟹ $S_5$ is not solvable

There is a quintic with Galois group $S_5$ ⟹ Quintics are not, in general, solvable by radical formula